HIGHER RANK POLYHEDRAL GEOMETRY I: GENERAL THEORY

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Abstract. With the aim of starting a systematic development of higher rank tropical geometry, we develop a theory of *higher rank polyhedral geometry* over the ordered ring of *generalized dual numbers* $\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^k)$. We generalize several classical results to this context, including, but not limited to, Fourier-Motzkin Elimination, Farkas' Lemma, the Minkowski-Weyl decomposition and the basic results on the duality theory of cones and the theory of normal fans of polyhedra.

We use this theory to endow tropical hypersurfaces of higher rank with the structure of a polyhedral complex over $\mathbb{D}$. As a first application, we show how tropical hypersurfaces of higher rank are dual to *layered regular subdivision* of their Newton polytope. This answers a question of Joswig and Smith. This duality is obtained naturally from the duality machinery for polyhedra over $\mathbb{D}$ developed in the article.

Contents

1. Elementary Concepts and Key Theorems

In the following we will work with the *ring of generalized dual numbers of rank k* defined by $\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^k)$. For $k = 2$ it recovers the usual ring of dual numbers. Elements of $\mathbb{D}$ have the form

$$x = x^{(0)} + x^{(1)}\varepsilon + \cdots + x^{(k-1)}\varepsilon^{k-1} \tag{1.1}$$

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with $x^{(0)}, \dots, x^{(k-1)} \in \mathbb{R}$. They are manipulated as usual power series with coefficients in $\mathbb{R}$ imposing that $\varepsilon^k = 0$, in the same way as one works with Taylor expansions of the form $a_0 + a_1 z + \cdots + a_{k-1} z^{k-1} + o(z^{k-1})$. If the rank of $\mathbb{D}$ has to be made explicit, we use a lower index $\mathbb{D}_k$.

For an element $x \in \mathbb{D}$ as in (1.1) and for $0 \leq i \leq k-1$, we use the notation

$$x^{[i]} := x^{(0)} + x^{(1)}\varepsilon + \cdots + x^{(i)}\varepsilon^i \in \mathbb{D}$$

Moreover, we introduce the *order* of x by $\operatorname{ord}(x) := \min\{j \in \{0, \dots, k-1\} \mid x^{(j)} \neq 0\}$ if $x \neq 0$ and $\operatorname{ord}(0) = k$.

Notice that $\mathbb{D}$ is isomorphic to $\mathbb{R}^k$ as an additive group. We endow $\mathbb{D}$ with the lexicographic order

$$a^{(0)} + \varepsilon a^{(1)} + \cdots + \varepsilon^{k-1} a^{(k-1)} < b^{(0)} + \varepsilon b^{(1)} + \cdots + \varepsilon^{k-1} b^{(k-1)} \tag{1.2}$$

$$\iff \quad a^{(i)} < b^{(i)} \text{ for the first } i \text{ such that } a^{(i)} \neq b^{(i)}. \tag{1.3}$$

In this way, we obtain an order on $\mathbb{D}$ that we simply denote by $\leq$. This order turns out to be compatible with the additive and multiplicative structure of $\mathbb{D}$ turning it into an ordered ring.

Remark 1.1. The following observation is useful and will be used sometimes in the arguments. Given $a, b \in \mathbb{D}$, we have $a \leq b$ with the lexicographic order introduced in (1.2) iff we have

$$a^{(0)} + \delta a^{(1)} + \cdots + \delta^{k-1} a^{(k-1)} \leq b^{(0)} + \delta b^{(1)} + \cdots + \delta^{k-1} b^{(k-1)}$$

for every $\delta \in \mathbb{R}_{>0}$ small enough.

Given a lattice $N \cong \mathbb{Z}^n$ with dual lattice $M = \operatorname{Hom}(N, \mathbb{Z})$ we consider the base changes $N_{\mathbb{D}} = N \otimes \mathbb{D}$ and $M_{\mathbb{D}} = M \otimes \mathbb{D}$. The pairing $M \otimes N \to \mathbb{Z}$ naturally extends to a pairing $M_{\mathbb{D}} \otimes N_{\mathbb{D}} \to \mathbb{D}$ which we denote by $\langle \cdot \,, \cdot \rangle$.

Remark 1.2. Under the pairing $\langle \cdot \,, \cdot \rangle$, the $\mathbb{D}$-linear functions from $N_{\mathbb{D}}$ to $\mathbb{D}$ correspond exactly to the elements of $M_{\mathbb{D}}$. For this reason, we decide to write y instead of $\langle y \,, \cdot \rangle$ when there is no risk of confusion. More generally, the affine functions from $N_{\mathbb{D}}$ to $\mathbb{D}$ are all of the form $\langle y \,, \cdot \rangle + a$ for some $y \in M_{\mathbb{D}}$ and $a \in \mathbb{D}$.

Using the ordered ring structure on $\mathbb{D}$ we can introduce several geometric concepts over the module $N_{\mathbb{D}}$.

Definition 1.3.

(1) A set $P \subseteq N_{\mathbb{D}}$ is *convex* if for any $x, y \in P$ and any $t \in \mathbb{D}$ such that $0 \leq t \leq 1$, we have

$$tx + (1-t)y \in P.$$

(2) A set $\sigma \subseteq N_{\mathbb{D}}$ is a *cone* if for any $x, y \in \sigma$ and any $t \in \mathbb{D}_{\geq 0}$, we have

$$tx \in \sigma \text{ and } x + y \in \sigma.$$

Notice that, by definition, every cone in this document is convex. As an example of how to work with this ring, we will show that for subsets of the ordered ring $\mathbb{D}$, the notion of convexity agrees with its counterpart from order theory.

Proposition 1.4. *A set $C \subseteq \mathbb{D}$ is convex iff it has the following property:*

$(*)$ *For each $x, y, z \in \mathbb{D}$ such that $x \leq y \leq z$ and $x, z \in C$, we have $y \in C$.*

Proof. If $x \leq y$ then $x \leq tx + (1-t)y \leq y$ for every $0 \leq t \leq 1$. Hence, if C satisfies property $(*)$, then, for every $x, y \in C$ we have $tx + (1-t)y \in C$. Therefore C is convex.

On the other hand, suppose that $x \leq y \leq z$ and $x, z \in C$. If $z - x$ is invertible we can consider the expression

$$y = \frac{y-x}{z-x} z + \left(1 - \frac{y-x}{z-x}\right) x.$$

More generally, we have $0 \leq y - x \leq z - x$. Hence $\operatorname{ord}(z-x) \leq \operatorname{ord}(y-x)$, so we can take elements $a, b \in \mathbb{D}$ with b invertible such that

$$z - x = b\varepsilon^{\operatorname{ord}(z-x)} \quad \text{and} \quad y - x = a\varepsilon^{\operatorname{ord}(z-x)}.$$

If we define $t = a/b$ we have $0 \leq t \leq 1$ and $tb = a$, hence

$$tz + (1-t)x = t(z-x) + x = tb\varepsilon^{\operatorname{ord}(z-x)} + x = a\varepsilon^{\operatorname{ord}(z-x)} + x = (y-x) + x = y.$$

Therefore, $y \in C$. □

Some elementary examples of convex sets and cones in any dimension are given by the half-spaces which we introduce as follows.

Definition 1.5. A *half-space* is a subset of $N_{\mathbb{D}}$ of the form

$$H := \{x \in N_{\mathbb{D}} \mid \langle y\,, x\rangle \geq a\}$$

for some $y \in M_{\mathbb{D}}$ and $a \in \mathbb{D}$. To simplify notations, we frequently write this as $H = \{y \geq a\}$. For a given subring $R \subseteq \mathbb{D}$, if we can take $y \in M_R$ we say that H is *R-rational.* If moreover we can take $a \in R$ we say that H is *strongly R-rational.* If $a = 0$, then H is a half-space *going through the origin.*

Of special interest for us are the convex sets and cones which are defined in terms of finitely many data. One approach to this is to represent them from *outside* as an intersection of half-spaces. This leads to the following definition.

Definition 1.6.

(1) A *polyhedron* is a finite intersection of half-spaces. We say that a polyhedron is *R-rational* (resp. *strongly R-rational*) for a subring $R \subseteq \mathbb{D}$ if we can take each half-space in the intersection to be R-rational (resp. strongly R-rational).

(2) A *polyhedral cone* is a finite intersection of half-spaces going through the origin. A polyhedral cone is *R-rational* for a subring $R \subseteq \mathbb{D}$ if we can take each half-space in the definition to be R-rational itself.

In order to manage the data defining a polyhedron we consider the following.

Definition 1.7. Given a polyhedron $P \subseteq N_{\mathbb{D}}$, a *representation* of P is an equality of the form

$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}. \tag{1.4}$$

for some $y_1, \dots, y_r \in M_{\mathbb{D}}$ and $a_1, \dots, a_r \in \mathbb{D}$. This representation is *non-redundant* if it is not possible to obtain a different representation by removing an inequality of the form $\{y_i \geq a_i\}$ from the intersection.

If we allow ourselves to use affine functions instead of linear functions, Equation (1.4) can be written as

$$P = \{f_1 \geq 0, \dots, f_r \geq 0\} \quad \text{for } f_i = \langle y_i\,, \cdot \rangle - a_i.$$

Proposition 1.8. *Given a non-empty polyhedron with a non-redundant representation as in (1.4). For any $1 \leq i \leq r$, the function y_i attains its minimum on P and this minimum is a_i.*

Proof. Of course a_i is a lower bound for the values of y_i over P. We will show that this lower bound is attained. For this consider the set

$$\bigcap_{\substack{1 \leq j \leq r \\ j \neq i}} \{x \in N_{\mathbb{D}} \mid \langle y_j\,, x\rangle \geq a_j\}\,.$$

This is a convex set and hence its image under $\langle y_i\,, \cdot\rangle$ is a convex set as well that we denote by $C \subseteq \mathbb{D}$. As P is non-empty we have $C \cap [a_i, \infty) \neq \varnothing$, and as the representation is non-redundant we have $C \cap (-\infty, a_i) \neq \varnothing$. So, by Proposition 1.4 we have $a_i \in C$. This shows that $\min_P \langle y_i\,, \cdot\rangle = a_i$. $\square$

One can alternatively construct cones and convex sets from *inside* by means of generators as follows.

Definition 1.9. For a non-empty subset $X \subseteq N_{\mathbb{D}}$,

(1) The *convex hull* of X, denoted by $\operatorname{conv}_{\mathbb{D}}(X)$, is the smallest convex set containing X. Alternatively, this set equals

$$\left\{ \sum_{i=1}^{r} t_i x_i \in N_{\mathbb{D}} \;\middle|\; r \geq 1, x_1, \dots, x_r \in X \text{ and } t_1 \dots, t_r \in \mathbb{D}_{\geq 0} \text{ s.t } \sum_{i=1}^{r} t_i = 1 \right\}.$$

A convex set P is said to be a *polytope* if there is a finite set $X \subseteq N_{\mathbb{D}}$ such that $\operatorname{conv}_{\mathbb{D}}(X) = P$.

(2) The *cone generated by* X, also known as the *cone hull of* X, denoted by $\operatorname{cone}_{\mathbb{D}}(X)$, is the smallest cone containing X. Alternatively,

$$\operatorname{cone}_{\mathbb{D}}(X) = \left\{ \sum_{i=1}^{r} t_i x_i \in N_{\mathbb{D}} \;\middle|\; r \geq 1, x_1, \dots, x_r \in X \text{ and } t_1 \dots, t_r \in \mathbb{D}_{\geq 0} \right\}.$$

A cone σ is said to be *finitely generated* if there is a finite set $X \subseteq N_{\mathbb{D}}$ such that $\operatorname{cone}_{\mathbb{D}}(X) = \sigma$.

Definition 1.10 (Face)**.** Let P be a polyhedron in $N_{\mathbb{D}}$ and consider an element $y \in M_{\mathbb{D}}$ such that

$$\min_{x \in P} \langle y\,, x\rangle$$

exists. Then, the *face* of P determined by y is the subset consisting of all elements in P for which y attains its minimum, that is,

$$\operatorname{face}_y P := \left\{ x \in P \;\middle|\; \langle y\,, x\rangle = \min_{x' \in P} \langle y\,, x'\rangle \right\}.$$

A face of P is a set of the form $\operatorname{face}_y P$ for some $y \in M_{\mathbb{D}}$.[1] We write $F \preceq P$ if F is a face of P.

Remark 1.11.

(1) A linear function can be bounded below and still not attain its minimum. For example, consider
$$P = \{x \in \mathbb{D} \mid \varepsilon x \geq 0\}.$$
and the linear function $x \mapsto x$. This function is bounded below in P but does not attain its minimum. This phenomenon has to be kept in mind as minimizing functions plays an important role along the theory.

(2) If σ is a polyhedral cone, then $y \in M_{\mathbb{D}}$ attains its minimum over σ iff y is non-negative over σ, and in this case, the minimum has to be zero.

(3) A face of P is given by adding one equality, hence two inequalities, to the expression defining P. Therefore, it is a polyhedron itself.

(4) If P has a representation of the form
$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\},$$
then it is not true in general that all faces are obtained by adding equalities in this expression of the form $y_i = a_i$. To obtain some faces we may have to add equalities of the form $\varepsilon^\alpha y_i = \varepsilon^\alpha a_i$. In other words, we not only need to consider the locus of points $x \in P$ in which $\langle y_i \, , x \rangle$ is a minimum, but also the locus of points $x \in P$ in which the first $\alpha - k$ coordinates of $\langle y_i \, , x \rangle$ coincide with the minimum. This is a core reason why the combinatorics of faces in this theory is more subtle and captures more information about the minimization of functions with lexicographic values. See Proposition 2.1 for a precise statement about this.

In order to use results about polyhedral cones over polytopes, one can go from the perfect pairing $M_{\mathbb{D}} \times N_{\mathbb{D}} \to \mathbb{D}$ to the *extended perfect pairing* defined as follows.

Definition 1.12. The *extended perfect pairing* is given by
$$\begin{aligned} (M_{\mathbb{D}} \times \mathbb{D}) \times (N_{\mathbb{D}} \times \mathbb{D}) &\longrightarrow \mathbb{D} \\ ((y,a),(x,b)) &\longmapsto \langle (y,a) \, , (x,b) \rangle \coloneqq \langle y \, , x \rangle + ab. \end{aligned}$$

In this context, a *lower face* (resp. upper face) of a polyhedron $P \subseteq N_{\mathbb{D}} \times \mathbb{D}$, is a face of the form $\operatorname{face}_{(y,1)(P)}$ (resp. $\operatorname{face}_{(y,-1)(P)}$) for some $y \in M_{\mathbb{D}}$.

Definition 1.13 (Face Poset)**.** The *face poset* of P is the partially ordered set
$$\mathfrak{F}(P) \coloneqq \{F \subseteq P \mid F \text{ is a face of } P\} \cup \{\varnothing\}$$
where the order is given by the inclusion of sets. Moreover, we denote by $\mathfrak{F}(P)^*$ the *reduced face poset* of P given by $\mathfrak{F}(P) \setminus \{\varnothing\}$.

Remark 1.14.

(1) If we consider $F, G \in \mathfrak{F}(P)^*$ such that $G \subseteq F$, then G is a face of F. Indeed, if $G = \operatorname{face}_y P$ for some $y \in M_{\mathbb{D}}$ then, $G = \operatorname{face}_y F$ for the same y. This shows that we can replace $\subseteq$ by $\preceq$ as the order relation in the definition of $\mathfrak{F}(P)^*$.

[1] Notice that with our definition, the empty-set is not considered to be a face. This differ with the definitions of some authors.

(2) In Corollary 2.3 we will see that if F is a face of P and G is a face of F, then G is a face of P. Therefore, $\mathfrak{F}(F) \subseteq \mathfrak{F}(P)$ for any face F of P.
(3) Also in Corollary 2.3 we will show that a non-empty intersection of faces of P is a face of P. This shows that $\mathfrak{F}(P)$ is an order lattice. That is, every pair $\{F, G\} \subseteq \mathfrak{F}(P)$ has an infimum given by $F \wedge G := F \cap G$ and a supremum given by

$$F \vee G := \bigcap_{\substack{H \in \mathfrak{F}(P) \\ H \supseteq F \cup G}} H.$$

We will work with more general families of polyhedra besides the set of faces of a given polyhedron. The properties of these families are captured in the following definition.

Definition 1.15. A *polyhedral complex* in $N_{\mathbb{D}}$ is a collection of polyhedra Σ in $N_{\mathbb{D}}$ with the following two properties:

(1) Given $F, G \in \Sigma$, the intersection $F \cap G$ is either empty or a face of both F and G.
(2) If F is a face of G, and $G \in \Sigma$, then $F \in \Sigma$.

The elements of Σ are called the *cells* or *faces* of Σ. Given a polyhedral complex Σ in $N_{\mathbb{D}}$, its *support* is the set

$$|\Sigma| := \bigcup_{F \in \Sigma} F \subseteq N_{\mathbb{D}}.$$

If $P = |\Sigma|$ is itself a polyhedron, we say that Σ is a *subdivision* of P. More generally, if Σ_1 and Σ_2 are polyhedral complexes such that, $|\Sigma_1| = |\Sigma_2|$ and for every $F \in \Sigma_2$ there is a $G \in \Sigma_1$ such that $F \subseteq G$. Then, Σ_2 is said to be a *refinement* of Σ_1 and we write $\Sigma_1 \preceq \Sigma_2$. If every face of Σ is a polyhedral cone, we say that Σ is a *fan* in $N_{\mathbb{D}}$.

Remark 1.16. Some basic results concerning the definitions will come naturally after developing the theory.

(1) In Corollary 3.12 we will prove that a polyhedron that is a cone in the sense of Definition 1.3 is a polyhedral cone.
(2) In Proposition 1.27 we prove that polytopes are polyhedra and finitely generated cones are polyhedral cones.
(3) Conversely, in Proposition 3.2 we show that polyhedral cones are finitely generated cones. Hence, the concept of finitely generated cones and polyhedral cones coincide.
(4) In Proposition 3.28 we obtain a criterion to determine which polyhedra are polytopes: A polyhedron P is a polytope iff every linear function achieve its minimum in P.

1.1. **The Fibration Point of View.** Notice that for positive integers $i < j$, there is an order preserving surjective ring morphism

$$\mathbb{D}_j = \mathbb{R}[\varepsilon]/(\varepsilon^j) \to \mathbb{D}_i = \mathbb{R}[\varepsilon]/(\varepsilon^i)$$

given by modding out by the ideal (ε^i). For a given rank k, we can fit all the projections to the lower rank rings together in the sequence

$$\mathbb{D} := \mathbb{D}_k \to \mathbb{D}_{k-1} \to \cdots \to \mathbb{D}_1 = \mathbb{R}. \tag{1.5}$$

We propose to study this sequence, and many different sequences that can be deduced from it, geometrically. To do this we introduce the following concept.

Definition 1.17. For a given lattice N, an *iterated fibration of subsets of* $N_{\mathbb{R}}$ or simply, an *iterated fibration*, is a diagram of sets of the form

$$X^{[r]} \xrightarrow{\pi_{r-1}} X^{[r-1]} \xrightarrow{\pi_{r-2}} \dots \xrightarrow{\pi_1} X^{[0]}$$

where each map is surjective, $X^{[0]} \subseteq N_{\mathbb{R}}$ and for each $x \in X^{[i]}$ the fiber $\pi_i^{-1}(x)$ can be identified with a subset of $N_{\mathbb{R}}$, denoted by $X_x^{[i+1]}$.

In this sense, the sequence in (1.5) is an iterated fibration of subsets of $\mathbb{R}$ in which each fiber is equal to $\mathbb{R}$ itself.

More generally, by extension of scalars, the diagram in (1.5) induce the sequence of projections

$$N_{\mathbb{D}} = N_{\mathbb{D}_k} \to N_{\mathbb{D}_{k-1}} \to \cdots \to N_{\mathbb{D}_1} = N_{\mathbb{R}} \tag{1.6}$$

Given a subset $X \subseteq N_{\mathbb{D}}$ and an integer $0 \leq r \leq k$, we define the set $X^{[r]}$ as the image of X under the projection to $N_{\mathbb{D}_r}$. In this way, there is a sequence of projections

$$X = X^{[k]} \to X^{[k-1]} \to \cdots \to X^{[0]}$$

which allows us to regard X as an iterated fibration of subsets of $N_{\mathbb{R}}$. Given $x \in X^{[i]}$ its fiber at x is the set

$$X_x^{[i+1]} = \{y \in N_{\mathbb{R}} \mid x + \varepsilon^i y \in X^{[i+1]}\}.$$

In order to get an idea of the objects involved, let us start with a small example.

Example 1.18. Consider $k = 2$, $N = \mathbb{Z}^2$ and the polyhedral cone

$$\sigma = \{(x_1, x_2) \in \mathbb{D}^2 \mid x_1, x_2 \geq 0\}.$$

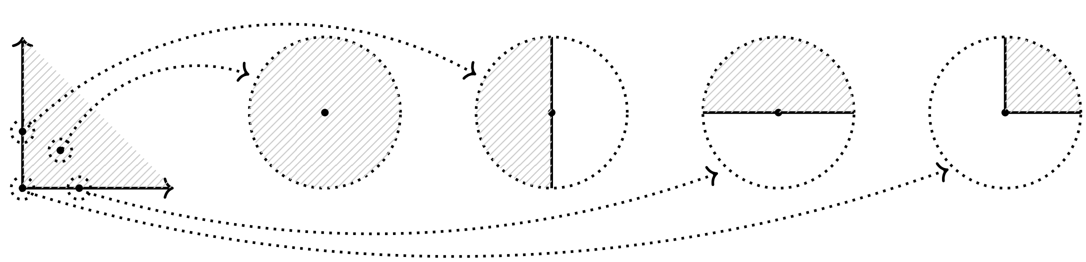
Notice that in order for $x = x^{(0)} + \varepsilon x^{(1)}$ to be positive we should have either $x^{(0)} > 0$ or $x^{(0)} = 0$ and $x^{(1)} \geq 0$. Therefore, if we regard σ as a fibration, its base is

$$\sigma^{[0]} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0\}$$

and the possible fibers are

$$\mathbb{R}^2,\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0\},\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\},\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0\}$$

depending on the position of the base-point as the following picture represents.



Another way to describe this fibration is as follows: The point $(x_1^{(0)} + \varepsilon x_1^{(1)}, x_2^{(0)} + \varepsilon x_2^{(1)})$ belongs to σ iff $(x_1^{(0)}, x_2^{(0)})$ belongs to $\sigma^{[0]}$ and $(x_1^{(1)}, x_2^{(1)})$ is a tangent point at $(x_1^{(0)}, x_2^{(0)})$ *pointing inside* to $\sigma^{[0]}$.

The example above is a special case of the notion of tangent cone bundle which we now introduce. This object has been already defined for polyhedral cone complexes in [AI21] where it plays a major role.

Definition 1.19. Given a set $A \subseteq N_{\mathbb{R}}$ and a point $x \in A$ we define the *tangent cone* of A at x as the set $T\mathcal{C}_x A$ of all vectors y in $N_{\mathbb{R}}$ such that $x + \delta y \in A$ for each $\delta \in \mathbb{R}_{>0}$ small enough. The *tangent cone bundle* of A is then the disjoint union

$$T\mathcal{C}A \coloneqq \bigsqcup_{x \in A} \{x\} \times T\mathcal{C}_x A$$

together with the projection $T\mathcal{C}A \to A$ given by $(x_0, x_1) \mapsto (x_0)$.

We can extend this definition inductively to an iterated fibration

$$T\mathcal{C}^r A \to T\mathcal{C}^{r-1} A \to \cdots \to T\mathcal{C}^1 A \to A,$$

by fixing $T\mathcal{C}^1 A \coloneqq T\mathcal{C}A$, and for $r \geq 1$ and $(x_0, \ldots, x_r) \in T\mathcal{C}^r A$

$$T\mathcal{C}^{r+1}_{(x_0 \ldots, x_r)} A \coloneqq T\mathcal{C}_{x_r}(T\mathcal{C}^r_{(x_0, \ldots, x_{r-1})} A).$$

Then, we have

$$T\mathcal{C}^{r+1} A \coloneqq \bigsqcup_{(x_0, \ldots, x_r) \in T\mathcal{C}^r A} \{(x_0, \ldots, x_r)\} \times T\mathcal{C}^{r+1}_{(x_0, \ldots, x_r)} A$$

together with the map $T\mathcal{C}^{r+1} A \to T\mathcal{C}^r A$ given by $(x_1, \ldots, x_{r+1}) \mapsto (x_1, \ldots, x_r)$.

Proposition 1.20. *If $A \subseteq N_{\mathbb{R}}$ is a convex set then a point $(x_0, \ldots, x_r) \in (N_{\mathbb{R}})^{r+1}$ belongs to $T\mathcal{C}^r A$ iff*

(1) For every $1 \leq i \leq r$ and for every $\delta > 0$ small enough we have

$$x_0 + \delta x_1 + \cdots + \delta^i x_i \in A.$$

(2) For every $1 \leq i \leq r$ and for every sequence of positive numbers $\{\delta_j\}_{j=1}^i$ small enough we have

$$x_0 + \delta_1 x_1 + \cdots + \delta_1 \cdots \delta_i x_i \in A.$$

Proof. If $(x_0, \ldots, x_{r-1}) \in T\mathcal{C}^{r-1} A$ then we have

$$\begin{aligned}
(x_0, \ldots, x_r) \in T\mathcal{C}^r A &\iff x_r \in T\mathcal{C}_{x_{r-1}}(T\mathcal{C}_{x_{r-2}}(\ldots(T\mathcal{C}_{x_0} A)\ldots)) \\
&\iff x_{r-1} + \delta_r x_r \in T\mathcal{C}_{x_{r-2}}(\ldots(T\mathcal{C}_{x_0} A)\ldots) \text{ for } \delta_r > 0 \text{ small} \\
&\;\;\vdots \\
&\iff x_0 + \delta_1 x_1 + \cdots + \delta_1 \cdots \delta_r x_r \in A \text{ for } \delta_1, \ldots, \delta_r > 0 \text{ small.}
\end{aligned}$$

We can transform $(x_0, \ldots, x_{r-1}) \in T\mathcal{C}^{r-1} A$ in a similar statement, and in this way we can show that $(x_0, \ldots, x_r) \in T\mathcal{C}^r A$ is equivalent to condition (2) above. Moreover, it is clear than (2) implies (1) by taking $\delta_j = \min\{\delta_i\}$ for all j. To see that (1) implies (2) take $\delta = \max\{\delta_1, \dots, \delta_i\}$, by the convexity assumption for $t_1, \dots, t_i > 0$ small we have

$$\begin{aligned}&(1 - t_1 - \cdots - t_i)x_0 + t_1(x_0 + \delta x_1) + \cdots + t_i(x_0 + \delta x_1 + \dots \delta^i x_i)\\ &= x_0 + (t_1 + \cdots + t_i)\delta x_1 + \cdots + t_i \delta^i x_i.\end{aligned}$$

Then, by taking $t_1 + \cdots + t_i = \delta_1/\delta$, $t_2 + \cdots + t_i = \delta_1\delta_2/\delta^2$,...,$t_i = \delta_1 \cdots \delta_i/\delta^i$ we are done. □

We will identify $T\mathcal{C}^{k-1}A$ with a subset of $N_{\mathbb{D}}$ using the map

$$\begin{aligned}T\mathcal{C}^{k-1}A &\longrightarrow N_{\mathbb{D}}\\ (x_0, \dots, x_{k-1}) &\longmapsto x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1}x_{k-1}.\end{aligned}$$

In this map, ε is a formal variable which we regard as an infinitesimal, nonetheless by Proposition 1.20 above we can think of $x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1}x_{k-1}$ as morally lying on A.

