Weak continuity on the variation of Newton-Okounkov bodies

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Introduction: Flag valuations and Okounkov Bodies

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where X_i is a Cartier divisor on X_{i+1} we consider

$$\begin{aligned} \nu_{\mathcal{F}} \colon K(X) &\longrightarrow \mathbb{R}^d \\ f &\longmapsto (\mathrm{ord}_{X_{d-1}}(f), \mathrm{ord}_{X_{d-2}}(\tilde{f}_1), \ldots, \mathrm{ord}_{X_0}(\tilde{f}_d)). \end{aligned}$$

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Moreover, if we have a big divisor $E \subseteq X$ we can define

$$\Delta_{\mathcal{F}}(E) = \overline{\bigcup_{n \geq 1} \left\{ \frac{\nu(f)}{n} \,\middle|\, f \in H^0(nE) \setminus \{0\} \right\}}.$$

This is the Okounkov Body of E with respect to $\mathcal{F}$.

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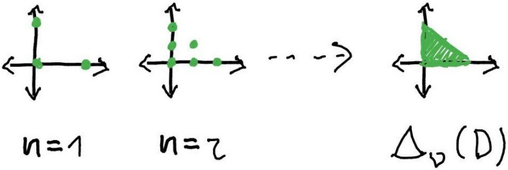
Example: $X = \mathbb{P}^2$ with coordinates $[x : y : z]$ and $E = V(z)$. Then $H^0(nE)$ corresponds to

$$\{p(x, y) \in \mathbb{C}[x, y] \mid \deg(p) \leq n\}$$

If $\mathcal{F}$ corresponds to the x axis followed by the origin, then



At each step we compute all the values of $H^0(nD)$ and normalize. The Okoukov body corresponds to the closure of the union of these points.

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For this kind of questions it is convenient to generalize the definition of Okounkov Body to an arbitrary valuation ν with values in $\mathbb{R}^d$.

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   Kiumars Kaveh, Christopher Manon

2. Understand the continuity: Mutations

   Newton-Okounkov bodies sprouting on the valuative tree

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3. Find the hidden information: Canonical Measures

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In the following we will describe some developments in the direction of questions 1 and 2.

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- Berkovich Analytification
- The valuative tree

Monomial Valuations

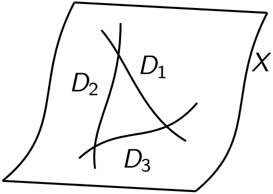
Consider $X = \mathbb{A}^2 = \mathrm{Spec}\ \mathbb{C}[x, y]$ and fix $\alpha, \beta \in \mathbb{R}_{\geq 0}$. Then, the map

$$\begin{aligned} \nu_{\alpha,\beta} \colon \mathbb{C}[x, y] &\longrightarrow \mathbb{R} \\ \sum_{i,j\geq 0} a_{ij} x^i y^j &\longmapsto \min\{i\alpha + j\beta \mid a_{ij} \neq 0\} \end{aligned}$$

extends to a valuation in $K(X)$. We cal this a Monomial valuation.

Quasi-monomial valuations

Let X be a variety and $D = \sum_{i=1}^{r} D_i$ a SNC divisor.



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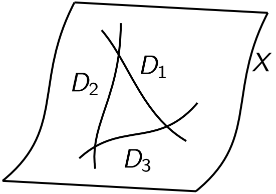

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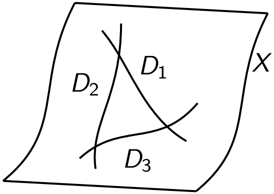
We can take local equations for each D_i to define monomial-like valuations.



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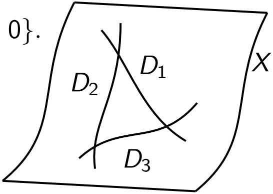
Example: If X is a surface, $D_1 = V(x)$ and $D_2 = V(y)$ locally around their intersection p. Then, by the transversality

$$\widehat{\mathcal{O}}_{X,p} \cong \mathbb{C}[[x, y]].$$

So, given $\alpha, \beta \in \mathbb{R}_{\geq 0}$ we can consider $\nu_{\alpha,\beta}$ defined by

$$\nu_{\alpha,\beta}\left(\sum_{ij} a_{ij} x^i y^j\right) := \min\{i\alpha + j\beta \mid a_{ij} \neq 0\}.$$



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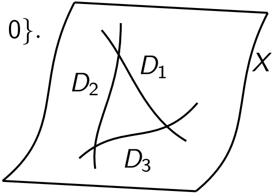
$\mathscr{M}(D) :=$ set of all quasi-monomial valuations w.r.t D.



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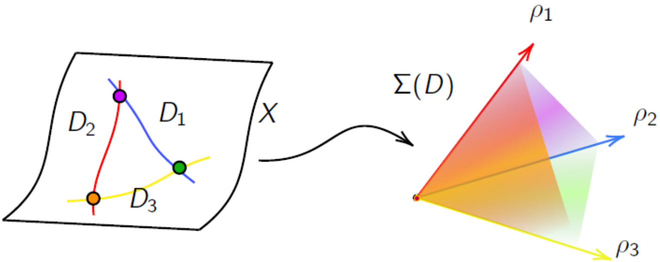

Monomial Valuations Geometrically

Putting coordinates on each cone of $\Sigma(D)$ gives the equality

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So geometrically, the monomial valuations defined in terms of D with weights in $\mathbb{R}_{\geq 0}$ are parametrized by the dual complex!

Now, by changing the weights to $(\mathbb{R}^k)_{\geq_{\mathrm{lex}} 0}$ with its lexicographic order, we can construct monomial valuations of higher rank. Let us denote by $\mathscr{M}^k(D)$ the set of all such valuations.

→ Is there any way to relate $\mathscr{M}^k(D)$ geometrically to the dual cone complex $\Sigma(D)$?

Answer: Yes, they are given by tangent directions in $\Sigma(D)$.

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In terms of coordinates, if $f = \sum_{ij} a_{ij} x^i y^j$ then

$$\mathrm{trop}(f)(x, y) = \min_{i,j} \{ix + jy \mid a_{ij} \neq 0\}.$$

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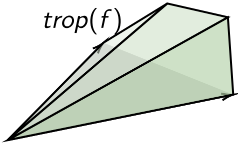


The map $\mathrm{trop}(f)$ is continuous, rational and piecewise linear.

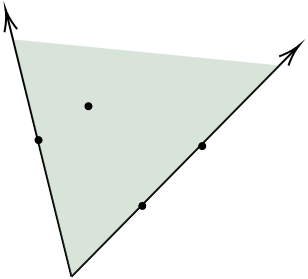
Theorem 1. Approximation Theorem (Amini - I '21)

Any continuous, rational, piecewise linear function in $\Sigma(D)$ is the tropicalization of some rational function.

Tangent Cone Bundles

Given a polyhedral complex Σ, we define its tangent cone bundle of order k, denoted by $\mathcal{TC}^k\,\Sigma$ as the set of all elements $(x; w_1, \ldots, w_k)$ such that:

1. $x \in \Sigma$
2.
3.
4.



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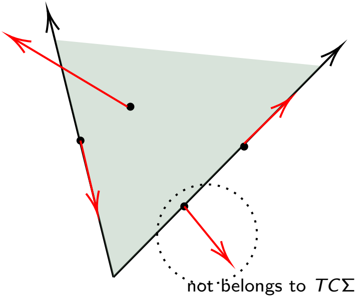

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