Remark 1.21. In Example 1.18 we can consider $\sigma = \{(x_1, x_2) \in \mathbb{D}^2 \mid x_1, x_2 \geq 0\}$ as the extension of scalars of $\sigma^{[0]} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0\}$ from $\mathbb{R}$ to the dual numbers $\mathbb{D}$. In this regard the equation $\sigma = T\mathcal{C}\sigma^{[0]}$ should be considered as a polyhedral version of the equality $X(k[\varepsilon]/(\varepsilon^2)) = TX(k)$, for a variety X over a field k, from algebraic geometry. In Corollary 4.3 below we extend this statement to a general real polyhedron. Moreover, in Section 14 we give another manifestation on the extension of scalars and we discuss how the elements of $T\mathcal{C}^r A$ can be seen dually as tangent derivative operators.

Using Proposition 1.20 we can generalize the notion of tangent cone to flag of subsets.

Definition 1.22. Let us consider a flag of convex subsets in $N_{\mathbb{D}}$ of the form

$$\mathcal{A} : A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r.$$

We define the *tangent cone of* $\mathcal{A}$ as the set $T\mathcal{C}\mathcal{A}$ of all tuples $(x_0, x_1, \dots, x_r) \in (N_{\mathbb{R}})^{r+1}$ such that $x_0 \in A_0$ and for each $1 \leq i \leq r$,

$$x_0 + \delta x_1 + \cdots + \delta^i x_i \in A_i$$

for each $\delta > 0$ small enough. If for $0 \leq i \leq r$ we denote by $\mathcal{A}|_i$ the restriction flag given by $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i$. Then, we have an iterated fibration

$$T\mathcal{C}\mathcal{A} = T\mathcal{C}\mathcal{A}|_r \to T\mathcal{C}\mathcal{A}|_{r-1} \to \cdots \to T\mathcal{C}\mathcal{A}|_1 \to A_0.$$

Again, for a flag of length k we identify $T\mathcal{C}\mathcal{A}$ with a subset of $N_{\mathbb{D}}$ with the map

$$\begin{aligned}T\mathcal{C}\mathcal{A} &\longrightarrow N_{\mathbb{D}}\\ (x_0, \dots, x_{k-1}) &\longmapsto x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1}x_{k-1}\end{aligned}$$

Remark 1.23.

(1) For the constant flag $\mathcal{A}$ equals to A we recover the iterated fibration of $T\mathcal{C}^r A$ as $T\mathcal{C}\mathcal{A}$.

(2) The tangent cone behaves well with intersections: If $\mathcal{A} = (A_i)_{i=0}^r$ and $\mathcal{B} = (B_i)_{i=0}^r$ are flags of the same length, then $T\mathcal{C}\mathcal{A} \cap T\mathcal{C}\mathcal{B} = T\mathcal{C}(\mathcal{A} \cap \mathcal{B})$, where $\mathcal{A} \cap \mathcal{B} = (A_i \cap B_i)_{i=0}^r$. In particular, $T\mathcal{C}^r A \cap T\mathcal{C}^r B = T\mathcal{C}^r A \cap B$.

(3) The tangent cone behaves well with subdivisions: Given a flag of polyhedra
$$\mathcal{P} : P_0 \subseteq P_1 \subseteq \cdots \subseteq P_r$$
consider polyhedral complexes $\Sigma_0, \Sigma_1, \ldots, \Sigma_r$ with supports $P_0, P_1, \ldots, P_r$ respectively, and such that each cell of Σ_i is also a cell of Σ_{i+1}. Then
$$TC\mathcal{P} = \bigcup_{\mathcal{Q}} TC\mathcal{Q}$$
where the union goes over all flags
$$\mathcal{Q} : Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_r$$
where $Q_i \in \Sigma_i$ and Q_i is a face of Q_{i+1} for each i.

1.2. **The Fourier-Motzkin Elimination and Farkas' Lemma.** In the following, we will provide a generalization of the Fourier-Motzkin elimination (Theorem 1.26) to reduce the number of variables in systems of linear inequalities. Although more technical than its classical counterpart, this result is at the heart of the theory from a technical standpoint. As an immediate consequence, we get that the projection of a polyhedron is a polyhedron, which implies that both polytopes and finitely generated cones are polyhedral. A more subtle application is Farkas' Lemma (Theorem 1.28) on the structure of affine functions that are positive on a given polyhedron.

Let us start with a result about the intersection of convex sets in linear orders which, although very simple, we could not find a reference for it in the literature. As in Proposition 1.4, given a linear order L, a subset $C \subseteq L$ is called *order-convex* if for every $x, z \in C$ and every $y \in L$ such that $x \leq y \leq z$ we have $y \in C$. In the style of Helly's theorem, we have the next lemma.

Lemma 1.24. *Consider a linear order L and a finite family of non-empty order-convex sets $\{C_i\}_{i\in I}$ in L. If we have $C_i \cap C_j \neq \varnothing$ for every $i, j \in I$, then $\bigcap_{i\in I} C_i \neq \varnothing$.*

Proof. For each unordered pair $\{i, j\} \subseteq I$ take an element $x_{ij} \in C_i \cap C_j$ and consider $a_i = \min_{j\in I} x_{ij}$ and $b_i = \max_{i\in I} x_{ij}$. In this way, we have $[a_i, b_i] \subseteq C_i$ for each $i \in I$ and $a_i \leq x_{ij} \leq b_j$ for each $i, j \in I$. Therefore $\max_{i\in I} a_i \leq \min_{i\in I} b_i$ from where
$$\bigcap_{i\in I} C_i \supseteq \bigcap_{i\in I} [a_i, b_i] = [\max_{i\in I} a_i, \min_{i\in I} b_i] \neq \varnothing$$
□

As the order-convex subsets of $\mathbb{D}$ coincide with its convex subsets, the convex subsets of $\mathbb{D}$ satisfy the Helly property above. This tells us that, to understand if an arbitrary finite intersection of convex sets in $\mathbb{D}$ is non-empty, we can restrict ourselves to study that each convex set is independently non-empty and each intersection of pairs of convex sets is non-empty. The convex sets of $\mathbb{D}$ in which we will be interested are the solutions to linear inequalities like
$$c + dx \geq 0 \quad \text{or} \quad c + dx > 0.$$
After multiplying by an invertible element, we can suppose that these inequalities are of one of the forms
$$-a + \varepsilon^\alpha x \geq 0, \quad -a + \varepsilon^\alpha x > 0, \quad b - \varepsilon^\beta x \geq 0, \quad b - \varepsilon^\beta x > 0.$$

The next lemma gives us conditions in terms of the coefficients a and b for which a single inequality has a solution or a pair of inequalities have a common solution.

Lemma 1.25 (Fourier-Motzkin, base case)**.** *Let $a, b \in \mathbb{D}$ and consider the inequalities*

(i) $$-a + \varepsilon^{\alpha} x \geq 0$$

(i*) $$-a + \varepsilon^{\alpha} x > 0$$

(ii) $$b - \varepsilon^{\beta} x \geq 0$$

(ii*) $$b - \varepsilon^{\beta} x > 0$$

Then,

(1) *The inequality* (i) *has a solution iff the inequality* (i*) *has a solution iff*
$$-\varepsilon^{k-\alpha} a \geq 0.$$
Analogously, the inequality (ii) *has a solution iff the inequality* (ii*) *has a solution iff*
$$\varepsilon^{k-\beta} b \geq 0.$$

(2) *The inequalities* (i) *and* (ii) *have a common solution iff each of them has a solution in their own and*
$$b - \varepsilon^{\beta-\alpha} a \geq 0 \text{ if } \beta \geq \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a \geq 0 \text{ if } \alpha \geq \beta.$$
Similarly, (i*) *and* (ii) *have a common solution iff each of them has a solution in their own and*
$$b - \varepsilon^{\beta-\alpha} a \geq 0 \text{ if } \beta > \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a > 0 \text{ if } \alpha \geq \beta.$$
The inequalities (i) *and* (ii*) *have a common solution iff each of them has a solution in their own and*
$$b - \varepsilon^{\beta-\alpha} a > 0 \text{ if } \beta \geq \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a \geq 0 \text{ if } \alpha > \beta.$$
Finally, (i*) *and* (ii*) *have a common solution iff each of them has a solution in their own and*
$$b - \varepsilon^{\beta-\alpha} a > 0 \text{ if } \beta \geq \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a > 0 \text{ if } \alpha \geq \beta.$$

Proof.

(1) If there is an x satisfying (i) or (i*) then by multiplying the inequality on both sides by $\varepsilon^{k-\alpha}$ we get $-\varepsilon^{k-\alpha} a \geq 0$. Conversely, If $-\varepsilon^{k-\alpha} a \geq 0$ then either $-\varepsilon^{k-\alpha} a > 0$ or $-\varepsilon^{k-\alpha} a = 0$. In the first case $-a + \varepsilon^{\alpha} x > 0$ for any x and we are done. In the second case a is of the form $\varepsilon^{\alpha} a'$ and we have $-\varepsilon^{\alpha} a' + \varepsilon^{\alpha} x \geq 0$ (resp. $-\varepsilon^{\alpha} a' + \varepsilon^{\alpha} x > 0$) iff $-a' + x \geq 0$ (resp. $-a' + x > 0$) which always have a solution. The statement about (ii) and (ii*) follows from the previous one by replacing x with $-x$ and a with $-b$.

(2) Suppose that both inequalities (i) and (ii) have a solution and moreover $\beta \geq \alpha$. By multiplying (i) by $\varepsilon^{\beta-\alpha}$ and adding (ii) we get $b - \varepsilon^{\beta-\alpha} a \geq 0$. Conversely, suppose that both (i) and (ii) have a solution independently and $b - \varepsilon^{\beta-\alpha} a \geq 0$. Then, by the first part we have $-\varepsilon^{k-\alpha} a \geq 0$ and $\varepsilon^{k-b} \geq 0$. Moreover, if $-\varepsilon^{k-\alpha} a > 0$ then (i) is satisfied for every x and any solution for (ii) works for both inequalities. Hence, we can assume $-\varepsilon^{k-\alpha} a = 0$, and for a similar reason, we can assume $\varepsilon^{k-\alpha} b = 0$. Then, $a = \varepsilon^{\alpha} a'$ and $b = \varepsilon^{\beta} b'$ for some $a', b' \in \mathbb{D}$, which we can replace in the inequality $b - \varepsilon^{\beta-\alpha} a \geq 0$ to obtain $\varepsilon^{\beta} b' - \varepsilon^{\beta} a' \geq 0$. Then, there is an x such that

$$b = \varepsilon^{\beta} b' \geq \varepsilon^{\beta} x \geq \varepsilon^{\beta} a' = \varepsilon^{\beta-\alpha} a. \tag{1.7}$$

If $\beta = \alpha$ we are done. If $\beta > \alpha$ then, notice that for every $x' \in \mathbb{D}$ we have $\varepsilon^{\beta} x = \varepsilon^{\beta}(x + \varepsilon^{k-\beta} x')$. Hence, we can modify the last β coordinates of x and (1.7) remains true. By making them big enough we have $\varepsilon^{\alpha} x > a$, that is, x satisfy (ii*). In particular, x satisfy both (i) and (ii) simultaneously and we are done. The case in which $\beta \geq \alpha$ is done similarly.

Notice that in the argument above we proved that if $\beta > \alpha$ then (i*) and (ii) are satisfied together iff they are satisfied individually and $b - \varepsilon^{\beta-\alpha} a \geq 0$. This is the next part of the proposition. For the other part, if $\alpha \geq \beta$ then, if both (i*) and (ii) are satisfied we can multiply the first equation by $\varepsilon^{\alpha-\beta}$ and add it to the second one to obtain $\varepsilon^{\alpha-\beta} b - a > 0$. Conversely, working in the same way as to obtain (1.7) we get $a', b', x \in \mathbb{D}$ such that

$$b = \varepsilon^{\beta} b' > \varepsilon^{\beta} x > \varepsilon^{\beta} a' = \varepsilon^{\beta-\alpha} a.$$

Again, if $\beta = \alpha$ we are done and if $\beta > \alpha$ we can modify the last β coordinates of x as to get $\varepsilon^{\alpha} x > a$, and such an x satisfy both (i*) and (ii*) (in particular (i*) and (ii)). The remaining cases can be done in the same way.

□

For the following result, it will be necessary to use coordinates, hence we will work with spaces of the form $\mathbb{D}^n$ for $n \geq 0$, instead of $N_{\mathbb{D}}$ for a general lattice N. A *linear inequality in* $\mathbb{D}^n$ is an inequality of one of the forms

$$a_1 x_1 + \cdots + a_n x_n \geq a \qquad a_1 x_1 + \cdots + a_n x_n > a \tag{1.8}$$

with $a, a_1, \ldots, a_n \in \mathbb{D}$. If it is of the first form we say it is *closed*, if it is of the second form, we say it is *open*, and if $a = 0$, we say it is *homogeneous*. A finite family of linear inequalities is called a *system of linear inequalities*. Such a system is said to be *closed*, *open* or *homogeneous* if each inequality is of this form.

Theorem 1.26 (Fourier-Motzkin over $\mathbb{D}$). *Given an integer $n \geq 1$ and a system of linear inequalities $\mathcal{L}$ in $\mathbb{D}^{n+1}$, there is another system of linear inequalities $\mathcal{L}'$ in $\mathbb{D}^n$ such that an element $(x_1, \ldots, x_{n+1})$ is a solution of $\mathcal{L}$ iff $(x_1, \ldots, x_n)$ is a solution of $\mathcal{L}'$. Moreover, if $\mathcal{L}$ is closed or homogeneous then $\mathcal{L}'$ can also be taken to be closed or homogeneous, respectively. If $\mathcal{L}$ is open, it may not be possible to take $\mathcal{L}'$ open.*

Proof. Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \mathbb{D}^n$. There is an element $x_{n+1} \in \mathbb{D}$ such that $(\tilde{x}, x_{n+1})$ is a solution to $\mathcal{L}$ iff the system of inequalities $\mathcal{L}_{\tilde{x}}$ has a non-empty set of solutions, where

$\mathcal{L}_{\tilde{x}}$ consists of the inequalities

$$a_1\tilde{x}_1 + \cdots + a_n\tilde{x}_n + a_{n+1}x \geq a \quad \text{or} \quad a_1\tilde{x}_1 + \cdots + a_n\tilde{x}_n + a_{n+1}x > a \tag{1.9}$$

where $a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1} \geq a$ and $a_1x_1 + \cdots + a_nx_n + a_{n+1}x_{n+1} > a$ go over all the inequalities of $\mathcal{L}$. After multiplying by a positive invertible element of $\mathbb{D}$ we can suppose that each inequality in $\mathcal{L}_{\tilde{x}}$ is of one of the forms

(i) $$-a(\tilde{x}) + \varepsilon^{\alpha}x \geq 0$$
(i*) $$-a(\tilde{x}) + \varepsilon^{\alpha}x > 0$$
(ii) $$b(\tilde{x}) - \varepsilon^{\beta}x \geq 0$$
(ii*) $$b(\tilde{x}) - \varepsilon^{\beta}x > 0$$
(iii) $$c(\tilde{x}) \geq 0$$
(iii*) $$c(\tilde{x}) > 0$$

By Lemma 1.25 we know that each of these conditions can be translated into a linear inequality in the variable $\tilde{x}$, giving rise to a system of linear inequalities $\mathcal{L}'$ in the variable $\tilde{x}$. Hence, there is an x_{n+1} such that $(\tilde{x}, x_{n+1})$ is a solution to $\mathcal{L}$ iff $\mathcal{L}_{\tilde{x}}$ has at least one solution iff $\tilde{x}$ is a solution to $\mathcal{L}'$, as we wanted. Moreover, the explicit linear equations obtained in Lemma 1.25 give us that if each inequality in $\mathcal{L}$ is closed or homogeneous, then each inequality in $\mathcal{L}'$ is also closed or homogeneous as well. Finally, in the case in which $\mathcal{L}$ consists in the single equation $x_1 + \varepsilon x_2 > 0$, then $\mathcal{L}'$ should have as solution set $\varepsilon^{k-1}x_1 \geq 0$, so $\mathcal{L}'$ cannot consist in a finite set of open inequalities in $\mathbb{D}$. □

As an immediate consequence of this, we get that the projection of a polyhedron in $\mathbb{D}^k$ to some of its coordinates is still a polyhedron, and if it is a polyhedral cone then the projection is also a polyhedral cone. Less immediate consequences are summarized in the following proposition.

Proposition 1.27.

(1) *The image of a polyhedron P under a linear map is a polyhedron. If P is a polyhedral cone, then the image is also a polyhedral cone.*
(2) *The sum of two polyhedra is a polyhedron. The sum of two polyhedral cones is a polyhedral cone.*
(3) *Every polytope is a polyhedron.*
(4) *Every finitely generated cone is a polyhedral cone.*

Proof.

(1) Without loss of generality we can suppose that P is a polyhedron inside $\mathbb{D}^n$ and the linear map f goes from $\mathbb{D}^n$ to $\mathbb{D}^k$. Now consider
$$\Gamma(f|_P) = \left\{(x, y) \in \mathbb{D}^n \times \mathbb{D}^k \mid x \in P, y = f(x)\right\}.$$
This is a polyhedron in $\mathbb{D}^n \times \mathbb{D}^k$ and $f(P)$ is the projection to the second component. Hence, by Fourier-Motzkin this is a polyhedron. In the same way, if P is a polyhedral cone, then so is $\Gamma(f|_P)$ and then its projection $f(P)$.
(2) Given $P, Q \subseteq \mathbb{D}^n$, consider
$$R = \{(x, y, z) \in \mathbb{D}^n \times \mathbb{D}^n \times \mathbb{D}^n \mid x \in P, y \in Q, z = x + y\}.$$

By (1), this is a polyhedron and $P+Q$ is the projection to the last component. Hence, it is a polyhedron as well. As R is a polyhedral cone if each of P and Q are, then so is $P+Q$ in this case.

(3) The polytope P is equal to $\mathrm{conv}_{\mathbb{D}}(a_1,\dots,a_r)$ for some $a_1,\dots,a_r \in \mathbb{D}^n$. Then, $P=f(Q)$ where Q is the polyhedron
$$Q=\{(x_1,\dots,x_r)\in\mathbb{D}^r \mid x_1,\dots,x_r\geq 0, x_1+\cdots+x_r=1\}$$
and f is the linear map defined by
$$\begin{aligned} f:\mathbb{D}^r &\longrightarrow \mathbb{D}^n \\ e_i &\longmapsto a_i \end{aligned}$$
By part (2) we get that P is a polyhedron as well.

(4) As above, if $\sigma=\mathrm{cone}_{\mathbb{D}}(a_1,\dots,a_r)$ for some $a_1,\dots,a_r\in\mathbb{D}^n$. Then, $\sigma=f(\tau)$ for the same function f and
$$\tau=\{(x_1,\dots,x_r)\in\mathbb{D}^r \mid x_1,\dots,x_r\geq 0\}.$$
$\square$

Next, we will prove the analogous statement to Farkas' Lemma which works over the $\mathbb{D}$. Its proof is based on the Fourier-Motzkin elimination in developed the previous section.

Theorem 1.28 (Farkas' Lemma over $\mathbb{D}$)**.** *Let $f_1,\dots,f_r : N_{\mathbb{D}}\to\mathbb{D}$ be a family of affine functions such that*
$$P=\{f_1\geq 0,\dots,f_r\geq 0\}$$
is non-empty. Then, any affine function $f:N_{\mathbb{D}}\to\mathbb{D}$ that achieves its minimum in P can be written in the form
$$f-\min_P f=\lambda_1 f_1+\cdots+\lambda_r f_r$$
for some $\lambda_1,\dots,\lambda_r\in\mathbb{D}_{\geq 0}$.

Remark 1.29. One can state Farkas' lemma over $\mathbb{R}$ as follows. If $f,f_1,\dots,f_r:N_{\mathbb{R}}\to\mathbb{R}$ is a family of affine functions such that
$$\{f\geq 0\}\supseteq\{f_1\geq 0,\dots,f_r\geq 0\}.$$
Then, there are $c,\lambda_1,\dots,\lambda_r\in\mathbb{R}_{\geq 0}$ such that
$$f=\lambda_1 f_1+\cdots+\lambda_r f_r+c.$$
The exact translation of this statement to $\mathbb{D}$ is false. For example, we can take $N=\mathbb{Z}$, $f(x)=x$ and $f_1(x)=\varepsilon x-1$. Then,
$$\{f\geq 0\}\supseteq\{f_1\geq 0\}=\varnothing,$$
but there are no $c,\lambda\in\mathbb{D}_{\geq 0}$ such that
$$x=\lambda(\varepsilon x+1)+c,\ \forall x\in\mathbb{D}.$$
In this way, we see that the hypothesis that P is not empty is unavoidable. Similarly, the hypothesis that f achieves its minimum over P is unavoidable as, we can take $f(x)=\varepsilon x+1$ and $f_1(x)=\varepsilon^2 x$. Then
$$\{\varepsilon x+1\geq 0\}=N_{\mathbb{R}}\supseteq\{\varepsilon^2 x\geq 0\}.$$

But, there are no $c, \lambda \in \mathbb{D}_{\geq 0}$ such that

$$\varepsilon x = \lambda(\varepsilon^2 x) + c, \ \forall x \in \mathbb{D}.$$

We will deduce the theorem above from the following more general technical lemma.

Lemma 1.30. *Consider affine functions $f_1, \dots, f_s, f_{s+1}, ..., f_t$ in $N_{\mathbb{D}}$ such that*

(A) $\{f_1 \geq 0, \dots, f_s \geq 0, f_{s+1} \geq 0, \dots, f_t \geq 0\} \neq \varnothing$

(B) $\{f_1 \geq 0, \dots, f_s \geq 0, f_{s+1} > 0, \dots, f_t > 0\} = \varnothing.$

Then, there are $\lambda_1, \dots, \lambda_t \in \mathbb{D}_{\geq 0}$ such that

$$\lambda_1 f_1(x) + \cdots + \lambda_t f_t(x) = 0, \quad \forall\, x \in N_{\mathbb{D}}$$

and at least one element between $\lambda_{s+1}, \dots, \lambda_t$ is invertible.

Proof of Theorem 1.28. We have

$$\{f_1 \geq 0, \dots, f_r \geq 0, -f + \min_P f \geq 0\} \neq \varnothing \text{ and,}$$
$$\{f_1 \geq 0, \dots, f_r \geq 0, -f + \min_P f > 0\} = \varnothing.$$

So, by Lemma 1.30, there are $\lambda_1, \dots, \lambda_r, \lambda \in \mathbb{D}_{\geq 0}$ with λ invertible such that

$$\lambda_1 f_1 + \cdots + \lambda_r f_r + \lambda \left(-f + \min_P f \right) = 0.$$

That is,

$$f - \min_P f = \frac{\lambda_1}{\lambda} f_1 + \cdots + \frac{\lambda_r}{\lambda} f_r$$

as we wanted. □

Proof of Lemma 1.30. After composing with an isomorphism we can suppose $N = \mathbb{Z}^n$. The proof is by induction on n.

Base case $n = 1$:

After multiplying by a positive invertible element in $\mathbb{D}_k$ if necessary, we can suppose that each f_i is of one of the forms

$$\varepsilon^{\alpha_i} x - a_i, \quad b_i - \varepsilon^{\beta_i} x, \qquad c_i \quad \text{for some } 0 \leq \alpha_i,\ \beta_i \leq k-1.$$

Then, by Lemma 1.25 the system of inequalities in (B) has no solutions iff at least one of the following conditions fail

(1) For every i we have
  (1.1) $-\varepsilon^{k-\alpha_i} a \geq 0$ if $f_i = \varepsilon^{\alpha_i} x - a_i$
  (1.1) $\varepsilon^{k-\beta_i} b \geq 0$ if $f_i = b_i - \varepsilon^{\beta_i} x$.
  (1.2) Either

$$c_i \geq 0 \text{ if } i \leq s \text{ or,}$$
$$c_i > 0 \text{ if } i > s$$

  if $f_i = c_i$.

(2) Whenever $f_i = \varepsilon^{\alpha_i} x - a_i$ and $f_j = b_j - \varepsilon^{\beta_j} x$ we have

(2.1) If $i, j \leq s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i \geq 0 \text{ if } \beta_j \geq \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i \geq 0 \text{ if } \alpha_i \geq \beta_j$$

(2.2) If $i > s$ and $j \leq s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i \geq 0 \text{ if } \beta_j > \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i > 0 \text{ if } \alpha_i \geq \beta_j$$

(2.3) If $i \leq s$ and $j > s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i > 0 \text{ if } \beta_j \geq \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i \geq 0 \text{ if } \alpha_i > \beta_j$$

(2.4) If $i, j > s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i > 0 \text{ if } \beta_j \geq \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i > 0 \text{ if } \alpha_i \geq \beta_j$$

As the system (A) does have a solution, the only conditions that can fail are the ones with strict inequalities. Moreover, this can fail only by getting an equality.

- If condition (1.2) fail then for some $i > s$ we have $f_i = c_i = 0$ so in this case we can take $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$.
- If condition (2.2), (2.3), or (2.4) fail then we have either
$$f_i + \varepsilon^{\alpha_i - \beta_j} f_j = \varepsilon^{\alpha_i - \beta_j} b_j - a_i = 0$$
with $i > s$ or
$$\varepsilon^{\beta_j - \alpha_i} f_i + f_j = b_j - \varepsilon^{\beta_j - \alpha_i} a_i = 0$$
with $j > s$. In the former case we can take $\lambda_i = 1$, $\lambda_j = \varepsilon^{\alpha_i - \beta_j}$ and everything else 0. In the later case we can take $\lambda_j = 1$, $\lambda_i = \varepsilon^{\beta_j - \alpha_i}$ and everything else 0.

Induction step:

Assuming the result for $\mathbb{D}^n$ we will prove it for $\mathbb{D}^{n+1}$. For this let $x' = (x_1, \ldots, x_n)$. As in the base case, after multiplying by a positive invertible scalar we can assume that each f_i is of one of the forms

$$x_{n+1} - a_i(x'), \quad b_i(x') - x_{n+1}, \quad c_i(x').$$

Now, for a given $x' \in \mathbb{D}^n$ fixed, there is an $x_{n+1} \in \mathbb{D}$ such that (x', x_{n+1}) belongs to the set (A) iff the same conditions that we use in the base case are satisfied. As the set (A) is empty, this cannot happen for any x'. Hence, the system of inequalities on the variable x' which is formed by all these conditions has an empty set of solutions. On the other hand, the same system but in which all the inequalities are closed does have a solution, because the set (B) is not empty. This allows us to use the induction hypothesis on this new system of inequalities.

Applying the induction hypothesis we get a positive linear combination of the new affine functions involved which is equal to zero. Now, by doing the following replacements

- $-\varepsilon^{k-\alpha_i} a_i(x') = \varepsilon^{k-\alpha_i} f_i(x)$

- $\varepsilon^{k-\alpha_j} b_j(x') = \varepsilon^{k-\alpha_j} f_j(x)$
- $c_l(x') = f_l(x)$
- $b_j(x') - \varepsilon^{\beta_j-\alpha_i} a_i(x') = f_j(x) + \varepsilon^{\beta_j-\alpha_i} f_i(x)$
- $\varepsilon^{\alpha_i-\beta_j} b_j(x') - a_i(x') = \varepsilon^{\beta_j-\alpha_i} f_j(x) + f_i(x)$,

we turn the linear combination into one involving the original affine functions. By the induction hypothesis we get that at least one of the coefficients of the linear combination in either $c_i(x')$ with $i > s$, $b_j(x') - \varepsilon^{\beta_j-\alpha_i} a_i(x')$ with $j > s$ or $\varepsilon^{\alpha_i-\beta_j} b_j(x') - a_i(x')$ with $i > s$ is invertible. Hence, at least one of the coefficients in f_i for $i > s$ is invertible. This finishes the induction step. □

2. The Structure of Faces

In this section we develop tools to explicitly describe a given face of a polyhedron. These descriptions depend on the data used to present the polyhedron. In Proposition 2.1 we study the case in which the polyhedron is defined in terms of a representation by inequalities. After this, we introduce the concept of *weighted convex hull* which allows us to introduce any polyhedron in terms of generators. In Propositions 2.4 and Proposition 2.9 we describe faces in terms of these generators.

Proposition 2.1. *Let $P \subseteq N_{\mathbb{D}}$ be a polyhedron with a representation*

$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}.$$

(1) For each $y \in M_{\mathbb{D}}$ achieving its minimum in P there are $\lambda_i \in \mathbb{D}_{\geq 0}$ for $1 \leq i \leq r$ such that

$$y = \lambda_1 y_1 + \cdots + \lambda_r y_r \text{ and}$$
$$\min_{x \in P} \langle y , x \rangle = \lambda_1 a_1 + \cdots + \lambda_r a_r.$$

Moreover, given such elements $\{\lambda_i\}_i$, we can write the face $F = \mathrm{face}_y(P)$ as

$$F = \bigcap_{i=1}^{r} \left\{ x \in P \,\middle|\, \varepsilon^{\alpha_i} \langle y_i , x \rangle = \varepsilon^{\alpha_i} a_i \right\} \tag{2.1}$$

where $\alpha_i = \mathrm{ord}(\lambda_i)$ for each i.

(2) Similarly, if F is a face of P, given $x_0 \in \mathrm{int}(F)$ we have an equality of the form

$$F = \bigcap_{i=1}^{r} \left\{ x \in P \,\middle|\, \varepsilon^{\beta_i} \langle y_i , x \rangle = \varepsilon^{\beta_i} a_i \right\} \tag{2.2}$$

for $\beta_i = k - \mathrm{ord}(\langle y_i , x_0 \rangle - a_i)$.

(3) Conversely, any choice of $0 \leq \alpha_i \leq k$ determines a set of the form (2.1) which is either empty or a face of P.

Remark 2.2. Given a linear function $y \in M_{\mathbb{D}}$ with $\min_{x \in P} \langle y , x \rangle = a$. The set

$$\left\{ x \in P \,\middle|\, \varepsilon^{k-\alpha} \langle y , x \rangle = \varepsilon^{k-\alpha} a \right\} = \left\{ x \in P \,\middle|\, \mathrm{ord}\{\langle y , x \rangle - a\} \geq \alpha \right\}$$

should be interpreted as the set of all elements $x \in P$ such that y achieves the minimum *at least in the first α coordinates.*

As an example, if we take $y \in M_{\mathbb{R}}$ to be real and

$$x = x^{(0)} + x^{(1)}\varepsilon + \cdots + x^{(k-1)}\varepsilon^{k-1} \in N_{\mathbb{D}},$$
$$a = a^{(0)} + a^{(1)}\varepsilon + \cdots + a^{(k-1)}\varepsilon^{k-1} \in \mathbb{D}.$$

Then, $\langle y\,, x\rangle = \sum_{i=0}^{k-1}\langle y\,, x^{(i)}\rangle\varepsilon^i$. So, we have that $\langle y\,, x\rangle\varepsilon^{k-\alpha} = a\varepsilon^{k-\alpha}$ iff $\langle y\,, x^{(i)}\rangle = a^{(i)}$ for each $0 \leq i < \alpha$, and this happens iff x minimize the vector

$$(\langle y\,, x^{(0)}\rangle, \ldots, \langle y\,, x^{(\alpha)}\rangle)$$

among all $x \in P$ with respect to the lexicographic order.

In this way, the equality in (2.1) can be read as: $\operatorname{face}_y P$ is the set of all $x \in P$ for which y_i achieves its minimum at least in the first $k - \alpha_i$ coordinates for each $1 \leq i \leq r$.

Proof of Proposition 2.1.

(1) If y achieves its minimum in P. By Farkas' Lemma, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$\langle y\,, \cdot\rangle - \min_{x\in P}\langle y\,, x\rangle = \lambda_1(\langle y_1\,, \cdot\rangle - a_1) + \cdots + \lambda_r(\langle y_r\,, \cdot\rangle - a_r) \tag{2.3}$$

By evaluating this at $x = 0$ we get $\min_{x\in P}\langle y\,, x\rangle = \lambda_1 a_1 + \cdots + \lambda_r a_r$ and, if we add this equation to the previous one, we get $y = \lambda_1 y + \cdots + \lambda_r y_r$. This shows the first part.

Now, if we evaluate (2.3) in an element $x \in F$, the left hand side vanishes and, as each term of the right hand side is positive, we get

$$\lambda_i\langle y_i\,, x\rangle = \lambda_i a_i \tag{2.4}$$

for each $1 \leq i \leq r$. After multiplication by an invertible element, this becomes $\varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i}a_i$. Which shows that

$$F \subseteq \bigcap_{i=1}^{r}\left\{x \in P \;\middle|\; \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i}a_i\right\}.$$

Conversely, if x is in the right hand side of (2.1) for every $1 \leq i \leq r$ then, the right hand side of (2.3) vanishes at x. Hence, so does the left hand side which implies $x \in F$. This shows the equality we wanted.

(2) Notice that $\varepsilon^{\operatorname{ord}(\langle y_i\,, x_0\rangle - a_i)}(\langle y_i\,, x_0\rangle - a_i) = 0$. Hence, by Proposition 2.13, as $x_0 \in \operatorname{int}(F)$ we have

$$F \subseteq \left\{x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i}a_i\right\},$$

and therefore

$$F \subseteq \bigcap_{i=1}^{r}\left\{x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i}a_i\right\}.$$

On the other hand, by (2.1) and as $x_0 \in F$ we have

$$\varepsilon^{\alpha_i}\langle y_i\,, x_0\rangle = \varepsilon^{\alpha_i}a_i.$$

Hence, $\varepsilon^{\alpha_i}(\langle y_i\,, x_0\rangle - a_i) = 0$ from where $\alpha_i \geq \operatorname{ord}(\langle y_i\,, x_0\rangle - a_i) = \beta_i$. This implies

$$\bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i} a_i \right\} \subseteq \bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i} a_i \right\} = F.$$

(3) Suppose now that F is a non-empty set of the form (2.1) and consider $y = \sum_{i=1}^{r} \varepsilon^{\alpha_i} y_i$. We will prove that $\operatorname{face}_y(P) = F$. For this, notice that as F is not empty, the function $\langle y, \cdot\rangle$ has as minimum over P the value $\sum_{i=1}^{r} \varepsilon^{\alpha_i} a_i$. Hence, we can consider $\operatorname{face}_y P$, and given $x \in P$, we have

$$\begin{aligned} x \in \operatorname{face}_y P \iff& \langle y\,, x\rangle = \sum_{i=1}^{r} \varepsilon^{\alpha_i} a_i \\ \iff& \sum_{i=1}^{r} \varepsilon^{\alpha_i}\left(\langle y_i\,, x\rangle - a_i\right) = 0 \\ \iff& \varepsilon^{\alpha_i}\langle y_i\,, x_i\rangle = \varepsilon^{\alpha_i} a_i \;\forall\, i \\ \iff& x \in F, \end{aligned}$$

as we wanted.

□

In particular, the proposition above implies that any polyhedron has finitely many faces. Another important consequence is the following.

Corollary 2.3. *Let P be a polyhedron.*

(1) If F is a face of P and G is a face of F, then G is a face of P.
(2) If F and G are faces of P and $F \cap G$ is non-empty, then it is a face of P.

Proof.

(1) Take a representation $P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$ of P. If F is a face of P by Proposition 2.1 part 1, there are α_i such that

$$\begin{aligned} F =& \bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i} a_i \right\} \\ =& \{y_1 \geq a_1, \ldots, y_r \geq a_r\} \cap \{\varepsilon^{\alpha_1} y_1 = \varepsilon^{\alpha_1} a_1, \ldots, \varepsilon^{\alpha_r} y_r = \varepsilon^{\alpha_r} a_r\}\,. \end{aligned}$$

Notice that this is expression gives a representation for F in terms of inequalities. Hence, if G is a face of F we can apply Proposition 2.1 part 1 again using this representation for F. In this way G can be written as

$$\begin{aligned} G =& \bigcap_{i=1}^{r} \left\{ x \in F \mid \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i} a_i \right\} \\ =& \bigcap_{i=1}^{r} \left\{ x \in P \mid \varepsilon^{\max\{\alpha_i,\beta_i\}}\langle y_i\,, x\rangle = \varepsilon^{\max\{\alpha_i,\beta_i\}} a_i \right\} \end{aligned}$$

for some integers β_i. Which, by Proposition 2.1 part 3, is a face of P.

(2) By Proposition 2.1 part 1 we have $F = \bigcap_{i=1}^r \{x \in P \mid \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i} a_i\}$ and $G = \bigcap_{i=1}^r \left\{x \in P \mid \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i} a_i\right\}$ for some α_i, β_i. Therefore,

$$F \cap G = \bigcap_{i=1}^r \left\{x \in P \mid \varepsilon^{\max\{\beta_i,\alpha_i\}}\langle y_i\,, x\rangle = \varepsilon^{\max\{\beta_i,\alpha_i\}} a_i\right\}$$

which is a face of P by Proposition 2.1 part 3.

□

We will now proceed to study the case in which the polyhedron is given in terms of finitely many generators. We start with the case of polyhedral cones which, by Proposition 3.2, are all given by the cone hull of finitely many elements.

Proposition 2.4. *For a polyhedral cone $\sigma = \operatorname{cone}_{\mathbb{D}}(x_1, \ldots, x_r)$ and an element $y \in \sigma^\vee$, the face of σ induced by y is given by*

$$\operatorname{face}_y \sigma = \operatorname{cone}_{\mathbb{D}}\left(\left\{x_i \varepsilon^{k-\beta_i}\right\}_{1\le i\le r}\right)$$

where $\beta_i = \operatorname{ord}\langle y\,, x_i\rangle$.

Proof. Given $x = \sum_i \lambda_i x_i \in \sigma$ we have

$$x \in \operatorname{face}_y \sigma \iff \langle y, x\rangle = 0 \iff \sum_i \lambda_i \langle y, x_i\rangle = 0.$$

As $\lambda_i\langle y\,, x_i\rangle \geq 0$ for each i, this last thing happens iff $\lambda_i\langle y\,, x_i\rangle = 0$ for each i. Now, for $\beta_i = \operatorname{ord}\langle y\,, x_i\rangle$, this is equivalent to $\lambda_i \varepsilon^{\beta_i} = 0$ for each i, which correspond to the existence of elements $\lambda_i' \in \mathbb{D}_{\geq 0}$ such that $\lambda_i = \lambda_i' \varepsilon^{k-\beta_i}$. That is, $x \in \operatorname{cone}_{\mathbb{D}}(x_i \varepsilon^{k-\beta_i})$. □

To work out the general case, we need a new notion of finitely generatedness which allow us to understand every polyhedral cone as a finitely generated object. For this reason, we introduce the weighted convex hull of a family of vectors.

Definition 2.5. Consider elements $x_1, \ldots, x_r \in N_{\mathbb{D}}$ and integers $\alpha_1, \ldots, \alpha_r \in \{0, \ldots, k\}$. The *weighted convex hull* of the elements $x_1, \ldots, x_r$ with respect to the weights $\alpha_1, \ldots, \alpha_r$ is the set

$$\operatorname{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r]) = \left\{ \sum_{i=1}^r \lambda_i x_i \;\middle|\; \lambda_1, \ldots, \lambda_r \geq 0, \sum_{i=1}^r \varepsilon^{\alpha_i}\lambda_i = 1 \right\}.$$

Remark 2.6.

(1) If no α_i is equal to zero then the weighted convex hull is empty.
(2) The weighted convex hull generalize the usual convex hull as we have

$$\operatorname{wconv}_{\mathbb{D}}([x_1; 0], \ldots, [x_r; 0]) = \operatorname{conv}_{\mathbb{D}}(x_1, \ldots, x_r).$$

(3) If $\alpha_i = k$ for some i, then there is no restriction for the corresponding coefficient λ_i other that being non-negative. This implies the equality

$$\operatorname{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r]) = \operatorname{wconv}_{\mathbb{D}}(\{[x_i; \alpha_i] \mid \alpha_i \neq k\}) + \operatorname{cone}_{\mathbb{D}}(\{x_i \mid \alpha_i = k\}).$$

In particular, for $x_1, \ldots, x_r \in N_{\mathbb{D}}$ we have the equality

$$\operatorname{wconv}_D\{[0; 0], [x_1; k], \ldots, [x_r; k]\} = \operatorname{cone}_D(x_1, \ldots, x_r).$$

Proposition 2.7. *For any polyhedron $P \subseteq N_{\mathbb{D}}$, there are $x_1, \ldots, x_r \in N_{\mathbb{D}}$ and $0 \leq \alpha_1, \ldots, \alpha_r \leq k$ such that*

$$P = \mathrm{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r]).$$

Conversely, any set of this form is a polyhedron.

Proof. Using the extended perfect pairing from Definition 1.12, if we have a representation

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\} \subseteq N_{\mathbb{D}}$$

we can consider the polyhedral cone

$$\widehat{P} = \{(y_1, -a_1) \geq 0, \ldots, (y_r, -a_r) \geq 0\} \subseteq N_{\mathbb{D}} \times \mathbb{D}.$$

Then, we have

$$\widehat{P} \cap N_{\mathbb{D}} \times \{1\} = P \times \{1\}. \tag{2.5}$$

As $\widehat{P}$ is a polyhedral cone, it is finitely generated, hence there are generators $(x_1, b_1), \ldots, (x_r, b_r) \in N_{\mathbb{D}} \times \mathbb{D}$ such that

$$\widehat{P} = \mathrm{cone}_{\mathbb{D}}\left((x_1, b_1), \ldots, (x_r, b_r)\right).$$

After multiplying by an invertible element, we can suppose that $b_i = \varepsilon^{\alpha_i}$ for each $i = 1, \ldots, r$. Hence, using (2.5) we get that

$$\begin{aligned} P \times \{1\} &= \mathrm{cone}_{\mathbb{D}}\left((x_1, \varepsilon^{\alpha_1}), \ldots, D(x_r, \varepsilon^{\alpha_r})\right) \cap N_{\mathbb{D}} \times \{1\} \\ &= \left\{ \sum_{i=1}^{r} \lambda_i x_i \in N_{\mathbb{D}} \;\middle|\; \lambda_1, \ldots, \lambda_r \geq 0, \sum_{i=1}^{r} \lambda_i \varepsilon^{\alpha_i} = 1 \right\} \times \{1\} \\ &= \mathrm{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r]) \times \{1\}. \end{aligned}$$

as we wanted. On the other hand, to see that $\mathrm{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r])$ is a polyhedron, notice that

$$\left\{ (x_1, \ldots, x_r) \in \mathbb{D}^r \;\middle|\; x_1 \geq 0, \ldots, x_r \geq 0, \sum_{i=1}^{r} \lambda_i \varepsilon^{\alpha_i} = 1 \right\}$$

is a polyhedron in $\mathbb{D}^r$ and $\mathrm{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r])$ is the image of this polyhedron under the map

$$\begin{aligned} \mathbb{D}^r &\longrightarrow N_{\mathbb{D}} \\ e_i &\longmapsto x_i, \quad \forall\, i \in \{1, \ldots, r\}. \end{aligned}$$

Where $\{e_1, \ldots, e_r\}$ denotes the standard basis in $\mathbb{D}^r$. Hence, it is a polyhedron by Proposition 1.27. □

Remark 2.8. In the usual polyhedral geometry over $\mathbb{R}$, every polyhedron can be written as the sum of a polytope and a polyhedral cone, this is called a *Minkowski-Weil decomposition* for the polyhedron. The proposition above is the closest we can get to that statement for general polyhedra over $\mathbb{D}$. For a detailed study of when one can actually write a polyhedron over $\mathbb{D}$ as a sum of a polytope and a polyhedral cone we refer to Section 3.4.

We will proceed to study the faces of a polyhedron from this new description in terms of generators.

Proposition 2.9. *Let $P = \operatorname{wconv}_{\mathbb{D}}\big([x_1, \alpha_1], \dots, [x_r; \alpha_r]\big)$ be a polyhedron and $y \in M_{\mathbb{D}}$ a linear function achieving its minimum in P. Then, if $a = \min_{x \in P}\langle y\,, x\rangle$, we have*

$$\operatorname{face}_y(P) = \operatorname{wconv}\Big([x_1\varepsilon^{k-\beta_1}; k+\alpha_1-\beta_1], \dots, [x_r\varepsilon^{k-\beta_r}; k+\alpha_r-\beta_r]\Big)$$

where $\beta_i = \operatorname{ord}\big(\langle y\,, x_i\rangle - \varepsilon^{\alpha_i}a\big)$.

Proof. As in the proof of Proposition 2.7, if we consider

$$\widehat{P} = \operatorname{cone}_{\mathbb{D}}\big((x_1, \varepsilon^{\alpha_1}), \dots, (x_r, \varepsilon^{\alpha_r})\big) \subseteq N_{\mathbb{D}} \times \mathbb{D}$$

then we have

$$P = \widehat{P} \cap N_{\mathbb{D}} \times \{1\}. \tag{2.6}$$

Now, we claim that if y achieves its minimum a in P, then $(y, -a) \in \widehat{P}^\vee$. Indeed, as $\langle y\,, x\rangle \geq a$ for any $x \in P$ we get

$$\langle (y, -a)\,, (x, 1)\rangle \geq 0 \text{ for any } x \in P.$$

Then, given $(x, b) \in \widehat{P}$, with $b \in \mathbb{D}^\times_{>0}$ invertible, by the equality in (2.6), we have $x/b \in P$. Hence,

$$\langle (y, -a)\,, (x, b)\rangle = b\langle (y, -a)\,, (x/b, 1)\rangle \geq 0.$$

Finally, let x' be an element in P achieving the minimum of y, that is $\langle (y, -a)\,, (x', 1)\rangle = 0$. Then, for an element of the form $(x, b) \in \widehat{P}$ with b no invertible we can consider $(x', 1) + (x, b) = (x' + x, 1 + b)$. Now $1 + b$ is invertible, so from the previous step

$$0 \leq \langle (y, -a)\,, ((x' + x, 1 + b))\rangle = \langle (y, -a)\,, (x', 1)\rangle + \langle (y, -a)\,, (x, b)\rangle = \langle (y, -a)\,, (x, b)\rangle.$$

Hence, $(y, -a)$ is positive in (x, b) for any $(x, b) \in \widehat{P}$. Therefore, $(y, -a) \in \widehat{P}^\vee$, which proves the claim.

After this, we can consider $\operatorname{face}_{(y,-a)} \widehat{P}$ and, by Proposition 2.4 above, if $\beta_i = \operatorname{ord}(\langle (y, -a)\,, (x_i, \varepsilon^{\alpha_i})\rangle) = \operatorname{ord}(\langle y\,, x_i\rangle - \varepsilon^{\alpha_i}a)$, then we have that

$$\operatorname{face}_{(y,-a)}\big(\widehat{P}\big) = \operatorname{cone}_{\mathbb{D}}\left(\left\{(x_i, \varepsilon^{\alpha_i})\varepsilon^{k-\beta_i}\right\}_{1\leq i\leq r}\right).$$

Moreover, we have the equality

$$\operatorname{face}_{(y,-a)}\big(\widehat{P}\big) \cap N_{\mathbb{D}} \times \{1\} = \operatorname{face}_y(P) \times \{1\}. \tag{2.7}$$

This gives

$$\begin{aligned}
\operatorname{face}_y(P) &= \left\{x \in N_{\mathbb{D}} \;\middle|\; (x, 1) \in \operatorname{cone}_{\mathbb{D}}\left(\left\{(x_i, \varepsilon^{\alpha_i})\varepsilon^{k-\beta_i}\right\}_{1\leq i\leq r}\right)\right\} \\
&= \left\{\sum_{i=1}^r \lambda_i\varepsilon^{k-\beta_i}x_i \;\middle|\; \lambda_1, \dots, \lambda_r \geq 0, \sum_{i=1}^r \varepsilon^{k+\alpha_i-\beta_i}\lambda_i = 1,\right\} \\
&= \operatorname{wconv}\Big([x_1\varepsilon^{k-\beta_1}; k+\alpha_1-\beta_1], \dots, [x_r\varepsilon^{k-\beta_r}; k+\alpha_r-\beta_r]\Big),
\end{aligned}$$

as we wanted. □

Corollary 2.10. *For a polytope $P = \mathrm{conv}_{\mathbb{D}}(x_1, \dots, x_r)$ and any element $y \in M_{\mathbb{D}}$, the face of P induced by y is given by*

$$\mathrm{face}_y(P) = \mathrm{wconv}_{\mathbb{D}}\left([x_1\varepsilon^{k-\beta_1}; k-\beta_1], \dots, [x_r\varepsilon^{k-\beta_r}; k-\beta_r]\right)$$

where $\beta_i = \mathrm{ord}(\langle y\,, x_i\rangle - a)$ with $a = \min_{x\in P}\langle y\,, x\rangle$.

Proof. It follows from the previous proposition by considering the equation

$$\mathrm{wconv}_{\mathbb{D}}([x_1, 0], \dots, [x_r, 0]) = \mathrm{conv}_{\mathbb{D}}(x_1, \dots, x_r).$$

□

Remark 2.11. In general, a face of a polytope is not necesarily a polytope. For example, consider the polytope $P = \mathrm{conv}_{\mathbb{D}}\{0, 1\} = [0, 1] \subseteq \mathbb{D}$ and the face

$$\mathrm{face}_{\varepsilon^{k-1}}(P) = \mathrm{cone}_{\mathbb{D}}(\varepsilon).$$

This is not a polytope as, on a polytope, any linear function always attains its minimum in at least one of its vertices, in fact, this characterize a polytope as we will see in Corollary 3.28. However, in $\mathrm{cone}_{\mathbb{D}}(\varepsilon)$ the linear function $y = -\varepsilon$ does not achieve a minimum at all.

Finally, we introduce the relative interior of a polyhedron. We cannot introduce this concept as a topological interior, as we do not have appropriate topological tools over the ring $\mathbb{D}$. For this reason, we introduce it combinatorially by means of the structure of faces, and we show that, in the case of polyhedral cones, this coincides with an algebraic construction in terms of generators.

Definition 2.12. Let P be a polyhedron. The *relative interior* of P is the set

$$\mathrm{int}(P) \coloneqq P \setminus \bigcup_{\substack{F \preceq P \\ F \neq P}} F$$

where the union goes over all the proper faces F of P.

The next proposition summarize some basic proprieties of this concept.

Proposition 2.13. *For a given polyhedron $P \subseteq N_{\mathbb{D}}$:*

(1) There is a decomposition

$$P = \bigsqcup_{F \preceq P} \mathrm{int}(F),$$

where the disjoint union goes over all the different faces of P.

(2) For a face F of P and an element $x \in \mathrm{int}(F)$. A face G of P contains x iff $F \subseteq G$.

(3) We have $\mathrm{int}(P) \neq \varnothing$.

(4) Given a finitely generated cone $\sigma = \mathrm{cone}_{\mathbb{D}}(x_1, \dots, x_r)$, the relative interior can be computed as

$$\mathrm{int}\,\sigma = \left\{ \sum_{i=1}^{r} \lambda_i x_i \;\middle|\; \lambda_1, \dots, \lambda_r \in \mathbb{D}^{\times}_{>0} \right\}. \tag{2.8}$$

Proof.

(1) Given $x \in P$. Let F be the smallest face of P containing x, this exists because of Corollary 2.3(2)[2] and the fact that there are only finitely many faces, a consequence of Proposition 2.1 part 1. Given this face, we have

$$x \in F \setminus \bigcup_{\substack{G \preceq F \\ G \neq F}} F$$

therefore $x \in \operatorname{int}(F)$. This shows that $\sigma = \bigcup_{\tau \leq \sigma} \operatorname{int}(\tau)$.

Now, to see that the union is disjoint, notice that if $x \in \operatorname{int}(F) \cap \operatorname{int}(G)$, then $x \in F \cap G$, which is a face by Corollary 2.3. But as $x \in \operatorname{int}(F)$, this is only possible if $F \cap G = F$, that is $F \supseteq G$. Similarly $G \supseteq F$ so $F = G$.

(2) If $F \not\subseteq G$, then $x \in F \cap G$ and $F \cap G$ is a face of P contained in F, hence it is a proper face of F. Then, $x \notin F \setminus F \cap G$ but as $\operatorname{int}(F) \subseteq F \setminus F \cap G$ we get a contradiction.

(3) For each polyhedron P consider

$$\operatorname{length}(P) = \max \left\{ s \in \mathbb{N} \mid \exists\, F_1, \ldots, F_s \in \mathfrak{F}(P)^* \text{ s.t } \varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_s = P \right\}.$$

We show that $\operatorname{int}(P) \neq \varnothing$ by induction on $\operatorname{length}(P)$.

If $\operatorname{length}(P) = 1$, then P does not have any proper face, hence $\operatorname{int}(P) = P \neq \varnothing$. Now, suppose that $\operatorname{length}(P) = s + 1$ and the result is true whenever the length is smaller or equal to s. Consider a maximal chain of faces of P of the form

$$\varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_s \subsetneq F_{s+1} = P.$$

Then, $\operatorname{length}(F_s) = s$, and so, by the induction hypothesis, we have $\operatorname{int}(F_s) \neq \varnothing$. Take $x \in \operatorname{int}(F_s)$ and $x' \in P \setminus F_s$. Then, we claim that $\frac{1}{2}(x + x') \in \operatorname{int}(P)$.

Indeed, if $\frac{1}{2}(x + x') \notin \operatorname{int}(P)$, then there is a $y \in M_{\mathbb{D}}$ such that

$$\frac{1}{2}(x + x') \in \operatorname{face}_y(P) \subsetneq P.$$

In particular, y achieves its minimum in $\frac{1}{2}(x + x')$. Moreover, as we have

$$\begin{aligned} \langle y \,, \tfrac{1}{2}(x + x') \rangle &= \tfrac{1}{2}(\langle y \,, x \rangle + \langle y \,, x' \rangle) \\ &\geq \min\{\langle y \,, x \rangle, \langle y \,, x' \rangle\} \end{aligned}$$

with equality iff $\langle y \,, \frac{1}{2}(x + x') \rangle = \langle y \,, x \rangle = \langle y \,, x' \rangle$. Therefore, we have that y also achieves its minimum in x and x', that is $x, x' \in \operatorname{face}_y(P)$. As $x \in \operatorname{int}(F_s)$, by part (2) of this proposition we have $F_s \subseteq \operatorname{face}_y(P)$, and as $x' \notin F_s$ we also have $F_s \neq \operatorname{face}_y(P)$. Hence, by the maximality of s, we get $\operatorname{face}_y(P) = P$ which is a contradiction. Therefore, $\frac{1}{2}(x + x') \in \operatorname{int}(P)$, so $\operatorname{int}(P) \neq \varnothing$ as we wanted.

(4) We will prove the equality by a double inclusion.

First, take an x in the right-hand side of (2.8). For every face $\tau \subseteq \sigma$ there is $y \in M_{\mathbb{D}}$ attaining its minimum in σ such that $\tau = \operatorname{face}_y \sigma$. As σ is a polyhedral

[2] This result is proved in Section 2 and it is based on Proposition 2.1, parts (1) and (3). It might be worth mentioning that this does not produce any loop in the logic.

cone, y must be non-negative over σ to achieve its minimum. Now, suppose that $x = \sum_{i=1}^{r} \lambda_i x_i \in \tau$. Then, we must have

$$\langle y\,, x\rangle = \sum_{i=1}^{r} \lambda_i \langle y\,, x_i\rangle = 0.$$

As each term of the sum is non-negative, this happens iff $\lambda_i\langle y\,, x_i\rangle = 0$ for each i, and as all λ_i are invertible, this is the same as $\langle y\,, x_i\rangle = 0$ for each i. Then, we have $x_1, \dots, x_r \in \tau$ so $\sigma = \mathrm{cone}_D(x_1, \dots, x_r) \subseteq \tau$, and hence $\tau = \sigma$. This shows that

$$x \in \sigma \setminus \bigcup_{\substack{\tau \preceq \sigma \\ \tau \neq \sigma}} \tau = \mathrm{int}(\sigma).$$

On the other hand, take $x \in \mathrm{int}(\sigma)$ and fix some $1 \leq i \leq r$. We claim that there is a $\lambda \in \mathbb{D}^{\times}_{>0}$ such that

$$x - \lambda x_i \in \sigma.$$

If this is not the case, as σ is a polyhedral cone by Proposition 1.27 part 4, there are $y_1, \dots, y_s \in M_{\mathbb{D}}$ such that

$$\sigma = \{y_1 \geq 0, \dots, y_s \geq 0\}.$$

Then, there must be a y_j such that for every $\lambda \in \mathbb{D}^{\times}_{>0}$ small enough we have

$$\langle y_j\,, x - \lambda x_i\rangle < 0.$$

That is, $0 \leq \langle y_j\,, x\rangle < \lambda\langle y_j\,, x_i\rangle$ for every $\lambda \in \mathbb{D}^{\times}_{>0}$ small enough, which implies that $\langle y_j\,, x\rangle$ is *infinitesimally smaller* than $\langle y_j\,, x_i\rangle$, that is, $\langle y_j\,, x\rangle = \varepsilon\mu\langle y_j\,, x_i\rangle$ for some $\mu \in \mathbb{D}_{\geq 0}$. Then, for some $l \geq 0$ we have $\varepsilon^l\langle y_j\,, x\rangle = 0$ but $\varepsilon^l\langle y_j\,, x_i\rangle \neq 0$. Therefore, $y_j\varepsilon^l \in M_{\mathbb{D}}$ defines a face $\mathrm{face}_{y_j\varepsilon^l}\,\sigma$ containing x but not x_i which contradicts the fact that x is in the relative interior. This finishes the proof of the claim. Therefore, for each $1 \leq i \leq r$ there is a λ such that

$$x = \lambda x_i + x'$$

for some $x' \in \sigma$. By writting x' in terms of $x_1, \dots, x_r$ we get a representation of x in the form

$$x = \sum_{i=1}^{r} \lambda_i x_i$$

with $\lambda_i \in \mathbb{D}^{\times}_{>0}$ and $\lambda_j \geq 0$ for $j \neq i$. By taking an average of all this representations for all i, we get a representation with all $\lambda_i \in \mathbb{D}^{\times}_{>0}$. This finishes the proof.

□

3. Duality Theory

3.1. **Cone Duality.** In this section we will study polyhedral cones and their duals. After introducing the dual we show how to find generators for it from a representation of the original cone, or how to find a representation for it from generators of the original cone. In particular, this implies that finitely generated cones are the same as polyhedral cones. The main result of this section is the duality theorem, which gives an explicit order reversing bijection between the faces of a cone and the faces of its dual.

Definition 3.1. Let $\sigma \subseteq N_{\mathbb{D}}$ be a cone. Its *dual cone* is the set of all linear functionals non-negative over it, that is,

$$\sigma^{\vee} := \{y \in M_{\mathbb{D}} \mid \langle y\,, x\rangle \geq 0,\ \forall\, x \in \sigma\}\,.$$

Proposition 3.2.

(1) Given $y_1, \ldots, y_r \in M_{\mathbb{D}}$ we have

$$\left(\bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i\,, x\rangle \geq 0\}\right)^{\vee} = \operatorname{conv}_{\mathbb{D}}(y_1, \ldots, y_r).$$

(2) Given $a_1, \ldots, a_r \in N_{\mathbb{D}}$, we have

$$\operatorname{conv}_{\mathbb{D}}(a_1, \ldots, a_r)^{\vee} = \bigcap_{i=1}^{r} \{y \in M_{\mathbb{D}} \mid \langle y\,, a_i\rangle \geq 0\}\,.$$

(3) For any polyhedral cone σ, we have $\left(\sigma^{\vee}\right)^{\vee} = \sigma$.

(4) Polyhedral cones are the same as finitely generated cones.

Proof.

(1) Let $\sigma = \bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i\,, x\rangle \geq 0\}$. We have that $\langle \lambda_1 y_1 + \cdots + \lambda_r y_r\,, \cdot\rangle$ is positive over σ, hence $\lambda_1 y_1 + \cdots + \lambda_r y_r \in \sigma^{\vee}$ for every $\lambda_1, \ldots, \lambda_r \geq 0$, then $\operatorname{cone}_{\mathbb{D}}(y_1, \ldots, y_r) \subseteq \sigma^{\vee}$. On the other hand, given $y \in \sigma^{\vee}$, we have that $\langle y, \cdot\rangle$ is positive over σ and its minimum is 0 on it. Hence, by Farkas' Lemma, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$\lambda_1 y_1 + \cdots + \lambda_r y_r = y.$$

Therefore, $y \in \operatorname{conv}_{\mathbb{D}}(y_1, \ldots, y_r)$, so $\sigma^{\vee} \subseteq \operatorname{conv}_{\mathbb{D}}(y_1, \ldots, y_r)$.

(2) We have

$$\begin{aligned} y \in \operatorname{cone}_{\mathbb{D}}(a_1, \ldots, a_r)^{\vee} &\iff \langle y\,, x\rangle \geq 0 \quad \forall\, x \in \operatorname{cone}_{\mathbb{D}}(a_1, \ldots, a_r) \\ &\iff \langle y\,, a_i\rangle \geq 0 \quad \forall\, 1 \leq i \leq r \\ &\iff y \in \bigcap_{i=1}^{r} \{y \in M_{\mathbb{D}} \mid \langle y\,, a_i\rangle \geq 0\}. \end{aligned}$$

(3) By part (1) and (2), given a polyhedral cone as $\bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i\,, x\rangle \geq 0\}$ we have

$$\begin{aligned} \left(\left(\bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i\,, x\rangle \geq 0\}\right)^{\vee}\right)^{\vee} &= \operatorname{conv}_{\mathbb{D}}(y_1, \ldots, y_r)^{\vee} \\ &= \bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i\,, x\rangle \geq 0\}\,. \end{aligned}$$

(4) In Proposition 1.27, we saw that that finitely generated cones are polyhedral. On the other hand, let us suppose that a cone $\sigma \subseteq N_{\mathbb{D}}$ is polyhedral. By part (1) then $\sigma^{\vee}$ is finitely generated and hence polyhedral by Proposition 1.27. So by part (1) again $(\sigma^{\vee})^{\vee} = \sigma$ is finitely generated.

□

Now, we will prove a duality result for cones, which states that the faces of a cone are in an order reversing correspondence with the faces of its dual cone.

Theorem 3.3 (Cone duality). *Given a polyhedral cone σ and its dual $\sigma^\vee$, there is an order reversing bijection between the reduced face poset of σ and the reduced face poset of its dual $\sigma^\vee$ given by*

$$\begin{aligned}\mathfrak{F}(\sigma)^* &\longrightarrow \mathfrak{F}(\sigma^\vee)^* \\ \tau &\longmapsto \tau^* := \tau^\perp \cap \sigma^\vee,\end{aligned}$$

where

$$\tau^\perp := \{y \in M_{\mathbb{D}} \mid \langle y\,, x\rangle = 0,\ \forall\, x \in \tau\}.$$

Proof. First, notice that, as τ is a polyhedral cone, it is finitely generated. Hence, $\tau = \operatorname{cone}_{\mathbb{D}}(x_1, \dots, x_r)$ for some $x_1, \dots, x_r \in \sigma$. Then, we have

$$\tau^* = \{y \in \sigma^\vee \mid \langle y\,, x_i\rangle = 0,\ \text{for } i = 1, \dots, r\} = \operatorname{face}_{x_1+\cdots+x_r}(\sigma^\vee)$$

Therefore, τ^* is a face of $\sigma^\vee$ and the map is well defined.

Now, let us see that the map is surjective: An arbitrary face of $\sigma^\vee$ is of the form $\operatorname{face}_{x_0}(\sigma^\vee)$ for some $x_0 \in \sigma$. By Proposition 2.13, there is a face τ of σ such that $x' \in \operatorname{int}(\tau)$. If we consider generators $\tau = \operatorname{cone}_D(x_1, \dots, x_r)$ then, by Proposition **??**, we have $x_0 = \sum_i \lambda_i x_i$ with $\lambda_i \in \mathbb{D}^\times_{>0}$. Hence, for $y \in \sigma^\vee$ we have

$$y \in \operatorname{face}_{x_0} \sigma^\vee \Leftrightarrow \langle y\,, x_0\rangle = 0 \Leftrightarrow \langle y\,, x_i\rangle = 0\;\forall\, 0 \le i \le r \Leftrightarrow \langle y, x\rangle = 0\;\forall\, x \in \tau.$$

Therefore, $\operatorname{face}_{x_0} \sigma^\vee = \tau^*$.

Finally, we will prove that the map is its own inverse: As the map is surjective it is enough to prove that $((\tau^*)^*)^* = \tau^*$. For this, we will prove that $(\tau^*)^* \subseteq \tau$. Suppose $\tau = \operatorname{face}_{y'} \sigma$ for some $y' \in \sigma^\vee$. Then,

$$\begin{aligned}(\tau^*)^* &= \{x \in \sigma \mid \langle y, x\rangle = 0\ \forall y \in \tau^*\} \\ &\subseteq \{x \in \sigma \mid \langle y', x\rangle = 0\} \\ &= \tau.\end{aligned}$$

This finishes the proof. □

3.2. **The Support of the Normal Fan.** In this section we introduce the *support of the normal fan* of a polyhedron P. This is the set of all elements $y \in M_{\mathbb{D}}$ for which $\operatorname{face}_y(P)$ is well defined. We regard it as a generalization of the dual cone of a polyhedral cone introduced in Section 3.1. The support of the normal fan for polyhedra over $\mathbb{D}$ happens to be a more subtle concept than its counterpart over $\mathbb{R}$, for instance, see Example 3.6 below. Moreover, for a polyhedron P we introduce its *recession cone* as the set of all directions for which, any point in the polyhedron that moves along this direction remain in the polyhedron. In Proposition 3.11 we show how the dual of the support of the normal cone coincides with its recession cone.

Definition 3.4. Let $P \subseteq N_{\mathbb{D}}$ be a non-empty polyhedron. The *support of the normal fan* of P, denoted by $|\mathrm{NF}(P)|$, is the set of all $y \in M_{\mathbb{D}}$ such that $\langle y\,,\cdot\rangle$ achieves its minimum over P. That is,

$$|\mathrm{NF}(P)| \coloneqq \left\{ w \in M_{\mathbb{D}} \;\middle|\; \min_P \langle y\,,\cdot\rangle \;\text{ exists} \right\}.$$

Remark 3.5. If $\sigma \subseteq N_{\mathbb{D}}$ is a polyhedral cone, then $|\mathrm{NF}(\sigma)|$ recovers the dual cone $\sigma^\vee$.

Example 3.6. Given a polyhedron P, the set $|\mathrm{NF}(P)|$ is always closed under positive scalar multiplications, but is not convex in general. Indeed, consider $N = M = \mathbb{Z}^3$ together with the polyhedron

$$P = \{(x, y, z) \in \mathbb{D}^3 \mid x \geq 0,\, y \geq 0,\, z \geq 0,\, x + y + \varepsilon z = 1\}.$$

Then, we have $(1, 0, 0), (0, 1, 0) \in \mathrm{NF}(P)$ as both of these elements achieve 0 as their minimum over P. Nonetheless, their sum $(1, 1, 0)$ does not achieve its minimum over P, as for $(x, y, z) \in P$, we have

$$\langle (1, 1, 0)\,, (x, y, z)\rangle = x + y = 1 - \varepsilon z,$$

and the set $\{1 - \varepsilon z \in \mathbb{D} \mid z \geq 0\}$ does not have a minimum. Therefore, $|\mathrm{NF}(P)|$ is not a convex cone in this case.

Question 3.7. Is there a simple characterization for the sets of the form $|\mathrm{NF}(P)| \subseteq M_{\mathbb{D}}$ for some polyhedron $P \subseteq N_{\mathbb{D}}$?

Although we do not know the answer to the question above, in the following proposition we provide an understanding of $\mathrm{cone}_{\mathbb{D}} |\mathrm{NF}(P)|$ in terms of a representation of P. Moreover, in Section 3.4 we give a characterization for the polyhedra P for which $|\mathrm{NF}(P)|$ is convex.

Proposition 3.8. *Let P be a non-empty polyhedron and suppose that*

$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}$$

is a non-redundant representation of P. Then,

$$\mathrm{cone}_{\mathbb{D}} |\mathrm{NF}(P)| = \mathrm{cone}_{\mathbb{D}}(y_1, \dots, y_r).$$

Remark 3.9.

(1) As $|\mathrm{NF}(P)|$ is closed under positive scalar multiplications, we can replace $\mathrm{cone}_{\mathbb{D}} |\mathrm{NF}(P)|$ by $\mathrm{conv}_{\mathbb{D}} |(\mathrm{NF}(P))|$ in the statement above.

(2) The assumption that the representation is not-redundant is unavoidable in the hypothesis. For example, for $N = M = \mathbb{Z}$ consider

$$P = \{x \in \mathbb{D} \mid \langle \varepsilon\,, x\rangle \geq 0, \langle 1\,, x\rangle \geq -1\}.$$

Then, $1 \notin \mathrm{cone}_{\mathbb{D}} |\mathrm{NF}(P)| = \mathrm{cone}_{\mathbb{D}}(\varepsilon)$ and this does not contradict the statement of the result as the inequality $\langle 1\,, x\rangle \geq -1$ is redundant in the representation.

Proof of Proposition 3.8. As the representation is non-redundant, by Proposition 1.8, each y_i attains its minimum in P. Hence, $y_i \in |\mathrm{NF}(P)|$ for each i. This shows $\mathrm{cone}_{\mathbb{D}}(y_1, \dots, y_r) \subseteq \mathrm{cone}_{\mathbb{D}} |\mathrm{NF}(P)|$. On the other hand, given $y \in |\mathrm{NF}(P)|$, as y achieves its minimum in P, by Proposition 2.1 there are $\lambda_1 \dots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$y = \lambda_1 y_1 + \cdots + \lambda_r y_r$$

Hence, $y \in \operatorname{cone}_{\mathbb{D}}(y_1, \dots, y_r)$. This shows $\operatorname{cone}_{\mathbb{D}} |\mathrm{NF}(P)| \subseteq \operatorname{cone}_{\mathbb{D}}(y_1, \dots, y_r)$. □

Definition 3.10. The recession cone of P is the set

$$\operatorname{recc}(P) := \{x \in N_{\mathbb{D}} \mid P + x \subseteq P\}.$$

Proposition 3.11. *Let $P \subseteq N_{\mathbb{D}}$ be a non-empty polyhedron. Then,*

(1) given a non-redundant representation of P

$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}$$

we have

$$\operatorname{recc}(P) = \{y_1 \geq 0, \dots, y_r \geq 0\}$$

(2) The dual of the cone hull of the support of the normal fan of P is the recession cone of P, that is,

$$\operatorname{cone}_{\mathbb{D}}(|\mathrm{NF}(P)|)^{\vee} = \operatorname{recc}(P).$$

In particular, $\operatorname{recc}(P)$ *is a polyhedral cone. Moreover, by duality* $\operatorname{recc}(P)^{\vee} = \operatorname{cone}_{\mathbb{D}} |\mathrm{NF}(P)|$.

Proof. Notice that, by Proposition 3.8 together with Proposition 3.2 we have

$$\operatorname{cone}_D(|\mathrm{NF}(P)|)^{\vee} = \{y_1 \geq 0, \dots, y_r \geq 0\}.$$

Hence, it is enough to prove that $\operatorname{recc}(P)$ is equal to any of these sets. If $x' \in N_{\mathbb{D}}$ satisfies $\langle y_i, x' \rangle \geq 0$ for every $1 \leq i \leq r$, then for any $x \in P$, we have

$$\langle y_i, x + x' \rangle = \langle y_i, x \rangle + \langle y, x' \rangle \geq a_i \quad \forall 1 \leq i \leq r.$$

Thus, $P + x' \subseteq P$. This shows $\operatorname{cone}_{\mathbb{D}}(|\mathrm{NF}(P)|)^{\vee} \subseteq \operatorname{recc}(P)$. On the other hand, by Proposition 1.8, for any $1 \leq i \leq r$, there is an $x \in P$ such that $\langle y_i, x \rangle = a_i$. Then, given $x' \in \operatorname{recc}(P)$, we must have $x + x' \in P$. In particular, $\langle y_i, x + x' \rangle \geq a_i$, from which we infer that $\langle y_i, x' \rangle \geq 0$. As this happens for each $1 \leq i \leq r$, we must have $x' \in \operatorname{cone}_{\mathbb{D}}(|\mathrm{NF}(P)|)^{\vee}$, and so $\operatorname{recc}(P) \subseteq \operatorname{cone}_{\mathbb{D}}(|\mathrm{NF}(P)|)^{\vee}$. □

Corollary 3.12. *If P is simultaneously a polyhedron and a convex cone (in the sense of Definition 1.3), then P is a polyhedral cone.*

Proof. By the previous proposition, $\operatorname{recc}(P)$ is a polyhedral cone, so it is enough to prove that $P = \operatorname{recc}(P)$. As $0 \in P$ we have $0 + \operatorname{recc}(P) = \operatorname{recc}(P) \subseteq P$. On the other hand, as P is a convex cone, we have $P + P \subseteq P$, hence $P \subseteq \operatorname{recc}(P)$. □

3.3. **Normal Fan Duality.** In this section we introduce the *normal fan* of a polyhedron P. This is an arrangement of polyhedral cones in $M_{\mathbb{D}}$ encoding the behavior of the function $y \mapsto \operatorname{face}_y P$. Its construction provides a generalization of the cone duality in Theorem 3.3, and it gives us an important tool to study the combinatorial type of a polyhedron, as we do in Section 4.2 for $\mathbb{R}$-rational polyhedra.

Definition 3.13. Let $P \subseteq N_{\mathbb{D}}$ be a polyhedron. For each face F of P, its *normal cone* is the set

$$\mathrm{C}(F) := \{y \in M_{\mathbb{D}} \mid \operatorname{face}_y P \supseteq F\}.$$

That is, the set of all $y \in M_{\mathbb{D}}$ such that $\operatorname{face}_y(P)$ exists and contains F.

Proposition 3.14. *Given a face F of a polyhedron P. The normal cone $\mathrm{C}(F)$ is a polyhedral cone. More precisely, given a non-redundant representation*

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$$

and an element $x \in \operatorname{int} F$, we have

$$\mathrm{C}(F) = \operatorname{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} y_1, \ldots, \varepsilon^{k-\alpha_r} y_r\right)$$

where $\alpha_i = \operatorname{ord}(\langle y_i\,, x\rangle - a_i)$.

Proof. Given $y \in \mathrm{C}(F)$, if $\min_{x\in P}\langle y\,, x\rangle = a$, then, by Farkas' Lemma, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$\langle y\,, \cdot\rangle - a = \lambda_1(\langle y_1\,, \cdot\rangle - a_1) + \cdots + \lambda_r(\langle y_r\,, \cdot\rangle - a_r).$$

By evaluating this equality in $x \in \operatorname{int}(F)$, the left hand side is 0 and the right hand side is a sum of non-negative terms. Hence, each term of the sum must be zero and we get

$$\lambda_i(\langle y_i\,, x\rangle - a_i) = 0 \quad \forall 1 \leq i \leq r.$$

So, if $\alpha_i = \operatorname{ord}(\langle y_i\,, x\rangle - a_i)$ then there are $\lambda'_i \in \mathbb{D}_{\geq 0}$ such that $\lambda_i = \varepsilon^{k-\alpha_i}\lambda'_i$. Therefore,

$$y = \varepsilon^{k-\alpha_1}\lambda'_1 y_1 + \cdots + \varepsilon^{k-\alpha_r}\lambda'_r y_r. \tag{3.1}$$

On the other hand, if y is of the form (3.1) above, then $\langle y\,, x\rangle = a$. Therefore, $x \in \operatorname{face}_y(P)$. But as $x \in \operatorname{int}(F)$, by Proposition 2.13, we must have $F \subseteq \operatorname{face}_y(P)$. That is, $y \in \mathrm{C}(F)$.

With this we conclude that

$$\mathrm{C}(F) = \operatorname{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} y_1, \ldots, \varepsilon^{k-\alpha_r} y_r\right)$$

as we wanted. □

The normal cone $\mathrm{C}(F)$ encodes the local shape of P around F. To make this concrete we introduce the following notion.

Definition 3.15. Let $P \subseteq N_{\mathbb{D}}$ be a polyhedron and let F be a face of P. The *star* of F with respect to P is the set

$$\operatorname{Star}_P(F) \coloneqq \{\lambda(x - x') \in N_{\mathbb{D}} \mid x \in P, x' \in F, \lambda \in \mathbb{D}^{\times}_{>0}\}.$$

Lemma 3.16. *Let P be a polyhedron and F a face of P, then*

$$\mathrm{C}(F)^{\vee} = \operatorname{Star}_P(F).$$

Proof. Fix elements $x \in P$, $x' \in F$ and $\lambda \in \mathbb{D}^{\times}_{>0}$. For any $y \in \mathrm{C}(F)$, as y achieves its minimum at x', we have $\langle y\,, x\rangle \geq \langle y\,, x'\rangle$. Hence, $\langle y\,, \lambda(x-x')\rangle \geq 0$, so $\lambda(x-x') \in \mathrm{C}(F)^{\vee}$. This shows that $\operatorname{Star}_P(F) \subseteq \mathrm{C}(F)^{\vee}$.

We will now prove the other direction, for this fix a representation

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}.$$

Then, by Proposition 2.1, there are $0 \leq \alpha_i \leq k$ such that

$$F = \bigcap_{i=1}^{r} \{x \in P \mid \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i} a_i\}. \tag{3.2}$$

Hence, an element $x' \in F$ satisfies

$$\varepsilon^{\alpha_i}\langle y_i\,, x'\rangle = \varepsilon^{\alpha_i} a_i \quad \forall\, 1 \leq i \leq r. \tag{3.3}$$

Moreover, we can take x' in such a way that

$$\varepsilon^{\alpha_i - 1}\langle y_i\,, x'\rangle > \varepsilon^{\alpha_i - 1} a_i \quad \forall\, 1 \leq i \leq r \text{ with } \alpha_i \geq 1 \tag{3.4}$$

Indeed, if for a certain i, we have

$$\varepsilon^{\alpha_i - 1}\langle y_i\,, x'\rangle = \varepsilon^{\alpha_i - 1} a_i, \quad \forall\, x' \in F.$$

Then, we can replace α_i with $\alpha_i - 1$ in (3.2) without altering the set F. We can proceed in this way and eventually there will be an $x_i' \in F$ such that

$$\varepsilon^{\alpha_i - 1}\langle y_i\,, x_i'\rangle > \varepsilon^{\alpha_i - 1} a_i.$$

After doing this for every i we can take $x' = \frac{1}{n}\sum_{i=1}^{r} x_i'$.

We will now prove that for every $w \in \mathrm{C}(F)^\vee$ there is a $\lambda \in \mathbb{D}^\times_{>0}$ such that $\lambda w + x' \in P$. This will finish the proof because then $w = \lambda^{-1}(x - x') \in \mathrm{Star}_P(F)$, so $\mathrm{C}(F)^\vee \subseteq \mathrm{Star}_P(F)$ as we needed. To prove this, notice that there is a $\lambda \in \mathbb{D}^\times_{>0}$ such that $\lambda w + x' \in P$ iff for each $1 \leq i \leq r$ there is a $\lambda_i \in \mathbb{D}^\times_{>0}$ such that

$$\langle y_i\,, \lambda_i w + x'\rangle \geq a_i$$

as then we can take $\lambda = \min_i\{\lambda_i\}$. We will work now with a fixed i and show that such a λ_i exist in all the possible cases in which the element $\langle y_i\,, w\rangle$ can be. Notice that as $w \in \mathrm{C}(F)^\vee$ and $\varepsilon^{\alpha_i} y_i \in NC(F)$ (as it attains its minimum $\varepsilon^{\alpha_i} a_i$ on F) we have

$$\varepsilon^{\alpha_i}\langle y_i\,, w\rangle \geq 0.$$

- If $\varepsilon^{\alpha_i}\langle y_i\,, w\rangle > 0$ we are done, as this together with (3.3) give us $\varepsilon^{\alpha_i}\langle y_i\,, w + x'\rangle > \varepsilon^{\alpha_i} a_i$ which implies $\langle y_i\,, w + x'\rangle > a_i$ so we can take $\lambda_i = 1$.
- If $\varepsilon^{\alpha_i}\langle y_i\,, w\rangle = 0$ and $\alpha_i = 0$ we have $\langle y_i\,, \lambda_i w + x'\rangle = \langle y_i\,, x'\rangle \geq a_i$ so any $\lambda_i \in \mathbb{D}^\times_{>0}$ works.
- If $\varepsilon^{\alpha_i}\langle y_i\,, w\rangle = 0$ and $\alpha_i > 0$ then, it is enough to find λ_i small enough such that
$$\varepsilon^{\alpha_i}\langle y_i\,, \lambda_i w + x'\rangle > \varepsilon^{\alpha_i} a_i \iff \lambda_i \varepsilon^{\alpha_i - 1}\langle y_i\,, w\rangle > \varepsilon^{\alpha_i - 1} a_i - \varepsilon^{\alpha_i - 1}\langle y_i\,, x'\rangle.$$
In this last inequality, both $\varepsilon^{\alpha_i - 1}\langle y_i\,, w\rangle$ and $\varepsilon^{\alpha_i - 1} a_i - \varepsilon^{\alpha_i - 1}\langle y_i\,, x'\rangle$ are of the form $\varepsilon^{k-1} A$ with $A \in \mathbb{R}$. As the right hand side is negative by (3.4), by taking $\lambda_i \in \mathbb{R}_{>0}$ small enough we can always make the left hand side bigger.

□

Theorem 3.17 (Normal fan duality)**.** *Given a polyhedron $P \subseteq N_{\mathbb{D}}$, the family*

$$\mathrm{NF}(P) = \{\mathrm{C}(F) \subseteq M_{\mathbb{D}} \mid F \in \mathfrak{F}(P)^*\}$$

is a fan whose support is $|\mathrm{NF}(P)|$. Moreover, for a polyhedral cone σ we have $\mathrm{C}(\tau) = \tau^*$ for each face τ of σ.

Proof. By Proposition 3.14 each set $C(F)$ is a polyhedral cone. Also, for $F, G \in \mathfrak{F}(P)^*$ we have

$$y \in \mathrm{C}(F) \cap \mathrm{C}(G) \iff \mathrm{face}_y(P) \supseteq F \cup G \iff \mathrm{face}_y(P) \supseteq F \vee G \iff y \in \mathrm{C}(F \vee G).$$

Hence, $\mathrm{C}(F) \cap \mathrm{C}(G) = \mathrm{C}(F \vee G)$. Also, a face $\tau \preceq \mathrm{C}(F)$ is defined by an element $x_0 \in \mathrm{C}(F)^\vee$. By Lemma 3.16 we have $\mathrm{C}(F)^\vee = \mathrm{Star}_P(F)$, hence $x_0 = \lambda(x - x')$ for $\lambda \in \mathbb{D}^\times_{>0}$, $x \in P$ and $x' \in F$. Therefore,

$$\begin{aligned}\tau =& \operatorname{face}_{x_0}(C(F)) \\ =& \{y \in \mathrm{C}(F) \mid \langle y , x_0\rangle = 0\} \\ =& \{y \in \mathrm{C}(F) \mid \langle y , x\rangle = \langle y , x'\rangle\} \\ =& \{y \in \mathrm{C}(F) \mid \min_{w\in P}\langle y , w\rangle = \langle y , x\rangle\} \\ =& \{y \in P \mid \operatorname{face}_y(P) \supseteq F \cup \{x\}\} \\ =& \mathrm{C}(F \vee G)\end{aligned}$$

where G is the only face of P such that $x \in \operatorname{int} G$. With this we have shown that $\mathrm{NF}(P)$ is a fan. Finally, for a polyhedral cone σ and a face τ of σ we have that

$$\begin{aligned}C(\tau) =& \{y \in |\mathrm{NF}(\sigma)| \mid \operatorname{face}_y \supseteq \tau\} \\ =& \{y \in \sigma^\vee \mid \langle y , x\rangle = 0 \ \forall x \in \tau\} \\ =& \sigma^\vee \cap \tau^\perp \\ =& \tau^*.\end{aligned}$$

□

Remark 3.18.

(1) The name of the theorem comes from the fact that the normal fan gives us an order reversing bijection

$$\mathfrak{F}(P)^* \xrightarrow{\sim} \mathrm{NF}(P)$$

in which each face $F \preceq P$ is orthogonal to its corresponding face $\mathrm{C}(F) \in \mathrm{NF}(P)$.

(2) If σ is a polyhedral cone then, the bijection above is given by

$$\begin{aligned}\mathfrak{F}(\sigma)^* &\xrightarrow{\sim} \mathrm{NF}(\sigma) \\ \tau &\longmapsto \tau^*.\end{aligned}$$

Therefore, this maps correspond to the one from the cone duality in Theorem 3.3. In this way, we see that the normal fan duality in Theorem 3.17 is a strict generalization of the dual cone duality in Theorem 3.3.

We finalize with the following concept.

Definition 3.19. A fan Σ in $M_{\mathbb{D}}$ is said to be *regular* if there is a polyhedron P such that $\Sigma = \mathrm{NF}(P)$.

3.4. **The Support Function.** In this section we go one step further in our dual understanding of a polyhedron and consider the map $y \mapsto \min_{x\in P}\langle y , x\rangle$. This is a piecewise linear concave function called the *support function* of the polyhedron P. Under mild hypothesis, in Theorem 3.23 we use the support function to obtain an alternative description of the normal fan, and in Theorem 3.25 we show how the support function gives us a bijection between polyhedra and piecewise linear concave functions. We use this in Corollary 3.28 to understand when a given polyhedron has a *Minkowski-Weyl decomposition*. That is, an equation of the form $P = Q + \sigma$, where Q is a polytope and σ is a polyhedral cone. In particular, this characterization allow us to show that polytopes are exactly the polyhedra in which any linear function achieves its minimum

Definition 3.20. Given a polyhedron P, we define its *support function* as the map

$$\begin{aligned} h_P : |\mathrm{NF}(P)| &\longrightarrow \mathbb{D} \\ y &\longmapsto \min_{x \in P} \langle y \,, x \rangle. \end{aligned}$$

Remark 3.21.

(1) The function h_P is positive homogeneous in the sense that, for any $\lambda \in \mathbb{D}_{\geq 0}$, if $y \in |\mathrm{NF}(P)|$ then $h_P(\lambda y) = \lambda h_P(y)$.
(2) For each face $F \preceq P$, if we take a point $x_F \in \mathrm{int}(F)$, then

$$h_P(y) = \min_{x \in P} \langle y \,, x \rangle = \langle y \,, x_F \rangle$$

for each $y \in C(F)$. In particular, h_P is linear along $C(F)$.
(3) The minimum in the definition of h_P can be taken to be finite because, as above, if we take for each face $F \preceq P$ a point $x_F \in \mathrm{int}(P)$ then,

$$h_P(y) = \min_{F \in \mathfrak{F}(P)^*} \langle y \,, x_F \rangle.$$

(4) If $P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$ is a non-redundant representation of P, then by Proposition 1.8 we have $h_P(y_i) = a_i$.
(5) From the support of the normal fan and the support function, we can recover the polyhedron as

$$P = \bigcap_{y \in |\mathrm{NF}(P)|} \{x \in N_{\mathbb{D}} \mid \langle y \,, x \rangle \geq h_P(y)\}.$$

We can use the support function to give new characterizations of the normal fan. For this, we will use the concepts from Definition 1.12 together with the following one.

Definition 3.22. Given a polyhedron P, the lifted normal fan is the set

$$|\mathrm{NF}(P)|^h \coloneqq \mathrm{cone}_{\mathbb{D}} \left\{ (y, h_P(y)) \in M_{\mathbb{D}} \times \mathbb{D} \;\middle|\; y \in |\mathrm{NF}(P)| \right\}.$$

Theorem 3.23. *Let P be a polyhedron with a non-redundant representation $P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$. The lifted normal fan can be computed as*

$$|\mathrm{NF}(P)|^h = \mathrm{cone}_{\mathbb{D}} \big((y_1, a_1), \ldots, (y_r, a_r) \big).$$

Moreover, $\mathrm{NF}(P)$ *can be obtained as the family of all projections of the upper faces of* $|\mathrm{NF}(P)|^h$ *from* $M_{\mathbb{D}} \times \mathbb{D}$ *to* $M_{\mathbb{D}}$.

Proof. The proof goes in three steps.

(1) $|\mathrm{NF}(P)|^h = \mathrm{cone}_{\mathbb{D}}((y_1, a_1), \ldots, (y_r, a_r))$:

As $y_i \in |\mathrm{NF}(P)|$ for each $i = 1, \ldots, r$, we have $(y_i, h(y_i)) = (y_i, a_i) \in |\mathrm{NF}(P)|^h$, hence

$$|\mathrm{NF}(P)|^h \supseteq \mathrm{cone}_{\mathbb{D}}((y_1, a_1), \ldots, (y_r, a_r))$$

For the other inclusion, by Proposition 3.14, for any face F of P we have

$$\mathrm{C}(F) = \mathrm{cone}_{\mathbb{D}} \left(\varepsilon^{k - \alpha_1} y_1, \ldots, \varepsilon^{k - \alpha_r} y_r \right)$$

where $\alpha_i = \mathrm{ord}(\langle y_i \,, x\rangle - a_i)$. Hence, for $y \in \mathrm{C}(F)$ there are $\lambda_1, \dots, \lambda_r \in \mathbb{D}_{\geq} 0$ such that

$$y = \lambda_1 \varepsilon^{k-\alpha_1} y_1 + \cdots + \lambda_r \varepsilon^{k-\alpha_r} y_r.$$

Moreover, as h_P is positive homogeneous and linear over $C(F)$ we have

$$\begin{aligned} h_P(y) &= \lambda_1 h_P(\varepsilon^{k-\alpha_1} y_1) + \cdots + \lambda_r h_P(\varepsilon^{k-\alpha_r} y_r) \\ &= \lambda_1 \varepsilon^{k-\alpha_1} h_P(y_1) + \cdots + \lambda_r \varepsilon^{k-\alpha_r} h_P(y_r) \\ &= \lambda_1 \varepsilon^{k-\alpha_1} a_1 + \cdots + \lambda_r \varepsilon^{k-\alpha_r} a_r. \end{aligned}$$

Hence, $(y, h_P(y)) \in \mathrm{cone}_{\mathbb{D}}((y_1, a_1), \dots, (y_r, a_r))$ for any $y \in C(F)$. As $|\mathrm{NF}(P)| = \bigcup_F C(F)$ we conclude that $|\mathrm{NF}(P)|^h = \mathrm{cone}_{\mathbb{D}}((y_1, a_1), \dots, (y_r, a_r))$.

(2) $\mathrm{face}_{(x,-1)}(Q)$ can be considered iff $x \in P$:

We have that $\mathrm{face}_{(x,-1)}(Q)$ exists iff $(x, 1) \in Q^\vee$. Moreover, for $x \in N_{\mathbb{D}}$

$$\begin{aligned} (x, 1) \in Q^\vee &\iff \langle (y, h_P(y)) \,, (x, -1)\rangle \geq 0 \text{ for every } y \in |\mathrm{NF}(P)| \\ &\iff \langle y \,, x\rangle \geq h(y) \text{ for every } y \in \mathrm{NF}(P) \\ &\iff x \in P. \end{aligned}$$

(3) For $x \in P$, if $x \in \mathrm{int}(F)$ then $\mathrm{face}_{(x,-1)}(Q) = F$:

By Proposition 2.4, using the generators from part (1) we get

$$\mathrm{face}_{(x,-1)}(Q) = \mathrm{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} (y_1, a_1) \,, \dots, \varepsilon^{k-\alpha_s} (y_r, a_r) \right),$$

where

$$\alpha_i = \mathrm{ord}\langle (y_i, h_P(y_i)) \,, (x, -1)\rangle = \mathrm{ord}(\langle w_i \,, x\rangle - h_P(y_i)) = \mathrm{ord}(\langle w_i \,, x\rangle - a_i).$$

Hence, if π denotes the projection from $M_{\mathbb{D}}$ to $\mathbb{D}$ we have

$$\pi \left(\mathrm{face}_{(x,-1)}(Q) \right) = \mathrm{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} y_1, \dots, \varepsilon^{k-\alpha_s} y_r \right),$$

which is exactly equal to $C(F)$ by Proposition 3.14.

□

Definition 3.24. Given a polyhedral cone $\sigma \subseteq M_{\mathbb{D}}$, a function $l : \sigma \to \mathbb{D}$ is called *piecewise linear concave* if there is a finite subset $A \subseteq N_{\mathbb{D}}$ such that

$$l(y) = \min_{x \in A} \langle y \,, x\rangle, \quad \forall\, y \in \sigma.$$

Theorem 3.25 (Higher rank Minkowki theorem)**.** *There is a bijection between polyhedra with convex normal fan and polyhedral cones endowed with concave linear functions. Explicitly:*

(1) We associate to a polyhedron P with convex normal fan the pair

$$\Psi(P) = (|\mathrm{NF}(P)|, h_P).$$

(2) We associate to a pair (σ, h) the polyhedron

$$\Phi(\sigma, h) = \mathrm{conv}_{\mathbb{D}}(A) + \sigma^\vee$$

where $A \subseteq N_{\mathbb{D}}$ is a finite subset such that $h = \min_{x \in A} \langle \cdot \,, x\rangle$.

Proof. The map in Ψ is well define because, as $|\mathrm{NF}(P)|$ is convex, it is a polyhedral cone by Proposition 3.8. Moreover, the support function is pieciewise linear and concave as mentioned in part (3) of Remark 3.21.

Let us see now that the map Φ is well defined as well. For this, notice that an element $y \in M_{\mathbb{D}}$ achieves the minimum in $\mathrm{conv}_{\mathbb{D}}(A) + \sigma^\vee$ iff it achieves the minimum independently in $\mathrm{conv}_{\mathbb{D}}(A)$ and in $\sigma^\vee$. Moreover, y always achieves the minimum in $\mathrm{conv}_{\mathbb{D}}(A)$ in one element of A, and it achieves the minimum in $\sigma^\vee$ iff $y \in (\sigma^\vee)^\vee = \sigma$. Hence, the support of the normal fan of $\Phi(\sigma, h) = \mathrm{conv}_{\mathbb{D}}(A) + \sigma^\vee$ is σ, which is convex. Moreover, the support function of this polyhedron is

$$\begin{aligned} \sigma &\longrightarrow \mathbb{D} \\ y &\longmapsto \min_{x \in \Phi(\sigma,h)} \langle y\,, x\rangle = \min_{x\in A} \langle y\,, x\rangle \end{aligned}$$

which is exactly h. As, mentioned in Remark 3.21 part (5), the support function of a polyhedron determines the polyhedron. Hence, $\mathrm{conv}_{\mathbb{D}}(A) + \sigma^\vee$ does not depend on A and then the map is well defined.

Moreover, the maps Ψ and Φ are mutually inverse: If we start with a pair (σ, h), we get a polyhedron $\Phi(\sigma, h) = \mathrm{conv}_{\mathbb{D}}(A) + \sigma^\vee$ which, as we saw above, has σ as normal fan and h as support function. Hence,

$$\Psi \circ \Phi(\sigma, h) = (\sigma, h).$$

This shows that Ψ is surjective. Moreover, the map Ψ is already injective by Remark 3.21 part (5). Hence, it is bijective and then Ψ and Φ are mutually inverse. □

Definition 3.26. A Minkowski-Weyl decomposition for a polyhedron, is an equality of the form $P = Q + \sigma$ with Q a polytope and σ a polyhedral cone.

Remark 3.27. If $P = \mathrm{conv}_{\mathbb{D}}(A) + \sigma$ is a Minkowski-Weyl decomposition we get

$$(\sigma^\vee, \min_{x\in A}\langle \cdot\,, x\rangle) = \Psi \circ \Phi(\sigma^\vee, \min_{x\in A}\langle \cdot\,, x\rangle) = \Psi(\mathrm{conv}_{\mathbb{D}}(A) + \sigma) = \Psi(P) = (|\mathrm{NF}(P)|, h_P).$$

Hence, $\sigma^\vee = |\mathrm{NF}(P)|$ and then, by Proposition 3.11, we get $\sigma = |\mathrm{NF}(P)|^\vee = \mathrm{recc}(P)$. In particular, the polyhedral cone in the decomposition is uniquely determined. On the other hand, the polytope on the decomposition is not uniquely determine. For example,

$$\{0\} + \sigma = \mathrm{conv}_{\mathbb{D}}(A) + \sigma$$

for any polyhedron σ and any finite set $A \subseteq \sigma$, but $\{0\} \neq \mathrm{conv}_{\mathbb{D}}(A)$ in general.

Corollary 3.28.

1. *A polyhedron admits a Minkowski-Weyl decomposition iff the support of its normal fan is convex.*
2. *A polyhedron is a polytope iff any linear function attains its minimum over it.*

Proof.

(1) If a polyhedron admits a Minkowski-Weyl decomposition $P = Q + \sigma$, then as in Remark 3.27 we get $|\mathrm{NF}(P)| = \sigma^\vee$ which is a convex set. On the other hand, if the support of the normal fan is convex then we can apply Theorem 3.25 and we get

$$P = \Phi \circ \Psi(P) = \Phi(|\mathrm{NF}(P)|, h_P) = \mathrm{conv}_{\mathbb{D}}(A) + |\mathrm{NF}(P)|^\vee$$

which is a Minkowski-Weyl decomposition for P.

(2) If $P = \mathrm{conv}_{\mathbb{D}}(A)$ is a polytope then, any linear function achieves its minimum in one its generators from A. In particular the minimum exists.

On the other hand, if $|\mathrm{NF}(P)| = M_{\mathbb{D}}$, then $|\mathrm{NF}(P)|$ is convex. By the previous part then P admits a Minkowski-Weyl decomposition $P = Q + \sigma$. As in Remark 3.27 we have

$$\sigma = \mathrm{recc}(P) = |\mathrm{NF}(P)|^{\vee} = M_{\mathbb{D}}^{\vee} = \{0\}.$$

Hence, $P = Q + \{0\} = Q$ is a polytope.

□

Remark 3.29.

(1) In Example 3.6 we saw a polyhedron whose normal fan is not convex. Hence, by Corollary 3.28 this also gives an example of a polyhedron which does not accept a Minkowski-Weyl decomposition.

(2) Using part (2) of Corollary 3.28 together with Theorem 3.25 we get a bijection between piecewise linear concave functions $h : M_{\mathbb{D}} \to \mathbb{D}$ and polytopes in $N_{\mathbb{D}}$.

4. $\mathbb{R}$-Rational Polyhedra

4.1. **$\mathbb{R}$-Rational Polyhedra as tangent cones.** As we have seen, polyhedra over the generalized dual numbers $\mathbb{D}$ give rise to iterated fibrations. In general, these fibrations may be difficult to understand, but in some particular cases, it may be possible to give a complete description of them. We will study two situations in which this happens, the one given by strongly $\mathbb{R}$-rational polyhedra and the one given by Strongly $\varepsilon\mathbb{R}$-rational polyhedra.

Let us recall that, from Definition 1.6, a polyhedron P is called strongly $\mathbb{R}$-rational if it admits a representation of the form

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$$

with $y_i \in M_{\mathbb{R}}$ and $a_i \in \mathbb{R}$ for each $1 \leq i \leq r$. For the other concept we have the following definition.

Definition 4.1. A polyhedron is called *strongly $\varepsilon\mathbb{R}$-rational* if it is an intersection of semispaces of the form

$$H = \{x \in N_{\mathbb{D}} \mid \varepsilon^{\alpha}\langle y\,, x\rangle \geq \varepsilon^{\alpha} a\}$$

for some $y \in M_{\mathbb{R}}$, $a \in \mathbb{R}$ and $0 \leq \alpha \leq k-1$. That is, it admits a representation of the form

$$P = \{\varepsilon^{\alpha_1} y_1 \geq \varepsilon^{\alpha_1} a_1, \ldots, \varepsilon^{\alpha_r} y_r \geq \varepsilon^{\alpha_r} a_r\}$$

with $y_i \in M_{\mathbb{R}}$, $a_i \in \mathbb{R}$ and $0 \leq \alpha_i \leq k-1$. If we can take $a_i = 0$ for each i we say that P is an *strongly $\varepsilon\mathbb{R}$-rational polyhedral cone.*

We will start with the following results. Which in particular shows that the tangent cone of polyhedra produces *strongly $\mathbb{R}$-rational* polyhedra and, more generally, the tangent cone of a flag of polyhedra produces *strongly $\varepsilon\mathbb{R}$-rational* polyhedra.

Theorem 4.2.

(1) *Given a flag of polyhedra in $N_{\mathbb{R}}$ of the form*

$$\mathcal{P} : P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{k-1},$$

the tangent cone $T\mathcal{C}\mathcal{P}$ is a polyhedra in $N_{\mathbb{D}}$.

In concrete terms,

(a) *if for each $0 \leq i \leq r$ we have $P_i = \mathrm{conv}_{\mathbb{R}}(\{x_{ij}\}_j)$. Then*

$$T\mathcal{C}\mathcal{P} = \mathrm{wconv}_{\mathbb{D}}\left(\left\{[\varepsilon^i x_{ij}; i]\right\}_{ij}\right)$$

(b) *if for each $0 \leq i \leq r$ we have*

$$P_i = \bigcap_j \{x \in N_{\mathbb{R}} \mid \langle x\,, y_{ij}\rangle \geq a_{ij}\}.$$

Then

$$T\mathcal{C}\mathcal{P} = \bigcap_{i,j} \{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq \varepsilon^{k-i} a_{ij}\}.$$

(2) *Let*

$$\mathcal{S} : \sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_{k-1}$$

be a flag of polyhedral cones in $N_{\mathbb{R}}$. Then $T\mathcal{C}\mathcal{S}$ is a finitely generated polyhedral cone in $N_{\mathbb{D}}$. In concrete terms,

(a) *if for each $0 \leq i \leq r$ we have $\sigma_i = \mathrm{cone}_{\mathbb{R}}(\{x_{ij}\}_j)$. Then*

$$T\mathcal{C}\mathcal{S} = \mathrm{cone}_{\mathbb{D}}\left(\{\varepsilon^i x_{ij}\}_{ij}\right).$$

(b) *If for each $0 \leq i \leq r$ we have $\sigma_i = \bigcap_j \{x \in N_{\mathbb{R}} \mid \langle x\,, y_{ij}\rangle \geq 0\}$. Then*

$$T\mathcal{C}\mathcal{S} = \bigcap_{i,j} \{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq 0\}.$$

Proof. Let us start with the proof of (1) part (b). For this pick $x \in N_{\mathbb{D}}$ and $y \in M_{\mathbb{R}}$. If we write $x = x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1} x^{(k-1)}$, then we have

$$\varepsilon^{k-i}\langle x\,, y\rangle = \varepsilon^{k-i}\langle x^{(0)}\,, y\rangle + \varepsilon^{k-i+1}\langle x^{(1)}\,, y\rangle + \cdots + \varepsilon^{k}\langle x^{(i)}\,, y\rangle.$$

Hence, for $a \in \mathbb{R}$, $\varepsilon^{k-i}\langle x\,, y\rangle \geq \varepsilon^{k-i} a$ happens in $\mathbb{D}$ iff for each $\delta \in \mathbb{R}_{>0}$ small enough we have

$$\delta^{k-i}\langle x^{(0)}\,, y\rangle + \delta^{k-i+1}\langle x^{(1)}\,, y\rangle + \cdots + \delta^{k}\langle x^{(i)}\,, y\rangle \geq \delta^{k-i} a$$
$$\iff \langle x^{(0)}\,, y\rangle + \delta^{1}\langle x^{(1)}\,, y\rangle + \cdots + \delta^{i}\langle x^{(i)}\,, y\rangle \geq a$$

which is equivalent to $x \in T\mathcal{C}\mathcal{A}^{i,y,a}$, where $\mathcal{A}^{i,y,a}$ is the flag

$$\mathcal{A}^{i,y,a} : A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{k-1}$$

given by $A_j = \{x \in N_{\mathbb{R}} \mid \langle x\,, y\rangle \geq a\}$ for $j \leq i$ and $A_j = N_{\mathbb{R}}$ for $j \geq i+1$ This shows that

$$\{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y\rangle \geq \varepsilon^{k-i} a\} = T\mathcal{C}\mathcal{A}^{i,y,a}.$$

Then, by Remark 1.23 part (2) we have that

$$T\mathcal{C}\mathcal{P} = \bigcap_{i,j} T\mathcal{C}\mathcal{A}^{i,y_{ij},a_{ij}} = \bigcap_{i,j} \{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq \varepsilon^{k-i} a_{ij}\}.$$

This finishes the proof. By taking $a_{ij} = 0$ for each pair i, j we obtain (2) part (b).

Let us now prove (2) part (a). First, we can write each σ_i in the form

$$\sigma_i = \bigcap_j \{x \in N_{\mathbb{R}} \mid \langle x\,, y_{ij}\rangle \geq 0\}.$$

Then, if $i \leq i'$ we have $\sigma_i \subseteq \sigma_{i'}$ and hence $x_{ij} \in \sigma_{i'}$ for each j, so we get $\langle x_{ij}\,, y_{i'j'}\rangle \geq 0$ for each j', and we conclude that

$$\varepsilon^{k-i'}\langle \varepsilon^i x_{ij}\,, y_{i'j'}\rangle \geq 0 \quad \forall\, i, i', j, j'.$$

So, applying (2) part (b) we have

$$\varepsilon^i x_{ij} \in \bigcap_{i,j} \{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq 0\} = \mathcal{TCS}$$

which implies $\operatorname{cone}_{\mathbb{D}}\left(\{\varepsilon^i x_{ij}\}_{ij}\right) \subseteq \mathcal{TCS}$. We prove now the other inclusion. For this take $x \in \mathcal{TCS}$. We have to construct $\lambda_{ij} = \lambda_{ij}^{(0)} + \cdots + \varepsilon^{(k-1)}\lambda_{ij}^{(k-1)} \in \mathbb{D}_{>0}$ such that

$$x = \sum_{i,j} \lambda_{ij}\varepsilon^i x_{ij}.$$

Without loss of generality we can assume

$$\{x_{ij}\}_j \subseteq \{x_{i'j}\}_j \text{ for } i \leq i' \tag{4.1}$$

otherwise we add the generators of σ_i to $\sigma_{i'}$. Write $x = x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1}x^{(k-1)}$. For $\delta > 0$ small we have

$$x^{(0)} \in \sigma_0, \quad x^{(0)} + \delta x^{(1)} \in \sigma_1, \quad x^{(0)} + \cdots + \delta^{k-1}x^{(k-1)} \in \sigma_{k-1}.$$

Denote by $\tau_0, \tau_1, \ldots, \tau_{k-1}$ the faces of $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$ respectively containing $x^{(0)}, x^{(0)} + \delta x^{(1)}, x^{(0)} + \cdots + \delta^{k-1}x^{(k-1)} \in \sigma_{k-1}$ in their relative interior. As each vertex of τ_i is a vertex of σ_i we have

$$\tau_i = \operatorname{cone}_{\mathbb{R}}\left(\{x_{ij}\}_j \cap \tau_i\right).$$

Hence, as $x^{(0)} \in \mathring{\tau}_0$ and

$$\mathring{\tau}_0 = \left\{ \sum_{x_{0j} \in \tau_0} \lambda_{0j}x_{0j} \mid \lambda_{0j} \in \mathbb{R}_{>0} \right\}$$

there are $\lambda_{0j}^{(0)} \in \mathbb{R}_{>0}$ such that $x^{(0)} = \sum_{x_{0j}\in\tau_0} \lambda_{0j}^{(0)}x_{0j}$. Now as $\tau_0 \subseteq \tau_1$ we can consider $\tau_1/\tau_0 \coloneqq (\tau_1 + \operatorname{span}\tau_0)/\operatorname{span}\tau_0$ as a cone in $N_{\mathbb{R}}/\operatorname{span}\tau_0$. Then, as $x^{(0)} + \delta x^{(1)} \in \mathring{\tau}_1$ we get $[x^{(0)} + \delta x^{(1)}] \in (\tau_1/\tau_0)^\circ$ so $[x^{(1)}] \in (\tau_1/\tau_0)^\circ$ and as

$$(\tau_1/\tau_0)^\circ = \left\{ \sum_{x_{1j} \in \tau_1} \lambda_{1j}[x_{1j}] \mid \lambda_{0j} \in \mathbb{R}_{>0} \right\}$$

there are $\lambda_{1j}^{(0)} \in \mathbb{R}_{>0}$ such that $[x^{(1)}] = \sum_{x_{1j}\in\tau_1} \lambda_{1j}^{(0)}[x_{1j}]$. Lifting this equation to τ_1 there are $\lambda_{0j}^{(1)} \in \mathbb{R}$ such that

$$x^{(1)} = \sum_{x_{1j}\in\tau_1} \lambda_{1j}^{(0)}x_{1j} + \sum_{x_{0j}\in\tau_0} \lambda_{0j}^{(1)}x_{0j}.$$

In a similar way, $\tau_1 \subseteq \tau_2$ so we can consider τ_1/τ_2. As $x^{(0)} + \delta x^{(1)} + \delta^2 x^{(2)} \in \mathring{\tau}_2$ and $x^{(0)} + \delta x^{(1)} \in \tau_1$ we get $[x^{(2)}] \in (\tau_1/\tau_2)^\circ$ from where there are $\lambda_{2j}^{(0)} \in \mathbb{R}_{>0}$ such that $[x^{(2)}] = \sum_{x_{2j}\in\tau_2} \lambda_{2j}^{(2)} [x_{2j}]$. Lifting this equation to τ_2 we get $\lambda_{1j}^{(1)} \in \mathbb{R}$ and $\lambda_{0j}^{(2)} \in \mathbb{R}$ such that

$$x^{(2)} = \sum_{x_{2j}\in\tau_2} \lambda_{2j}^{(0)} x_{2j} + \sum_{x_{1j}\in\tau_1} \lambda_{1j}^{(1)} x_{1j} + \sum_{x_{0j}\in\tau_0} \lambda_{0j}^{(2)} x_{0j}.$$

Continuing in this way we have constructed $\lambda_{ij} = \lambda_{ij}^{(0)} + \cdots + \varepsilon^{(k-1)} \lambda_{ij}^{(k-1)} \in \mathbb{D}_{>0}$ such that

$$x = \sum_{i,j} \lambda_{ij} \varepsilon^i x_{ij}$$

as we wanted. This finishes the proof of (2) part (a).

Now, (1) part (a) follows from (2) part (a). For this, given the polytope $P_i = \operatorname{conv}_{\mathbb{R}}(\{x_{ij}\}_i) \subseteq N_{\mathbb{R}}$ consider the cone

$$\widehat{P}_i = \operatorname{cone}_{\mathbb{D}}(\{(x_{ij}, 1)\}_i) \subseteq N_{\mathbb{R}} \times \mathbb{R}.$$

In this way, we obtain a flag of polyhedral cones

$$\widehat{\mathcal{P}} : \widehat{P}_0 \subseteq \widehat{P}_1 \subseteq \cdots \subseteq \widehat{P}_{k-1}.$$

By (2) part (a) we have $T\mathcal{C}\widehat{\mathcal{P}} = \operatorname{cone}_{\mathbb{D}}\left(\{\varepsilon^i (x_{ij}, 1)\}_{ij}\right)$, hence

$$\begin{aligned}
T\mathcal{C}\mathcal{P} \times \{1\} &= T\mathcal{C}\widehat{\mathcal{P}} \cap N_{\mathbb{D}} \times \{1\} \\
&= \operatorname{cone}_{\mathbb{D}}\left(\{\varepsilon^i (x_{ij}, 1)\}_{ij}\right) \cap N_{\mathbb{D}} \times \{1\} \\
&= \left\{ x \in N_{\mathbb{D}} \;\middle|\; (x, 1) \in \operatorname{cone}_{\mathbb{D}}\left(\{\varepsilon^i (x_{ij}, 1)\}_{ij}\right) \right\} \times \{1\} \\
&= \left\{ \sum_{ij} \lambda_{ij} x_{ij} \varepsilon^i \in N_{\mathbb{D}} \;\middle|\; \lambda_{ij} \geq 0 \text{ for all } i,j \text{ and,} \sum_{i,j} \lambda_{ij} \varepsilon^i = 1 \right\} \times \{1\} \\
&= \operatorname{wconv}_{\mathbb{D}}\left(\left\{ [\varepsilon^i x_{ij}; i] \right\}_{ij} \right)
\end{aligned}$$

□

Two immediate corollaries are the following.

Corollary 4.3 (Base change principle)**.** *The real polyhedra (resp. real polyhedral cones) in $N_{\mathbb{D}}$ correspond exactly to the tangent cone of polyhedra (resp. polyheral cones) in $N_{\mathbb{R}}$. In explicit terms.*

(1) Given a finite subset $X \subseteq N_{\mathbb{R}}$ we have

$$\operatorname{cone}_{\mathbb{D}} X = T\mathcal{C}^{k-1} \operatorname{cone}_{\mathbb{R}} X.$$

(2) Given $y_1, \ldots, y_r \in M_{\mathbb{R}}$ and $a_1, \ldots a_r \in \mathbb{R}$ we have

$$\begin{aligned}
&\{x \in N_{\mathbb{D}} \mid \langle y_1 , x\rangle \geq a_1, \ldots, \langle y_r , x\rangle \geq a_r\} \\
&= T\mathcal{C}^{k-1} \{x \in N_{\mathbb{R}} \mid \langle y_1 , x\rangle \geq a_1, \ldots, \langle y_r , x\rangle \geq a_r\}.
\end{aligned}$$

Which in particular if $a_i = 0$ for all i, gives us an equality between polyhedral cones.

Proof. This is simply the case in which we take a constant flag in Theorem 4.2 above. □

Corollary 4.4. *Given a polyhedral P in $N_{\mathbb{D}}$, the following are equivalent.*

(1) P is strongly $\varepsilon\mathbb{R}$-rational.
(2) $P = T\mathcal{C}^{k-1}\mathcal{P}$ for $\mathcal{P} : P_1 \subseteq \cdots \subseteq P_k$ a sequence of real polyhedra in $N_{\mathbb{R}}$.

Moreover, if P is a polyhedral cone then, this are also also equivalent to the fact that P is finitely generated by elements of the form $\varepsilon^i x$ with $x \in N_{\mathbb{R}}$.

Proof. This is a restatement of Theorem 4.2 above. □

Semi-real polyhedra appear naturally as faces of real polyhedra as the next proposition shows.

Proposition 4.5. *Let P be a polyhedron in $N_{\mathbb{R}}$. Then, the faces of $T\mathcal{C}^{k-1}P$ in $N_{\mathbb{D}}$ are given exactly by the sets of the form $T\mathcal{C}\mathcal{F}$ for a flag*

$$\mathcal{F} : F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1}$$

where each F_i is a face of P in $N_{\mathbb{R}}$.

Proof. We start proving that each set of the form $T\mathcal{C}\mathcal{F}$ is a face of $T\mathcal{C}^{k-1}P$. For this let $\mathrm{NF}_{\mathbb{R}}P$ be the normal fan of P. The flag of faces $\mathcal{F}$ correspond to a flag of cones

$$\sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_{k-1}$$

in $\mathrm{NF}_{\mathbb{R}}P$, in which σ_{i+1} is a face of σ_i for each i. Now, take $y^{(0)}, y^{(1)}, \ldots, y^{(k-1)} \in M_{\mathbb{R}}$ such that for each $\delta > 0$ small enough we have

$$y^{(0)} \in \sigma_{k-1}, \quad y^{(0)} + \delta y^{(1)} \in \sigma_{k-2}, \quad \ldots \quad y^{(0)} + \cdots + \delta^{k-1} y^{(k-1)} \in \sigma_0 \tag{4.2}$$

and consider $y = y^{(0)} + \cdots + \varepsilon^{(k-1)} y^{(k-1)} \in N_{\mathbb{D}}$. We claim that y defines $T\mathcal{C}F$ as a face. For this given $x = x^{(0)} + \cdots + \varepsilon^{(k-1)} x^{(k-1)} \in N_{\mathbb{D}}$ consider

$$\langle y\,, x\rangle = \langle y^{(0)}\,, x^{(0)}\rangle + \varepsilon \left(\langle y^{(0)}\,, x^{(1)}\rangle + \langle y^{(1)}\,, x^{(0)}\rangle\right) + \cdots + \varepsilon^{k-1} \left(\sum_{i+j=k-1} \langle y^{(i)}\,, x^{(j)}\rangle\right).$$

In order to minimize this expression for $x \in T\mathcal{C}^{k-1}P$ we need to first find the $x^{(0)}$ which minimize $\langle y^{(0)}\,, x^{(0)}\rangle$, then between those $x^{(0)}$ we need to minimize $\langle y^{(0)}\,, x^{(1)}\rangle + \langle y^{(1)}\,, x^{(0)}\rangle$, and so on.

To minimize $\langle y^{(0)}\,, x^{(0)}\rangle$, as we have $y^{(0)} \in \sigma_{k-1}$ we have to take $x^{(0)} \in F_{k-1}$.

To minimize $\langle y^{(0)}\,, x^{(1)}\rangle + \langle y^{(1)}\,, x^{(0)}\rangle$, we will minimize simultaneously $\langle y^{(0)}\,, x^{(1)}\rangle$ and $\langle y^{(1)}\,, x^{(0)}\rangle$. Notice that we already minimized $\langle y^{(0)}\,, x^{(0)}\rangle$, so $\langle y^{(0)}\,, x^{(1)}\rangle$ attach its minimum iff

$$\delta\langle y^{(0)}\,, x^{(1)}\rangle + \langle y^{(0)}\,, x^{(0)}\rangle = \langle y^{(0)}\,, x^{(0)} + \delta x^{(1)}\rangle$$

achieves its minimum, which happens iff $x^{(0)} + \delta x^{(1)} \in \sigma_{k-1}$. In the same way $\langle y^{(1)}\,, x^{(0)}\rangle$ is minimized exactly when

$$\delta\langle y^{(1)}\,, x^{(0)}\rangle + \langle y^{(0)}\,, x^{(0)}\rangle = \langle y^{(0)} + \delta y^{(1)}\,, x^{(0)}\rangle$$

achieves its minimum, which happens iff $x^{(0)} \in \sigma_{k-2}$. Therefore, $\langle y^{(0)}\,, x^{(1)}\rangle + \langle y^{(1)}\,, x^{(0)}\rangle$ is minimized when $x^{(0)} + \delta x^{(1)} \in \sigma_{k-1}$ and $x^{(0)} \in \sigma_{k-2}$ simultaneously.

In general, we want to minimize $\sum_{i+j=r}\langle y^{(i)}, x^{(j)}\rangle$ given that we have minimized $\sum_{i+j=s}\langle y^{(i)}, x^{(j)}\rangle$ for every $s < r$, and even more, we know that the minimum in $\sum_{i+j=s}\langle y^{(i)}, x^{(j)}\rangle$ is achieved exactly when each term has been independently minimized. Let us prove that, under this conditions, $\sum_{i+j=r}\langle y^{(i)}, x^{(j)}\rangle$ is also minimized when each tearm is independently minimized, and the minimum in the term $\langle y^{(i)}, x^{(j)}\rangle$ is achieved exactly when

$$x^{(0)} + \delta x^{(1)} + \cdots + \delta^{(j)} \in F_{k-1-i} \text{ for every small enough } \delta > 0$$

For this, notice that $\langle y^{(i)}, x^{(j)}\rangle$ is minimized iff

$$\langle y^{(0)} + \delta y^{(0)} + \cdots + \delta^i y^{(i)}, x^{(0)} + \delta x^{(1)}\rangle + \cdots + \delta^j x^{(j)}$$

is minimized, because if one expand this, then each term is constant except the term $\delta^{i+j}\langle y^{(i)}, x^{(j)}\rangle$. As $y^{(0)} + \delta y^{(0)} + \cdots + \delta^i y^{(i)} \in \sigma_{k-1-i}$, we have that the minimum is achieved when $x^{(0)} + \delta x^{(1)} + \cdots + \delta^j x^{(j)} \in F_{k-1-i}$ as we wanted.

In conclusion, $\langle y, x\rangle$ is minimized iff we have $x^{(0)} + \delta x^{(1)} + \cdots + \delta^j x^{(j)} \in F_{k-1-i}$ for every i, j with $i + j \leq k-1$ and $\delta > 0$ small enough, which happens iff

$$x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1} x^{(k-1)} \in \mathcal{TCF}.$$

Hence, $\mathcal{TCF}$ is the face defined by y as we wanted.

Conversely, take an element $y = y^{(0)} + \varepsilon y^{(1)} + \cdots + \varepsilon^{k-1} y^{(k-1)} \in M_{\mathbb{D}}$. Then, it needs to defined a flag of cones in the normal fan of P as in equation (4.2). Then, the argument above shows that y defines the face $\mathcal{TCF}$. Hence, every face of $\mathcal{TC}^{k-1}P$ is of the form $\mathcal{TCF}$ for some flag of faces $\mathcal{F}$. This finishes the proof. □

This gives us an understanding of the combinatorial type of a real polyhedron: Its lattice of faces is the chain poset of length k (see [Joh18] for the definition) of the lattice of face of the underlying rank 1 polyhedron.

4.2. **$\mathbb{R}$-Rational Polyhedra.** Recall that an $\mathbb{R}$-Rational polyhedron P is a polyhedron for which there are $y_1, \ldots, y_r \in M_{\mathbb{R}}$ and $a_1, \ldots, a_r \in \mathbb{D}$ such that

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}.$$

The objective of this section is to give a new description for the normal fan of an $\mathbb{R}$-rational polyhedron, and use it to understand the combinatorial behavior of the iterated fibration determined by the polyhedron.

First, we will start by introducing the concept of a *layered polyhedral complex*, these are sequences of real polyhedral complexes in which each term subdivide the previous one. The first example of such a layered polyhedral complex we will present is the *layered normal fan* of an $\mathbb{R}$-rational polyhedron, which we introduce in Proposition 4.7. Later, in Theorem 4.9 we show how it is possible to recover the usuan normal fan of the polyhedron from its layered normal fan by a tangent cone construction.

Definition 4.6. A *layered polyhedral complex* is a sequence of real polyhedral complex of the form

$$\underline{\Sigma} : \Sigma_0 \preceq \Sigma_1 \preceq \cdots \preceq \Sigma_{k-1}$$

where all Σ_i are polyhedral complexes in $N_\mathbb{R}$ of the same support and Σ_{i+1} is a subdivision of Σ_i for each $0 \le i \le k-2$. A *layered face* of $\underline{\Sigma}$ is a flag of faces

$$\underline{F} : F_{k-1} \subseteq F_{k-2} \subseteq \cdots \subseteq F_0$$

with $F_i \in \Sigma_i$ for each i. The *support* of $\underline{\Sigma}$, denote by $|\underline{\Sigma}|$, is defined as the support of $|\Sigma_i|$ for any i. A layered polyhedral complex in which each term is a flag is called a *layered fan.*

Proposition 4.7 (Layered Normal Fan)**.** *Let P be an $\mathbb{R}$-rational polyhedron with a fixed non-redundant representation*

$$P = \{x \in N_\mathbb{D} \mid y_1 \ge a_1, \ldots, y_r \ge a_r\}.$$

We can construct a sequence of fans in $M_\mathbb{R}$, which we call the layered normal fan of P *and denote by*

$$\underline{\Delta}(P) \coloneqq \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1}, \tag{4.3}$$

in the following equivalent ways:

(1)
- *Δ_0 is the normal fan of the real polyhedron $P^{[0]}$.*
- *Δ_1 is constructed by the following process. Given a cell $\sigma \in \Delta_0$, there is a face F of $P^{[0]}$ such that σ is the normal cone $\mathrm{C}(F)$. Given a point $x_0 \in \mathrm{int}(F)$, the fiber $P^{[1]}_{x_0}$ is a real polyhedron such that $|\mathrm{NF}(P^{[1]}_{x_0})| = \mathrm{C}(F)$. Then, $\mathrm{NF}(P^{[1]}_{x_0})$ is independent of the x_0 chosen and Δ_1 is obtained by replacing $C(F)$ by $\mathrm{NF}(P^{[0]}_{x_0})$ for every face $\mathrm{C}(F) \in \Delta_0$.*
- *Similarly, Δ_2 is constructed as follows. Given a cell $\sigma \in \Delta_1$, there is a point $x_0 \in P^{[0]}$ such that σ is the normal cone $\mathrm{C}(F)$ of a face F of $P^{[1]}_{x_0}$. Given a point $x_1 \in \mathrm{int}(F)$, the fiber $P^{[1]}_{x_0+\varepsilon x_1}$ is a real polyhedron such that $|\mathrm{NF}(P^{[1]}_{x_0+\varepsilon x_1})| = \mathrm{C}(F)$. Then, $\mathrm{NF}(P^{[1]}_{x_0+\varepsilon x_1})$ is independent of the x_1 chosen and Δ_1 is obtained by replacing $\mathrm{C}(F)$ by $\mathrm{NF}(P^{[1]}_{x_0+\varepsilon x_1})$ for every face $\mathrm{C}(F) \in \Delta_1$.*

Continuing in this way we construct Δ_i for every integer $0 \le i \le k-1$.

(2) For $\delta \in \mathbb{R}_{>0}$, the normal fan of the polyhedron

$$P_i(\delta) \coloneqq \left\{ x \in N_\mathbb{R} \;\middle|\; \langle y\,, x\rangle \ge a_j^{(0)} + \delta a_j^{(1)} + \cdots + \delta^i a_j^{(i)}, \quad \forall\, 1 \le j \le r \right\}$$

is independent of δ if it is small enough. Then, we let Δ_i to be this fan.

(3) Δ_i is the fan in $N_\mathbb{R}$ whose faces are the sets of the form $\mathrm{cone}_\mathbb{R}(S_i(x'))$ as x' moves along $P^{[i]}$, where

$$S_i(x') = \left\{ y_j \in M_\mathbb{R} \;\middle|\; 1 \le j \le r \text{ and } \langle y_j\,, x'\rangle = a_j^{[i]} \right\}.$$

In particular, from (1) we see that Δ_i is independent of the representation of P and that Δ_{i+1} is a subdivision of Δ_i.

Moreover, given a sequence of normal fans in $N_\mathbb{R}$ of the form

$$\underline{\Delta} : \Delta_0 \preceq \cdots \preceq \Delta_{k-1},$$

all of them with the same support, there is an $\mathbb{R}$-rational polyhedron $P \subseteq N_{\mathbb{D}}$ such that $\underline{\Delta} = \underline{\Delta}(P)$.

Remark 4.8. Notice that, by the first definition that we present for the layered normal fan, we have an explicit algorithm to understand the combinatorial structure of the fibers $P_x^{[i]}$ from the layered normal fan of P.

Proof of Proposition 4.7. Let us denote by $\Delta_i^{(1)}$, $\Delta_i^{(2)}$ and $\Delta_i^{(3)}$ the fans constructed in (1), (2) and (3) respectively. We need to prove that all of them are equal. Let us start by showing that $\Delta_i^{(2)}$ equals $\Delta_i^{(3)}$. For this, given $\delta \in \mathbb{R}_{>0}$ we consider the map

$$\begin{aligned}\psi_\delta : \mathbb{D} &\longrightarrow \mathbb{R}\\ x &\longmapsto x^{(0)} + \delta x^{(1)} + \cdots + \delta^{k-1}x^{(k-1)}.\end{aligned}$$

As this is an $\mathbb{R}$-linear map, by extension of scalars and composition, this map naturally extends to a map $N_{\mathbb{D}_i} \to N_{\mathbb{R}}$ which we still denote by ψ_δ. We can now write

$$P_i(\delta) = \{x \in N_{\mathbb{R}} \mid \langle y_j\,, x\rangle \geq \psi_\delta\big(a_j^{[i]}\big),\ \forall\, 1 \leq j \leq r\},$$

and we have

$$\begin{aligned}x \in P^{[i]} &\iff \langle y_j\,, x^{[i]}\rangle \geq a^{[i]}, \quad \forall\, 1 \leq j \leq r\\ &\iff \psi_\delta\big(\langle y\,, x^{[i]}\rangle\big) \geq \psi_\delta\big(a^{[i]}\big), \quad \forall\, 1 \leq j \leq r,\ \forall\, \delta > 0 \text{ small enough} \quad \text{(By Remark 1.1)}\\ &\iff \langle y\,, \psi_\delta\big(x^{[i]}\big)\rangle \geq \psi_\delta\big(a^{[i]}\big), \quad \forall\, 1 \leq j \leq r,\ \forall\, \delta > 0 \text{ small enough} \quad \text{(By } \mathbb{R}\text{-linearity)}\\ &\iff \psi_\delta(x) \in P_i(\delta), \quad \forall\, \delta > 0 \text{ small enough.}\end{aligned}$$

Thus, for all $\delta \in \mathbb{R}_{>0}$ small enough we have $\psi_\delta\big(P^{[i]}\big) \subseteq P_i(\delta)$. Now, given a point $x \in P^{[i]}$, as $\psi_\delta(x) \in P_i(\delta)$ we can consider the cell of $\Delta_i^{(2)}$ of the form $\mathrm{C}(F)$, where F is the face of $P_i(\delta)$ such that $\psi_\delta(x) \in \mathrm{int}(F)$. By Proposition 3.14, if we consider

$$S_{i,\delta}(x) = \left\{ y \in A \;\middle|\; \langle y\,, x\rangle = \psi_\delta\big(h(y)^{[i]}\big) \right\}$$

then $\mathrm{C}(F) = \mathrm{conv}_{\mathbb{R}}\left(S_{i,\delta}(\psi_\delta(x))\right)$. Moreover, by Remark 1.1, for $\delta \in \mathbb{R}_{>0}$ small enough we have

$$S_{i,\delta}(\psi_\delta(x)) = \left\{ y \in A \;\middle|\; \langle y\,, x\rangle = \psi_\delta\big(h(y)^{[i]}\big) \right\} = \left\{ y \in A \;\middle|\; \langle y\,, x\rangle = h(y)^{[i]} \right\} = S_i(x).$$

This shows that each cell of $\Delta_i^{(3)}$ belongs to $\Delta_i^{(2)}$, as both fans have the same support we conclude that they are equal. In particular, $\Delta_i^{(2)}$ does not depend on δ when it is small enough.

Now, let us see that $\Delta_i^{(3)}$ is also equal to $\Delta_i^{(1)}$ and $\Delta_i^{(2)}$.

(1) If $i = 0$ then $\Delta_0^{(1)}$ equals $\Delta_0^{(2)}$ by definition.

(2) If $i = 1$ then, to construct $\Delta_1^{(3)}$ we need to take for each cell $\mathrm{C}(F) \in \Delta_0^{(3)}$ a point $x_0 \in \mathrm{int}(F)$ and consider

$$\begin{aligned} P_{x_0}^{[1]} &= \left\{ x \in N_{\mathbb{R}} \mid x_0 + \varepsilon x \in P^{[1]} \right\} \\ &= \left\{ x \in N_{\mathbb{R}} \mid \langle y_j\,, x_0\rangle + \varepsilon \langle y_j\,, x\rangle \geq a^{(1)} + \varepsilon a^{(1)} \ \ \forall\, 1 \leq j \leq r \right\} \\ &= \left\{ x \in N_{\mathbb{R}} \mid \langle y_j\,, x\rangle \geq a_j^{(1)} \ \ \forall\, 1 \leq j \leq r \text{ such that } \langle y_j\,, x_0\rangle = a_j^{(0)} \right\} \\ &= \left\{ x \in N_{\mathbb{R}} \mid \langle y_j\,, x\rangle \geq a_j^{(1)} \ \ \forall j \text{ such that } y_j \in S_0(x_0) \right\}. \end{aligned}$$

Then, by Proposition 3.8 we have $|\mathrm{NF}(P_{x_0}^{[1]})| = \mathrm{cone}_{\mathbb{R}}(S_0(x_0)) = \mathrm{C}(F)$. Moreover, for $x_1 \in P_{x_0}^{[1]}$, if $x_1 \in \mathrm{int}(G)$ for a face G then the normal cone $\mathrm{C}(G)$ with respect to $P_{x_0}^{[1]}$ is a cell of $\mathrm{NF}(P_{x_0}^{[1]})$. By Proposition 3.14 using the representation for $P_{x_0}^{[1]}$ we have found above we have

$$\begin{aligned} \mathrm{C}(G) &= \mathrm{cone}_{\mathbb{R}} \left\{ y_j \in M_{\mathbb{R}} \mid \langle y_j\,, x_1\rangle = a^{(1)} \text{ with } y_j \in S_0(x_0) \right\} \\ &= \mathrm{cone}_{\mathbb{R}}(S_1(x_0 + \varepsilon x_1)). \end{aligned}$$

Hence, each face of $\Delta_1^{(3)}$ is a face of $\Delta_1^{(2)}$ and as they have the same support they must be equal.

(3) The general case is similar. Suppose the result is true for i and let us check it is true for $i+1$. By the induction hypothesis, a cell of $\Delta_i^{(3)}$ is of the form $\mathrm{cone}_{\mathbb{R}}(S_i(x_0+\varepsilon x_1+\cdots+\varepsilon^i x_i))$ for some $x_0, \ldots, x_i \in N_{\mathbb{R}}$ such that $x_0+\cdots+\varepsilon^i x_i \in P^{[i]}$. Then,

$$P_{x_0+\cdots+\varepsilon^i x_i}^{[i+1]} = \left\{ x \in N_{\mathbb{R}} \mid \langle y_j\,, x\rangle \geq a_j^{i+1} \ \ \forall\, j \text{ such that } y_j \in S_i(x_0 + \cdots + \varepsilon^i x_i) \right\}.$$

Hence, $|\mathrm{NF}(P_{x_0+\cdots+\varepsilon^i x_i}^{[i]})| = \mathrm{cone}_{\mathbb{R}}(S_i(x_0 + \cdots + \varepsilon^i x_i))$ and for a point x_{i+1} in the fiber, if $x_{i+1} \in \mathrm{int}(G)$ for a face G of the normal cone $C(G)$ with respect to $P_{x_0+\cdots+\varepsilon^i x_i}^{[i]}$, then by Proposition 3.14 we have

$$\begin{aligned} \mathrm{C}(G) &= \mathrm{cone}_{\mathbb{R}} \left\{ y_j \in M_{\mathbb{R}} \mid \langle y_j\,, x_{i+1}\rangle = a^{(i+1)} \text{ with } y_j \in S_i(x_0 + \cdots + \varepsilon^i x_i) \right\} \\ &= \mathrm{cone}_{\mathbb{R}}(S_1(x_0 + \cdots + \varepsilon^i x_i + \varepsilon^{i+1} x_{i+1})). \end{aligned}$$

Which is a face of $\Delta_{i+1}^{(3)}$. Hence, every face of $\Delta_{i+1}^{(1)}$ is a face of $\Delta_{i+1}^{(3)}$ and as they have the same support they are equal.

Finally, given a sequence of normal fans in $N_{\mathbb{R}}$ of the form

$$\underline{\Delta} : \Delta_0 \preceq \cdots \preceq \Delta_{k-1},$$

for each $0 \leq i \leq k-1$ we can consider a polyhedron P_i in $N_{\mathbb{R}}$ such that $\mathrm{NF}(P_i) = \Delta_i$. If

$$P_i = \{x \in N_{\mathbb{R}} \mid y_j \geq a_j^{(i)},\ \forall\, 1 \leq j \leq r\},$$

for some $y_j \in M_{\mathbb{R}}$, $a_j^{(i)} \in M_{\mathbb{R}}$. Without loss of generality, we can suppose that $\{y_1, \dots, y_r\}$ is independent of i. Then, we can consider

$$P = \left\{ x \in N_{\mathbb{D}} \;\middle|\; y_j \geq a_j^{(0)} + \varepsilon a_j^{(1)} + \cdots + \varepsilon^{(k-1)} a_j^{(k-1)},\ \forall\, 1 \leq j \leq r \right\}.$$

Then, we have that $\Delta_i = \Delta_i(P)$ for each i. Indeed, it is enough to prove that if

$$\underline{\delta} : \delta_0 \subseteq \delta_1 \subseteq \cdots \subseteq \delta_{k-1}$$

is a sequence of faces with $\delta_i \in \Delta_i$, and $\delta_i = \operatorname{conv}_{\mathbb{R}}(S_{P_i}(x_i))$ for some $x_i \in P_i$. Then, for $x \coloneqq x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1} x_{k-1} \in P$ we have

$$\delta_i = \operatorname{conv}_{\mathbb{R}} \left(S_{i,P}(x^{[i]}) \right) \tag{4.4}$$

for each $0 \leq i \leq k-1$. Because, if we prove this, then each face of Δ_i is a face of $\Delta_i(P)$ and as they have the same support we are done.

We will prove the equality in (4.4) by induction on i. If $i = 0$ this is trivial. For $i > 0$, as

$$S_{P_{i+1}}(\psi_\delta(x^{[i+1]})) = \left\{ y_j \in S_{P_i}(\psi_\delta(x^{[i]})) \;\middle|\; \langle y_j \,, \psi_\delta(x^{[i+1]}) \rangle = a_j^{(i+1)} \right\}$$

for $\delta \in \mathbb{R}_{>0}$ small enough, we have

$$\begin{aligned}
S_{i+1,P}(x) &= \left\{ y_j \in M_{\mathbb{R}} \;\middle|\; 1 \leq j \leq r \text{ and } \langle y_j \,, x \rangle = a_j^{[i+1]} \right\} \\
&= \left\{ y_j \in M_{\mathbb{R}} \;\middle|\; 1 \leq j \leq r,\ y_j \in S_{i,P} \ \text{ and } \langle y_j \,, x \rangle^{(i+1)} = a_j^{(i+1)} \right\} \\
&= \left\{ y_j \in M_{\mathbb{R}} \;\middle|\; 1 \leq j \leq r,\ y_j \in S_{P_i}(\psi_\delta(x^{[i]})) \ \text{ and } \langle y_j \,, x \rangle^{(i+1)} = a_j^{(i+1)} \right\} \\
&= \left\{ y_j \in M_{\mathbb{R}} \;\middle|\; 1 \leq j \leq r,\ y_j \in S_{P_i}(\psi_\delta(x^{[i]})) \ \text{ and } \langle y_j \,, \psi_\delta(x^{[i+1]}) \rangle = a_j^{(i+1)} \right\} \\
&= S_{P_{i+1}}(\psi_\delta(x^{[i+1]}))
\end{aligned}$$

□

Theorem 4.9 (Local Duality)**.** *Given an $\mathbb{R}$-rational polyhedron P, we can recover the normal fan of P from the layered normal fan as*

$$\operatorname{NF}(P) = T\mathcal{C}\underline{\Delta}(P).$$

In the sense that, $\operatorname{NF}(P)$ *is the fan consisting of all the polyhedral cones of the form $T\mathcal{C}\underline{\delta}$ where*

$$\underline{\delta} \colon \sigma_{k-1} \subseteq \sigma_{k-2} \cdots \subseteq \sigma_0$$

is a layered face of $\underline{\Delta}$.

Proof. Fix a point $x \in P$ and a non-reduced representation $P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}$.

Using x, we can construct a face of $\operatorname{NF}(P)$ by considering the normal cone $\mathrm{C}(F)$ of the face F of P such that $x \in \operatorname{int}(F)$.

On the other hand, using the same x, by the definition in part (3) of Proposition 4.7 we can construct a layered face $\underline{\delta}(x)$ of $\underline{\Delta}(P)$ by

$$\underline{\delta}(x)\colon\ \operatorname{cone}_{\mathbb{R}} \left(S_{k-1}(x) \right) \subseteq \cdots \subseteq \operatorname{cone}_{\mathbb{R}} \left(S_0(x) \right).$$

To prove the theorem, it will be enough to show that

$$\mathrm{C}(F) = T\mathcal{C}\underline{\delta}(x).$$

In order to do this, notice that by Proposition 3.14 we can write $\mathrm{C}(F)$ as

$$\mathrm{C}(F) = \mathrm{cone}_{\mathbb{D}}\left(\varepsilon^{k-\alpha_1}y_1, \dots, \varepsilon^{k-\alpha_r}y_r\right)$$

where $\alpha_i = \mathrm{ord}\big(\langle y_i\,, x\rangle - a_i\big)$, i.e, α_i is the biggest integer in $\{0, \dots, k\}$ such that

$$\varepsilon^{k-\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{k-\alpha_i}a_i \iff \langle y_j\,, x\rangle^{[\alpha_i-1]} = a_j^{[\alpha_i-1]} \iff y_i \in S_{\alpha_i-1}(x).$$

Hence, we can write this normal cone as

$$\begin{aligned}\mathrm{C}(F) &= \mathrm{cone}_{\mathbb{D}}\left(\bigcup_{i=0}^{k-1}\left\{\varepsilon^{k-1-i}y_j \;\middle|\; y_j \in S_i(x)\right\}\right)\\ &= \mathrm{cone}_{\mathbb{D}}\left(\bigcup_{i=0}^{k-1}\left\{\varepsilon^{i}y_j \;\middle|\; y_j \in S_{k-1-i}(x)\right\}\right)\end{aligned}$$

which is exactly equal to $T\mathcal{C}\underline{\delta}(x)$ by Theorem 4.2 part (2). This finishes the proof. $\square$

Remark 4.10. In particular, we see that the normal type of an $\mathbb{R}$-rational polyhedron, that is, the information encoded in its normal fan, is equivalent to the data of a sequence of length k normal types of real polyhedra each of them refining the previous one.

4.3. **Regular Subdivisions.** In this section we will extend the notion of regular subdivision of a polytope to the polyhedral geometry over $\mathbb{D}$. Moreover, in a similar way as we did in the previous section, we will study how this concept relates to *layered regular subdivisions*, which are defined in analogy to the *layered normal fans* of the previous section.

Using the extended perfect pairing of Definition 1.12 we can introduce the following concept.

Definition 4.11 (Regular subdivisions over $\mathbb{D}$)**.** Consider a finite subset $A \subseteq M_{\mathbb{D}}$, a function $h : A \to \mathbb{D}$, which we refer to as a *height function* on A, and the polytope $P = \mathrm{conv}_{\mathbb{D}}(A)$.

(1) The *lifted convex hull* of A is the set

$$\mathrm{conv}^h_{\mathbb{D}}(A) := \mathrm{conv}_{\mathbb{D}}\{(a, h(a)) \in M_{\mathbb{D}} \times \mathbb{D} \mid a \in A\} \subseteq M_{\mathbb{D}} \times \mathbb{D}.$$

(2) The *regular subdivision of P with respect to h*, denoted by $\Delta^h(P)$, is the family

$$\begin{aligned}\Delta^h(P) &:= \left\{\pi\left(\mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(P))\right) \;\middle|\; x \in N_{\mathbb{D}}\right\}\\ &= \left\{\pi(F) \;\middle|\; F \textit{ is a lower face of } \mathrm{conv}^h_{\mathbb{D}}(A)\right\},\end{aligned}$$

where $\pi : M_{\mathbb{D}} \times \mathbb{D} \to M_{\mathbb{D}}$ denotes the projection to the second coordinate. That is, $\Delta^h(P)$ is the projection of all the lower faces of $\mathrm{conv}^h_{\mathbb{D}}(A)$ to $M_{\mathbb{D}}$.

Proposition 4.12. *The regular subdivision $\Delta^h(P)$ is a polyhedral complex and the restriction of $M_{\mathbb{D}} \times \mathbb{D} \to M_{\mathbb{D}}$ to the set of lower faces of $\Delta^h(P)$ is injective and has P as image.*

Proof. We start by proving that the set of lower faces of $\operatorname{conv}^h_{\mathbb{D}}(A)$ is a polyhedral complex. For this, consider two lower faces F and G of $\operatorname{conv}^h_{\mathbb{D}}(A)$. They are of the form

$$F = \operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A)) \text{ and } G = \operatorname{face}_{(x',1)}(\operatorname{conv}^h_{\mathbb{D}}(A))$$

for some $x, x' \in N_{\mathbb{D}}$. If $F \cap G \neq \varnothing$ then every element $(y, y_0) \in F \cap G$ minimize simultaneously $\langle (y, y_0), (x, 1)\rangle$ and $\langle (y, y_0), (x', 1)\rangle$. Hence, the minimum of $\langle \cdot, (x, 1) + (x', 1)\rangle$ is achieved if and only if both $\langle \cdot, (x, 1)\rangle$ and $\langle \cdot, (x', 1)\rangle$ achieve their minimum simultaneously. This shows that

$$F \cap G = \operatorname{face}_{(x,1)+(x',1)}(\operatorname{conv}^h_{\mathbb{D}}(A)) = \operatorname{face}_{\left(\frac{x+x'}{2},1\right)}(\operatorname{conv}^h_{\mathbb{D}}(A))$$

which is a lower face.

Similarly, if F is a lower face of $\operatorname{conv}^h_{D}(A)$ and G is a face of F, then

$$F = \operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A)) \text{ and } G = \operatorname{face}_{(x',x_0)}(\operatorname{conv}^h_{\mathbb{D}}(A))$$

for some $x', x \in M_{\mathbb{D}}$ and $x_0 \in \mathbb{D}$. If x_0 is invertible then $G = \operatorname{face}_{(x'/x_0,1)}(\operatorname{conv}^h_{\mathbb{D}}(A))$, so it is a lower face. If x_0 is not invertible then $1 + x_0$ is invertible and then

$$G = G \cap F = \operatorname{face}_{(x,1)+(x',x_0)}(\operatorname{conv}^h_{\mathbb{D}}(A)) = \operatorname{face}_{\left(\frac{x+x'}{1+x_0},1\right)}(\operatorname{conv}^h_{\mathbb{D}}(A)).$$

Hence, G is a lower face as well in this case. This finishes the proof that the set of lower faces defines a polyhedral complex.

Now, notice that the restriction of

$$\begin{aligned} \pi : M_{\mathbb{D}} \times \mathbb{D} &\longrightarrow M_{\mathbb{D}} \\ (x, a) &\longmapsto x \end{aligned}$$

to the set of lower faces gives a bijection onto $\operatorname{conv}^h_{\mathbb{D}}(A)$. Indeed, if we have a lower face $F = \operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A))$ containing an element $(y, y_0) \in F$, then

$$\langle (y, y_0), (x, 1)\rangle = \langle y, x\rangle + y_0$$

should be minimized among all $(y, y_0) \in \operatorname{conv}^h_{\mathbb{D}}(A)$, in particular we should have

$$y_0 = \min \left\{ y' \in \mathbb{D} \;\middle|\; (y, y') \in \operatorname{conv}^h_{\mathbb{D}}(A) \right\},$$

hence y_0 is uniquely determined in terms of y and the map is injective.

This shows that $\Delta^h(\operatorname{conv}_{\mathbb{D}}(A))$ is a polyhedral complex, as it is the injective image of another polyhedral complex and by Proposition 1.27 the image of a polyhedron is a polyhedron. □

In this way, we have introduced the concept of regular subdivisions for a polytope over $\mathbb{D}$. In the next proposition, we introduce the concept of a layered regular subdivision in several equivalent ways.

Proposition 4.13 (Layered Regular Subdivisions)**.** *Let $A \subseteq M_{\mathbb{R}}$ be a finite subset of real vectors and consider a height function*

$$\begin{aligned} h : A &\longrightarrow \mathbb{D} \\ a &\longmapsto h(y) = h^{(0)}(y) + \cdots + \varepsilon^{k-1} h^{(k-1)}(y). \end{aligned}$$

We can construct a sequence of subdivisions of $\mathrm{conv}_{\mathbb{R}}(A)$

$$\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A)) \colon \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1} \tag{4.5}$$

in the following equivalent ways:

(1) Δ_0 *is the regular subdivision of* $\mathrm{conv}_{\mathbb{R}}(A)$ *induced by* $h^{(0)}$ *and, for* $0 < i \leq k-1$, Δ_i *is the subdivision of* Δ_{i-1} *obtained by subdividing each cell* $\delta \in \Delta_i$ *by the regular subdivision induced by the height function*

$$\begin{aligned} h^{(i)}\,|_{\delta} \colon \delta \cap A &\longrightarrow \mathbb{R} \\ y &\longmapsto h^{(i)}(y) \end{aligned}$$

(2) Δ_i *is the regular subdivision defined by the height function*

$$h^{(0)} + \delta h^{(1)} + \cdots + \delta^i h^{(i)} : A \longrightarrow \mathbb{R}$$

for $\delta \in \mathbb{R}_{>0}$ *small enough.*

(3) Given an element $x \in N_{\mathbb{D}}$, *for each* $0 \leq i \leq k-1$ *consider the set*

$$S_i^h(x) \coloneqq \mathrm{arg.min}_{a \in A}\{\langle y\,, x^{[i]}\rangle + h^{[i]}(y)\} \subseteq A.$$

That is, $S_i^h(x)$ *is the set of all* $y \in A$ *for which the expression*

$$\langle y\,, x^{[i]}\rangle + h^{[i]}(y)$$

is minimal among all $y \in A$. *The subdivision* Δ_i *is the one whose cells are the real polyhedra of the form* $\mathrm{conv}_{\mathbb{R}}(S_i^h(x))$ *for some* $x \in N_{\mathbb{D}}$.

Notice that, by item (3) above, Δ_{i+1} *is a refinement of* Δ_i *and, by item (2) above,* Δ_i *is a regular subdivision for each* i. *Conversely, any sequence of regular subdivisions in which each term is a refinement of the previous one is a layered regular subdivision for some height function.*

Proof. Let us call Δ_i^1, Δ_i^2 and Δ_i^3 the subdivisions defined by (1), (2) and (3) respectively.

First, let us see that Δ_i^2 and Δ_i^3 coincide. For this, given $\delta \in \mathbb{R}_{>0}$ consider the map

$$\begin{aligned} \psi_\delta : \mathbb{D} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^{(0)} + \delta x^{(1)} + \cdots + \delta^{k-1} x^{(k-1)} \end{aligned}$$

This map is $\mathbb{R}$-linear and extend to a map $\psi_\delta : N_{\mathbb{D}} \to N_{\mathbb{R}}$. Moreover, for a given $x \in N_{\mathbb{D}}$ it satisfies

$$S_i^h(x) = S_1^{\psi_\varepsilon \circ h}(\psi_\delta(x)) \tag{4.6}$$

for $\delta \in \mathbb{R}_{>0}$ small enough. As we need only finitely many x to cover all the sets of the form $S_i^h(x)$, we can take $\delta > 0$ small enough so (4.6) holds for every set in Δ_i^2. As Δ_i^2 is exactly the regular subdivision induced by the height function $\psi_\delta \circ h$ we get $\Delta_i^1 = \Delta_i^2$.

We will now see that $\Delta_i^1 = \Delta_i^3$. By definition $\Delta_0^1 = \Delta_0^2$ so we are done. This is true by definition if $i = 0$. Now, if $i = 1$ we have that

$$\langle y\,, x^{[1]}\rangle + h(y)^{[1]}$$

is minimal with the lexicographic order among all $y \in A$ iff we have that

$$\langle y\,, x\rangle^{(0)} + h^{(0)}(y)$$

is minimal and, among the ones which are minimal, that is, among $S_h^{(0)}(x^{(0)})$, we have that

$$\langle y\,,x\rangle^{(1)} + h^{(1)}(y)$$

is also minimal. Hence, we get that

$$S_1^{h^{[1]}}(x^{[1]}) = S_0^{h^{(1)}}\;|_{S_0^{h^{(0)}(x^{(0)})}}\;(x^{(1)})$$

which shows $\Delta_2^1 = \Delta_2^3$. In a similar way, we have

$$S_{i+1}^{h^{[i+1]}}(x^{[i+1]}) = S_1^{h^{(i+1)}}\;|_{S_i^{h^{[i]}(x^{[i]})}}\;(x^{(i+1)})$$

so $\Delta_i^1 = \Delta_i^3$ follows by induction.

Finally, take a sequence $\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))$ of regular subdivisions of $\mathrm{conv}_{\mathbb{R}} A$ in which each term is a refinement of the previous one. By Theorem 2.4 in [GKZ08], if we consider $\Lambda = \{\sigma\}_\sigma$ to be the normal fan of the secondary polytope of A, then, the sequence of subdivision $\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))$ in (4.5) correspond to a flag of cones

$$\underline{\sigma} = \sigma_0 \succeq \sigma_2 \succeq \cdots \geq \sigma_k - 1$$

where each σ_{i+1} is a face of σ_i, and a height function $h^{(i)}$ defines Δ_i iff it is in the relative interior of σ_i. Hence, by taking $h^{(i)}$ in the relative interior of σ_i for each i, we get that

$$h \coloneqq h^{(0)} + \varepsilon h^{(1)} + \cdots + \varepsilon^{k-1} h^{(k-1)}$$

defines the layered regular subdivision $\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))$. $\square$

Now, given a set $A \subseteq M_{\mathbb{R}}$ of real vectors and a height function $h : A \to \mathbb{D}$ we can construct two different objects: A regular subdivision for $\mathrm{conv}_{\mathbb{D}}(A)$ and a layered regular subdivision for $\mathrm{conv}_{\mathbb{R}}(A)$. The exact connection between these two objects is given in the following theorem.

Theorem 4.14. *Consider a finite set of real points $A \subseteq M_{\mathbb{R}}$ and a height function $h : A \to \mathbb{D}$. Then, we have an equality of the form*

$$\Delta^h(\mathrm{conv}_{\mathbb{D}}(A)) = T\mathcal{C}\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A)).$$

In the sense that, the elements of $\Delta^h(\mathrm{conv}_{\mathbb{D}}(A))$ are exactly the polyhedra of the form $T\mathcal{C}(\underline{F})$ for

$$\underline{F}\colon F_{k-1} \subseteq F_{k-2} \subseteq \cdots \subseteq F_0$$

where F_i is a face of Δ_i for each i.

Lemma 4.15. *Given an element $x \in N_{\mathbb{D}}$ and $a \in A$. The integer*

$$\beta = \mathrm{ord}\Big(\langle a\,,x\rangle + h(a) - \min_{b\in A}\big(\langle b\,,x\rangle + h(b)\big)\Big)$$

is the maximal integer in $\{0, 1 \ldots, k\}$ for which $a \in S_{\beta-1}^h(x)$.

Proof. If $\beta = \operatorname{ord}\Big(\langle a\,,x\rangle + h(a) - \min_{b\in A}\left(\langle b\,,x\rangle + h(b)\right)\Big)$ then β is the maximal element in $\{0,1,\ldots,k\}$ such that

$$\begin{aligned}
&\varepsilon^{k-\beta}\left(\langle a\,,x\rangle + h(a)\right) = \varepsilon^{k-\beta}\Big(\min_{b\in A}\left(\langle b\,,x\rangle + h(b)\right)\Big)\\
\iff &\left(\langle a\,,x\rangle + h(a)\right)^{[\beta-1]} = \Big(\min_{b\in A}\left(\langle b\,,x\rangle + h(b)\right)\Big)^{[\beta-1]} = \min_{b\in A}\Big(\left(\langle b\,,x\rangle + h(b)\right)^{[\beta-1]}\Big)\\
\iff &\langle a\,,x\rangle^{[\beta-1]} + h(a)^{[\beta-1]} \text{ is minimal among all } a\in A\\
\iff &a \in S^h_{\beta-1}(x)
\end{aligned}$$

□

Proof of Theorem 4.14. Given $x \in N_{\mathbb{D}}$, we can define a face of $\Delta^h(\operatorname{conv}_{\mathbb{D}}(A))$ by

$$\pi\left(\operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A))\right),$$

on the other hand, the same x defines a layered face in $\underline{\Delta}^h(\operatorname{cone}_{\mathbb{R}}(A)$ by

$$\underline{F}(x) : \operatorname{conv}_{\mathbb{R}}(S^h_{k-1}(x)) \subseteq \cdots \subseteq \operatorname{conv}_{\mathbb{R}}(S^h_0(x)).$$

In this way, it is enough to prove that

$$\pi\left(\operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A))\right) = T\mathcal{C}\underline{F}(x) \tag{4.7}$$

For this, from Proposition 2.10 we have

$$\begin{aligned}
&\operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A)) = \operatorname{wconv}_{\mathbb{D}}\left([\varepsilon^{k-\beta_a}(a,h(a)); k-\beta_a] \,\Big|\, a\in A\right)\\
\implies &\pi\left(\operatorname{face}_{(x,1)}(\operatorname{conv}^h_{\mathbb{D}}(A))\right) = \operatorname{wconv}_{\mathbb{D}}\left([\varepsilon^{k-\beta_a}a; k-\beta_a] \,\Big|\, a\in A\right)
\end{aligned}$$

where

$$\beta_a = \operatorname{ord}(\langle (a,h(a))\,,(x,1)\rangle - c) = \operatorname{ord}(\langle a\,,x\rangle + h(a) - c)$$

with c the minimum of $\langle\cdot\,,(x,1)\rangle$ on $\operatorname{conv}^h_{\mathbb{D}}(A)$. As this minimum must be achieved in one of the generators of $\operatorname{conv}^h_{\mathbb{D}}(A)$, we have $c = \min_{b\in A}\left(\langle b\,,x\rangle + h(b)\right)$. Then, by Lemma 4.15, β_a is equal to the maximum integer such that $a \in S^h_{\beta_a-1}(x)$.

On the other hand, by Theorem 4.2 we can write explicitly $T\mathcal{C}\underline{F}(x)$ in term of the generators of $\underline{F}(x)$ and we get

$$
\begin{aligned}
T\mathcal{C}\underline{F}(x) &= \operatorname{wconv}\left([\varepsilon^i a\,;\,i] \;\middle|\; i\in\{0,\dots,k\}, a\in S^h_{k-1-i}(x)\right)\\
&= \left\{\sum_{i=0}^{k-1}\sum_{a\in S^h_i(x)}\lambda_{a,i}a\varepsilon^{k-1-i} \;\middle|\; \lambda_{a,i}\geq 0\;\forall a\,\forall i,\ \text{and}\ \sum_{i=0}^{k-1}\sum_{a\in S^h_i(x)}\lambda_{a,i}\varepsilon^{k-1-i}=1\right\}\\
&= \left\{\sum_{a\in A} a(\lambda_{a,\beta_a}\varepsilon^{k-\beta_a}+\lambda_{a,\beta_a-1}\varepsilon^{k-(\beta_a-1)}+\dots\lambda_{a,0}\varepsilon^{k-1})\right.\\
&\qquad\left.\;\middle|\; \lambda_{a,i}\geq 0\;\forall a\forall i,\ \text{and}\ \sum_{a\in A}\lambda_{a,\beta_a}\varepsilon^{k-\beta_a}+\lambda_{a,\beta_a-1}\varepsilon^{k-(\beta_a-1)}+\dots\lambda_{a,0}\varepsilon^{k-1}=1\right\}\\
&= \left\{\sum_{a\in A}\mu_a a\varepsilon^{k-\beta_a} \;\middle|\; \mu_a\geq 0\;\forall a,\ \text{and}\ \sum_{a\in A}\mu_a\varepsilon^{k-\beta_a}=1\right\}\\
&= \operatorname{wconv}_{\mathbb{D}}\left([\varepsilon^{k-\beta_a}a; k-\beta_a] \;\middle|\; a\in A\right)
\end{aligned}
$$

Where, for the third equality we factorized by a and for the fourth equality we did the change of variable

$$\mu_a = \lambda_{a,\beta_a} + \lambda_{a,\beta_a-1}\varepsilon + \dots \lambda_{a,0}\varepsilon^{k-1}.$$

In this way, we have shown the equality in (4.7) as we wrote both sets in the same way. □

4.4. **Higher Rank Tropical Hypersurfaces.** This section gives a first set of applications of the theory of polyhedral geometry of higher rank to tropical geometry of higher rank. After introducing the basics objects of the theory, in Proposition 4.20 we show that higher rank tropical hypersurfaces can naturally be regarded as iterated fibrations. This fibration is studied in the Hypersurface Duality (Theorem 4.27). Where we show that the base and each fiber of a higher rank tropical hypersurface consist of tropical hypersurfaces of rank one, moreover, the normal type of these tropical hypersurfaces is encoded in a layered regular subdivision of the Newton polytope. Finally, in Theorem 4.32 we put a polyhedral structure over $\mathbb{D}$ on higher rank tropical hypersurfaces which is compatible with the Hypersurface Duality previously presented.

Definition 4.16. The *tropical semifield of rank k* or *min-plus algebra of rank k*, is the semifield

$$\mathbb{T}_k = \big(\mathbb{D}\cup\{\infty\},\ \min,\ +\big),$$

were we consider by addition the map $(a,b)\mapsto \min\{a,b\}$ and by multiplication the map $(a,b)\mapsto a+b$.

An expression in $\mathbb{T}_k$ will be written between quotation marks and with the usual symbols $+$ and $\cdot$, for example,

$$\text{“}\sum_{i=1}^{n} x_i y_i\text{”} = \min\{x_i+y_i \mid i=1,\dots,n\}.$$

Remark 4.17.

(1) In general, for any ordered abelian group $(\Gamma, +)$ one can consider its associated tropical semifield
$$\mathbb{T}_\Gamma = \big(\Gamma \cup \{\infty\},\ \min,\ +\big).$$
In this way, $\mathbb{T}_k$ corresponds to the case in which the ordered group is $(\mathbb{D}, +)$ or, equivalently, $(\mathbb{R}^k, +)$ with its lexicographic order.

(2) In $\mathbb{T}_k$ the element ∞ becomes the additive identity as we have $\min\{\infty, a\} = a$ for every $a \in \mathbb{T}_k$. Similarly, 0 becomes the multiplicative identity in $\mathbb{T}_k$. For these reasons we have equalities of the form
$$\text{“}\,x + y\,\text{”} = \text{“}\,0x + 0y + \infty\,\text{”}.$$
In particular, the coefficient of x in “ $x + y$ ” is 0 and not 1.

Definition 4.18. Given a lattice M, the ring of *Laurent tropical polynomials* on M is the set $\mathbb{T}_k[M]$ of all formal sums of the form
$$f = \text{“}\sum_{m \in M} a_m T^m\text{”}$$
whose *support*
$$\operatorname{Supp}(f) \coloneqq \{m \in M \mid a_m \neq \infty\}$$
is a finite set. We endow $\mathbb{T}_k[M]$ with the semiring structure induced by $\mathbb{T}_k$.

Let N be the dual lattice of M. A non-zero tropical polynomial $f = \text{“}\sum_{m \in M} a_m T^m\text{”}$ defines a map from $N_{\mathbb{D}}$, the *tropical torus of rank* k, to $\mathbb{D}$ by
$$\begin{aligned} f \colon N_{\mathbb{D}} &\longrightarrow \mathbb{D} \\ x &\longmapsto f(x) = \min\{\langle m\,, x\rangle + a_m \mid m \in M\}\,. \end{aligned}$$

Definition 4.19. Consider a tropical polynomial $f = \text{“}\sum_{m \in M} a_m T^m\text{”} \in \mathbb{T}_k[M]$.

(1) A point $x \in N_{\mathbb{D}}$ is said to be a *zero* of f if the minimum in
$$f(x) = \min\{\langle m\,, x\rangle + a_m \mid m \in M\}$$
is achieved at least twice. The set of all zeros of f is denoted by $V(f)$ and is called the *vanishing set* of f.

(2) A *tropical hypersurface of rank* k is a set of the form $V(f) \subseteq N_{\mathbb{R}^k}$ for a nonzero tropical polynomial $f \in \mathbb{T}_k[M]$.

Notice that the projections $\mathbb{D} = \mathbb{D}_k \xrightarrow{\pi} \mathbb{D}_k \xrightarrow{\pi} \dots \xrightarrow{\pi} \mathbb{D}_1$ induce, by applying them in each coefficient, the projections
$$\begin{array}{ccccccc} \mathbb{T}_k[M] & \xrightarrow{\pi} & \mathbb{T}_{k-1}[M] & \xrightarrow{\pi} & \dots & \xrightarrow{\pi} & \mathbb{T}_1[M] \\ f =: f^{[k-1]} & \mapsto & f^{[k-2]} & \mapsto & \dots & \mapsto & f^{[0]}. \end{array}$$

Using these projection we can get a natural fibered structure on the tropical hypersurface $V(f)$.

Proposition 4.20. *For each tropical polynomial* $f = \text{“}\sum_{m\in M} a_m T^m\text{”} \in \mathbb{T}[M]$*, the image of* $V\big(f^{[i]}\big)$ *under* $N_{\mathbb{D}_{i+1}} \to N_{\mathbb{D}_i}$ *goes inside* $V\big(f^{[i-1]}\big)$*. In this way, we get an iterated fibration*

$$V(f) = V\big(f^{[k]}\big) \to V\big(f^{[k-1]}\big) \to \cdots \to V\big(f^{[1]}\big).$$

Given a point $x \in V\big(f^{[i]}\big)$ *we denote by* $V_x\big(f^{[i+1]}\big)$ *the fiber of* $V(f)$ *at* x *in this fibration.*

Proof. Notice that for a given $x \in N_{\mathbb{D}_{i+1}}$, if $x \in V\big(f^{[i]}\big)$ then $f^{[i]}(x)$ achieves its minimum in two elements $\langle m\,, x\rangle + a_m$ and $\langle n\,, x\rangle + a_n$. That is,

$$f^{[i]}(x) = \langle m\,, x\rangle + a_m^{[i]} = \langle n\,, x\rangle + a_n^{[i]}$$

from which

$$f^{[i-1]}(x^{[i-1]}) = \langle m\,, x^{[i-1]}\rangle + a_m^{[i-1]} = \langle n\,, x^{[i-1]}\rangle + a_n^{[i-1]}.$$

Hence, $f^{[i-1]}(x^{[i-1]})$ also achieves its minimum at least two times, so $x^{[i-1]} \in V\big(f^{[i-1]}\big)$. □

In order to understand this fibration we introduce the following elements.

Definition 4.21. Consider a tropical polynomial $f = \text{“}\sum_{m\in M} a_m T^m\text{”} \in \mathbb{T}_k[M]$.

(1) The $\mathbb{R}$-*Newton polytope* of f is the real polytope defined by

$$\mathrm{New}_{\mathbb{R}}(f) \coloneqq \mathrm{conv}_{\mathbb{R}}(\mathrm{Supp}(f)) \subseteq M_{\mathbb{R}}$$

Similarly, we introduce the $\mathbb{D}$-*Newton polytope* of f by

$$\mathrm{New}_{\mathbb{D}}(f) \coloneqq \mathrm{conv}_{\mathbb{D}}(\mathrm{Supp}(f)) \subseteq M_{\mathbb{D}}$$

(2) Given an integer $0 \le i \le k-1$ and an element $x \in N_{\mathbb{D}_i}$, the *i-initial part of f with respect to x* is the tropical polynomial

$$\mathrm{in}_x^i(f) = \text{“}\sum_{\substack{m\in M\\ \langle m\,,x\rangle + a_m^{[i]} = f^{[i]}(x)}} a_m^{(i+1)}\, T^m\text{”} \in \mathbb{T}[M].$$

Where, for $i+1 = k$ we will use the convention

$$a_n^{(k)} \coloneqq \begin{cases} 0 & \text{if } a_m \neq \infty \\ \infty & \text{if } a_m = \infty. \end{cases}$$

(3) The height function

$$\begin{aligned} h : \mathrm{Supp}(f) &\longrightarrow \mathbb{D} \\ m &\longmapsto a_m \end{aligned}$$

naturally induce a layered regular subdivision on $\mathrm{New}_{\mathbb{R}}(f)$ which we denote by $\underline{\Delta}(f)$ and a regular subdivision on $\mathrm{New}_{\mathbb{D}}(f)$ which we denote by $\Delta(f)$.

Remark 4.22.

(1) By definition (3) on Proposition 4.13, the layered regular subdivision $\underline{\Delta}(f)$ is defined to be the one whose layered faces are of the form

$$\underline{F}(x) \coloneqq \operatorname{conv}_{\mathbb{R}}(\operatorname{Supp}(\operatorname{in}_x^{k-1}(f))) \subseteq \cdots \subseteq \operatorname{conv}_{\mathbb{R}}(\operatorname{Supp}(\operatorname{in}_x^{0}(f))).$$

where x moves over all elements $x \in N_{\mathbb{D}}$. Hence, $\underline{\Delta}(f)$ encodes all the possible values for the vector

$$(\operatorname{in}_x^0(f), \operatorname{in}_x^1(f), \dots, \operatorname{in}_x^{k-1}(f))$$

as x moves around $N_{\mathbb{D}}$.

(2) By Corollary 4.3 we have

$$\operatorname{New}_{\mathbb{D}}(f) = T\mathcal{C}^{k-1} \operatorname{New}_{\mathbb{R}}(f).$$

Moreover, by Theorem 4.14 this equality can be lifted to an equality of subdivisions of the form

$$\Delta(f) = T\mathcal{C}\underline{\Delta}(f).$$

The first important result of this section is the *Higher Rank Hypersurface Duality Theorem* below, which states that the layered regular subdivision $\underline{\Delta}(f)$ obtained by using as height function the coefficients of f, allow us to obtain the normal type of both the base and the fibers in the iterated fibration of Proposition 4.20.

In order to introduce this, let us recall the following concept.

Definition 4.23. Given polyhedral complexes Σ in $M_{\mathbb{D}}$ and Σ' in $N_{\mathbb{D}}$. A duality between Σ and Σ' is a map $\Lambda : \Sigma \to \Sigma'$ such that

(1) The map Λ is a bijection.

(2) Given faces $F, G \in \Sigma$, whenever $F \vee G$ exists we have that $\Lambda(F) \wedge \Lambda(G)$ exists and

$$\Lambda(F \vee G) = \Lambda(F) \wedge \Lambda(G).$$

(3) Similarly, given faces $F, G \in \Sigma$, whenever $F \wedge G$ exists we have that $\Lambda(F) \vee \Lambda(G)$ exists and

$$\Lambda(F \wedge G) = \Lambda(F) \vee \Lambda(G).$$

(4) For each $F \in \Sigma$, $\Lambda(F)$ is orthogonal to F in the sense that

$$\langle x \,, y \rangle = 0, \; \forall\, x \in F, y \in F'$$

Because of properties (2) and (3) we say that Λ preserves *incidences*.

Remark 4.24.

(1) Given $F, G \in \Sigma$ we have $F \preceq G$ iff $\Lambda(G) \preceq \Lambda(F)$. Indeed,

$$F \preceq G \iff F \wedge G = G \iff \Lambda(G) \vee \Lambda(F) = \Lambda(G) \iff \Lambda(G) \preceq \Lambda(F).$$

(2) In the case in which Σ and Σ' are real polyhedral complex, i.e, in rank 1. We have that

$$\dim(F) = \operatorname{codim}(\Lambda(F)).$$

Indeed, a maximal flag

$$F_0 \preceq \cdots \preceq F_{dim(F)} \coloneqq F$$

gives rise to a maximal flag

$$\Lambda(F) = \Lambda(F_{dim(F)}) \preceq \cdots \preceq \Lambda(F_0).$$

Let us recall the following fact from the usual theory of tropical geometry. For a proof of this result, we refer to [MS15] Theorem .

Theorem 4.25 (Hypersurface Duality)**.** *Given $f \in \mathbb{T}[M]$, if we denote by $\Delta(f)$ the regular subdivision of* New(f) *induced by the coefficients of f. Then, there is a polyhedral complex* $\mathrm{GC}(f)$*, called its Gröbner complex, whose support is $N_{\mathbb{D}}$ and whose cells are parametrized by the faces $F \in \Delta(f)$. Explicitly they are given by*

$$\mathrm{GC}(F) = \{x \in N_{\mathbb{R}} \mid \mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x(f))) \supseteq F\}.$$

Moreover, the map

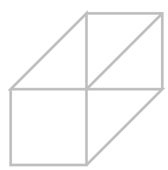
$$\begin{aligned} \Lambda : \Delta(f) &\longrightarrow \mathrm{GC}(f) \\ F &\longmapsto GC(F) \end{aligned}$$

is a duality in the sense of Definition 4.23. Furthermore, if we restrict Λ to the elements of $\Delta(f)$ that are not points we obtain a subcomplex $\Sigma(f)$ of $\mathrm{GC}(f)$ *whose support is $V(f)$.*

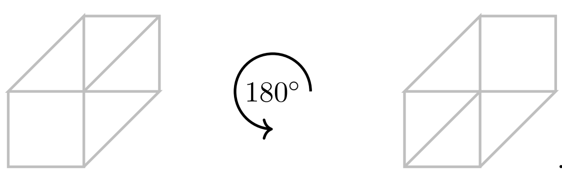
In a explicit way, to obtain the shape of the tropical hypersurface we have to

(1) Do a point reflection of $\Delta(f)$.
(2) Consider one point x_F for each facet F of the reflected $\Delta(f)$.
(3) Join the different points according to the incidence of $\Delta(f)$.
(4) Draw a cone pointed at x_F perpendicular to each face of F laying in the boundary of the Newton polytope.

Example 4.26. If $f =$ "$7x^2y^2 + 5x^2y + 5xy^2 + 4xy + 2x + 2y + 0$" then, the subdivision $\Delta(f)$ of New(f) looks like

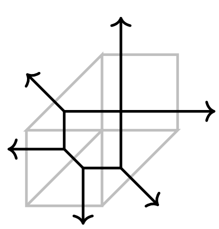
$\underline{\Delta}(f) =$ 

If we do a point reflection of it we get



.

Hence, the shape of the tropical hypersurface in this case is



.

Theorem 4.27 (Higher Rank Hypersurface Duality)**.** *Let $f \in \mathbb{T}_k[M]$ be a non-zero polynomial and consider*

$$\underline{\Delta}(f) = \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1}$$

the layered regular subdivision induce by f over $\operatorname{New} f$*. Then, we have that:*

(1) The base $V\big(f^{[0]}\big)$ is a rank one tropical hypersurface with the structure of a polyhedral complex dual to the first subdivision Δ_0.

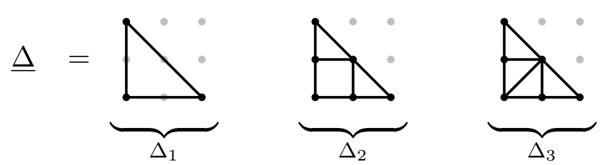
(2) For each $x \in V\big(f^{[0]}\big)$, the fiber $V_x\big(f^{[1]}\big)$ is also a rank one tropical hypersurface. Moreover, $V_x\big(f^{[1]}\big)$ remains constant as x varies over the interior of a cell $\mathrm{GC}(F) \subseteq V\big(f^{[0]}\big)$ for some $F \in \Delta_1$ and the normal type of $V_x\big(f^{[1]}\big)$ is dual to the subdivision Δ_1 restricted to F.

(3) More generally, for each $x \in V\big(f^{[i]}\big)$, the fiber $V_x\big(f^{[i+1]}\big)$ is also a rank one tropical hypersurface. It remains constant as x varies over the interior of a cell $\mathrm{GC}(F) \subseteq V_{x^{[i-1]}}\big(f^{[i]}\big)$ for some $F \in \Delta_{i-1}$ and the normal type of $V_x\big(f^{[i+1]}\big)$ is dual to the subdivision Δ_{i+1} restricted to F.

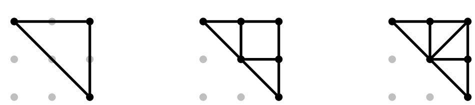
Example 4.28. Consider $k = 3$, $M = \mathbb{Z}^2$ and the polynomial

$$f(x, y) = (0, 1, 2) + (0, 1, 1)x + (0, 1, 1)y + (0, 1, 2)xy + (0, 0, 0)x^2 + (0, 0, 0)y^2$$

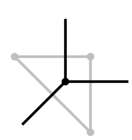
The Newton polytope of f is $\operatorname{New} f = \operatorname{conv}_{\mathbb{R}}((0, 0), (2, 0), (0, 2))$ and its associated layered subdivision is the following:



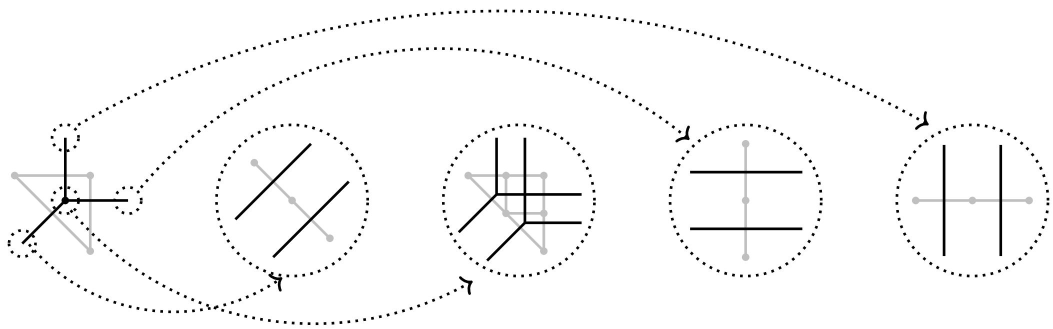
After a point reflection it becomes



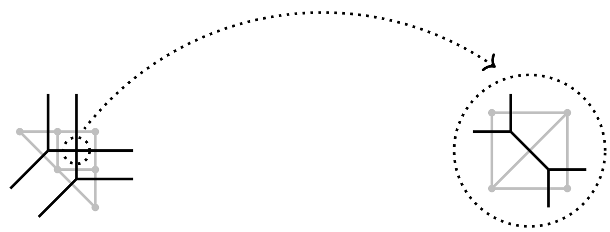
Therefore, the base of the fibration $V(f^{[1]})$ has the shape



And over each point of the base, there are 4 possible shapes for the fibers of $V(f^{[2]})$, represented in the following diagram:



Moreover, each of these fibers is the base for a fibration determined by $V(f^{[3]})$. All the fibers of this fibrations will have the shape of the corresponding tangent cone, with the exception of one fiber, the one corresponding to the subdivision of the square, which we sketch as follows:



Proof of Theorem 4.27. The subdivision Δ_0 corresponds to the regular subdivision induced by the coordinates of $f^{[0]}$. Hence, part (1) of the theorem follows directly from the hypersurface duality of rank one (Theorem 4.25).

In order to prove (2), let $x^{(0)} \in V(f^{[0]})$, then $V_{x^{(0)}}(f^{[1]})$ is the set of all $x^{(1)} \in N_{\mathbb{R}}$ such that

$$x^{(0)} + \varepsilon x^{(1)} \in N_{\mathbb{D}_2}.$$

Therefore, if we consider the polynomial

$$\mathrm{in}^0_{x^{(0)}}(f) = \text{“} \sum_{\substack{m \in M \\ f^{[0]}(x) = \langle m, x^{(0)} \rangle + a^{[0]}_m}} a^{(1)}_m T^m \text{”}$$

we see that $x^{(0)} + \varepsilon x^{(1)}$ is a zero of $f^{[1]}$ if and only if $x^{(0)}$ is a zero of $f^{[0]}$ and $x^{(1)}$ is a zero of $\mathrm{in}^1_{x^{(0)}}(f)$. Hence, we obtain

$$V_x(f^{[1]}) = V(\mathrm{in}^0_{x^{(0)}}(f)).$$

As $\mathrm{New}(\mathrm{in}^0_{x^{(0)}} f) = F$, again by the original hypersurface duality, we get that $V(\mathrm{in}^0_{x^{(0)}} f)$ is dual to the regular subdivision induced by the height function $m \mapsto a^{(1)}_m$, which, by Proposition 4.13, is exactly Δ_1.

The general case follows similarly as one can show that

$$V_{x^{[i]}}(f^{[i+1]}) = V(\mathrm{in}^i_{x^{[i]}} f).$$

□

The objective now is to put a polyhedral structure on $V(f)$ which is dual to the layered regular subdivision of its Newton polytope in a natural way. Generalizing to higher rank the polyhedral part of Theorem 4.25. For this, we will introduce the analog of the Gröbner complex in higher rank.

Definition 4.29. Given a layered face $\underline{F} \in \underline{\Delta}(f)$ of the form

$$\underline{F}\colon F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1}$$

where F_i is a face of Δ_i for each i. We consider its corresponding *Gröbner cell* as

$$\mathrm{GC}(\underline{F}) := \left\{ x \in N_{\mathbb{D}} \;\middle|\; \mathrm{conv}_{\mathbb{R}}(\mathrm{in}^i_x(f)) \supseteq F_i,\ \forall 1 \leq i \leq k \right\}.$$

We will prove that the family of all Gröner cells is a polyhedral complex. For this, the idea is to consider the *lifted Newton polytope*

$$\mathrm{New}_{\mathbb{D}}(f)^h := \mathrm{conv}^h_{\mathbb{D}}(\mathrm{Supp}(f)) \subseteq M_{\mathbb{D}} \times \mathbb{D}$$

used to define the regular subdivision as in Definition 4.11. We will show that the normal fan of this polytope intersected with $N_{\mathbb{D}} \times \{1\}$ is again a polyhedral complex and its cells are the Gröbner cells.

We start with the following lemmas.

Lemma 4.30. *Given a layered face $\underline{F} \in \underline{\Delta}(f)$, we have that*

$$x \in \mathrm{GC}(\underline{F}) \iff (x, 1) \in \mathrm{C}\left((T\mathcal{C}\underline{F})^h\right).$$

Where $(T\mathcal{C}\underline{F})^h$ represents the only lower face of $\mathrm{New}_{\mathbb{D}}(f)^h$ which projects to $T\mathcal{C}\underline{F}$ under $M_{\mathbb{D}} \times \mathbb{D} \to \mathbb{D}$, and $\mathrm{C}\left((T\mathcal{C}\underline{F})^h\right) \subseteq N_{\mathbb{D}} \times \mathbb{D}$ is the normal cone of this face. In other words, we have the equality

$$\mathrm{GC}(\underline{F}) \times \{1\} = \mathrm{C}\left((T\mathcal{C}\underline{F})^h\right) \cap N_{\mathbb{D}} \times \{1\}.$$

Proof. A point $x \in N_{\mathbb{D}}$ belongs to $\mathrm{GC}(\underline{F})$ if and only if the flag

$$\underline{F}(x) : F_0(x) \subseteq F_2(x) \subseteq \ldots F_{k-1}(x)$$

where $F_i(x) = \mathrm{conv}_{\mathbb{R}}\left(\mathrm{Supp}\left(\mathrm{in}^i_x(f)\right)\right)$ satisfies $F_i(x) \supseteq F_i$ for all $i = 0, \ldots, k-1$, and this happens iff

$$T\mathcal{C}\underline{F}(x) \supseteq T\mathcal{C}\underline{F}.$$

Moreover, if we consider the projection $\pi : M_{\mathbb{D}} \times \mathbb{D} \to M_{\mathbb{D}}$ then we have that

$$T\mathcal{C}\underline{F} = \pi\left((T\mathcal{C}F)^h\right)$$

and, by Theorem 4.14,

$$T\mathcal{C}\underline{F}(x) = \pi\left(\mathrm{face}_{(x,1)}\,\mathrm{New}_{\mathbb{D}}(f)^h\right).$$

Hence, as by Proposition 4.12 the map π restricted to the set of lower faces of $\mathrm{New}(f)^h$ is injective, we conclude that $T\mathcal{C}\underline{F}(x) \supseteq T\mathcal{C}\underline{F}$ happens iff

$$\mathrm{face}_{(x,1)} \mathrm{New}_{\mathbb{D}}(f)^h \supseteq (T\mathcal{C}\underline{F})^h.$$

Which by definition means $(x,1) \in \mathrm{C}\left((T\mathcal{C}\underline{F})^h\right)$. In this way we have seen that

$$x \in \mathrm{GC}(\underline{F}) \iff (x,1) \in \mathrm{C}\left((T\mathcal{C}\underline{F})^h\right)$$

as we wanted. $\square$

Lemma 4.31. *Let σ be a polyhedral cone in $N_{\mathbb{D}} \times \mathbb{D}$ and consider P to be the projection of $\sigma \cap N_{\mathbb{D}} \times \{1\}$ to $N_{\mathbb{D}}$. Then, the map*

$$\begin{aligned} \mathfrak{F}_\sigma &\longrightarrow \mathfrak{F}_P \\ \tau &\longmapsto \pi\left(\tau \cap N_{\mathbb{D}} \times \{1\}\right) \end{aligned}$$

is surjective where $\pi : N_{\mathbb{D}} \times \mathbb{D} \to N_{\mathbb{D}}$ is the usual projection.

Proof. A face of P is of the form $\mathrm{face}_y P$ for some $y \in M_{\mathbb{D}}$. Let us consider $a = \min_{x\in P}\langle y\,, x\rangle$. If we show that $(y,-a) \in \sigma^\vee$ we are done, as then we can consider $\mathrm{face}_{(y,-a)}\,\sigma$ and this satisfies

$$\mathrm{face}_{(y,-a)}\,\sigma \cap N_{\mathbb{D}} \times \{1\} = \mathrm{face}_y P \times \{1\}.$$

Let us see now that $(y,-a) \in \sigma^\vee$. For this, notice that

$$\sigma \cap N_{\mathbb{D}} \times \{1\} = P \times \{1\}. \tag{4.8}$$

Moreover, as $\langle y\,, x\rangle \geq a$ for any $x \in P$ we get

$$\langle (y,-a)\,, (x,1)\rangle \geq 0 \text{ for any } x \in P \times \{1\}.$$

Now, if we take $(x,b) \in \sigma$ with $b \in \mathbb{D}^\times_{>0}$ invertible, by the equality in (4.8), we have $x/b \in P$. Hence,

$$\langle (y,-a)\,, (x,b)\rangle = b\langle (y,-a)\,, (x/b,1)\rangle \geq 0.$$

On the other hand, take an element of the form $(x,b) \in \sigma$ with b not invertible and consider an element x' in P achieving the minimum of y, that is $\langle (y,-a)\,, (x',1)\rangle = 0$. Then, we can consider $(x',1)+(x,b) = (x'+x, 1+b)$. Now $1+b$ is invertible, so from the previous step

$$0 \leq \langle (y,-a)\,, ((x'+x,1+b))\rangle = \langle (y,-a)\,, (x',1)\rangle + \langle (y,-a)\,, (x,b)\rangle = \langle (y,-a)\,, (x,b)\rangle.$$

Hence, $(y,-a)$ is positive in (x,b) for any $(x,b) \in \sigma$. Therefore, $(y,-a) \in \sigma^\vee$. $\square$

Theorem 4.32 (Polyhedral Structure)**.** *The family*

$$\mathrm{GC}(f) = \{\mathrm{GC}(\underline{F}) \mid \underline{F} \in \underline{\Delta}(f)\}$$

is a polyhedral complex with support $N_{\mathbb{D}}$ called the Gröbner complex *of f.*

Moreover, if we consider only the layered faces of $\underline{\Delta}(f)$ in which F_{k-1} is not a point, that is,

$$\Sigma(f) = \{\mathrm{GC}(\underline{F}) \mid \underline{F} \in \underline{\Delta}(f) \text{ and } F_{k-1} \text{ is not a point}\},$$

we obtain a polyhedral complex with support $V(f)$.

Remark 4.33. The Gröbner complex $\mathrm{GC}(f)$ is exactly the subdivision of $N_{\mathbb{D}}$ under which the map

$$x \mapsto \left(\mathrm{in}_x^0(f), \ldots, \mathrm{in}_x^{k-1}(f)\right)$$

is constant over the interior of each cell. Moreover,

Proof of Theorem 4.32. We will start by showing that $\mathrm{GC}(f)$ is a polyhedral complex. First notice that, by Lemma 4.30, we have $\mathrm{GC}(\underline{F}) = \pi\left(\mathrm{C}\left((\mathcal{TC}\underline{\delta})^h\right) \cap N_{\mathbb{D}} \times \{1\}\right)$, which in particular implies that $\mathrm{GC}(\underline{F})$ is a polyhedron for each $\underline{F} \in \underline{\Delta}$. Moreover, given $\underline{F}, \underline{F}' \in \underline{\Delta}(f)$, we can consider $\underline{F} \vee \underline{F}'$ the layered face given by $F_i \vee F_i'$. Then,

$$\begin{aligned}
\mathrm{GC}(\underline{F}) \cap \mathrm{GC}(\underline{F}') &= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h\right)\right) \cap \pi\left(\mathrm{C}\left(\mathcal{TC}(\underline{F}')^h\right)\right)\\
&= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h\right) \cap \mathrm{C}\left(\mathcal{TC}(\underline{F}')^h\right)\right)\\
&= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h \vee \mathcal{TC}(\underline{F}')^h\right)\right)\\
&= \pi\left(\mathrm{C}\left(\left(\mathcal{TC}\left(\underline{F}\right) \vee \mathcal{TC}(\underline{F}')\right)^h\right)\right)\\
&= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F} \vee \underline{F}'\right)^h\right)\right)\\
&= \mathrm{GC}(\underline{F} \vee \underline{F}').
\end{aligned}$$

Therefore $\mathrm{GC}(\underline{F}) \cap \mathrm{GC}(\underline{F}') = \mathrm{GC}(\underline{F} \vee \underline{F}') \in \mathrm{GC}(f)$ and it is a face of both $\mathrm{GC}(F)$ and $\mathrm{GC}(F')$ because $\mathrm{C}\left(\mathcal{TC}\left(\underline{F} \vee \underline{F}'\right)^h\right)$ is a face of both $\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h\right)$ and $\mathrm{C}\left(\mathcal{TC}\left(\underline{F}'\right)^h\right)$.

Finally, if H is a face of $\mathrm{GC}(\underline{F})$ we will show that $H = \mathrm{GC}(\underline{F}')$ for some $\underline{F}' \in \underline{\Delta}(f)$. For this, notice that by Lemma 4.31 there is a face τ of $\mathrm{C}(\mathcal{TC}(\underline{F})^h)$ such that $\tau \cap N_{\mathbb{D}} \times \{1\} = H \times \{1\}$. Now, given $x \in \mathrm{int}(H)$ we have that $(x, 1) \in \mathrm{int}(\tau)$. Then, $\mathrm{face}_{(x,1)} \mathrm{New}(f)^h$ is a lower face of $\mathrm{New}(f)^h$ with

$$C\left(\mathrm{face}_{(x,1)} \mathrm{New}(f)^h\right) = \tau.$$

By Theorem 4.14, the projection of $\mathrm{face}_{(x,1)} \mathrm{New}(f)^h$ to $\mathrm{New}(f)$ is of the form $\mathcal{TC}\left(\underline{F}'\right)$ for some $\underline{F}' \in \underline{\Delta}(f)$. This $\underline{F}'$ satisfies $\mathrm{GC}(\underline{F}') = H$.

With this we have prove that $\mathrm{GC}(f)$ is a polyhedral complex with support

$$\pi\left(\left|\mathrm{NF}\left(\mathrm{New}(f)^h\right)\right| \cap N_{\mathbb{D}} \times \{1\}\right) = N_{\mathbb{D}}.$$

In order to see that $\Sigma(f)$ is a polyhedral complex, it is enough to notice that if $\mathrm{GC}(\underline{F}), \mathrm{GC}(\underline{F}) \in \mathrm{GC}(f)$ then F_{k-1} and F_{k-1}' are not points, hence $F_{k-1} \vee F_{k-1}'$ is not a point, so

$$\mathrm{GC}(\underline{F}) \cap \mathrm{GC}(\underline{F}) = \mathrm{GC}(\underline{F} \vee \underline{F}) \in \mathrm{GC}(f).$$

Similarly, if $\mathrm{GC}(\underline{F}) \in \mathrm{GC}(f)$ and $\mathrm{GC}(\underline{F}')$ is a face it, then $F_{k-1}' \supseteq F_{k-1}$. Hence, F_{k-1}' is also not a point, that is, $\mathrm{GC}(\underline{F}') \in \Sigma(f)$. This shows that $\Sigma(f)$ is a polyhedral complex.

Moreover, let us see that the support of $\Sigma(f)$ is $V(f)$. If $x \in V(f)$ then we can consider

$$\underline{F}(x) : \mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x^{k-1}(f)) \subseteq \cdots \subseteq \mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x^0(f))$$

and, as the minimum in $f(x)$ is attained at least twice, we have that $\operatorname{conv}(\operatorname{Supp}(\operatorname{in}_x^{k-1}(f))$ is not a point. Hence,

$$x \in \mathrm{GC}(\underline{F}(x)) \subseteq |\Sigma(f)|,$$

and we conclude that $V(f) \subseteq |\Sigma(f)|$. On the other hand, if $x \in |\Sigma(f)|$ then there is a face $\underline{F} \in \underline{\Delta}(f)$ such that $x \in \mathrm{GC}(\underline{F})$. Hence,

$$\operatorname{Supp}\left(\operatorname{in}_x^{k-1}(f)\right) \supseteq F_{k-1},$$

and as F_{k-1} is not a point, the minimum in $f(x)$ is attained at least twice, so $x \in V(f)$. $\square$

Sea $(X, \mathcal{F})$ un espacio de medida, si $f = \dfrac{d\nu}{d\lambda}$ entonces $\dfrac{d\nu}{d\mu} = \dfrac{f}{1-f}$.

Solución:

$$\frac{d\nu}{d\mu} = \frac{f}{1-f} \tag{1}$$

$$(1-f)d\nu = fd\mu \tag{2}$$

$$d\nu - fd\nu = fd\mu \tag{3}$$

$$d\nu = f(d\nu + d\mu) \tag{4}$$

$$d\nu = fd\lambda \tag{5}$$

$$\frac{d\nu}{d\lambda} = f \tag{6}$$

Hagamos esto preciso. Si $f = \dfrac{d\nu}{d\lambda}$ entonces por el teorema de Radon-Nikodym, para toda función $h \in L^1(X)$

$$\int hd\nu = \int h \cdot fd\lambda$$

Pero además

$$\int hfd\lambda = \int hfd\nu + \int hfd\mu \tag{$\star$}$$

La demostración de $(\star)$ es como sigue: Primero lo probamos para indicatrices de conjuntos. Luego para combinaciones de indicatrices de conjuntos (funciones simples). Luego para limite creciente de indicaciones de funciones simples (convergencia monotona). Por lo tanto es cierto para toda funcion medible y positiva. Luego tomando diferencias es cierto para toda función en $L^1(X)$.

Así tenemos que,

$$\int hd\nu = \int hfd\nu + \int hfd\mu$$

$$\int hd\nu - \int hfd\nu = \int hfd\mu$$

$$\int h(1-f)d\nu = \int hfd\mu$$

Tomando $h = \dfrac{g}{(1-f)}$

$$\int gd\nu = \int g\frac{f}{1-f}d\mu$$

Y así, por la unicidad de la derivada de Radon-Nikodym tenemos que

$$\frac{f}{1-f} = \frac{d\nu}{d\mu}.$$

asd

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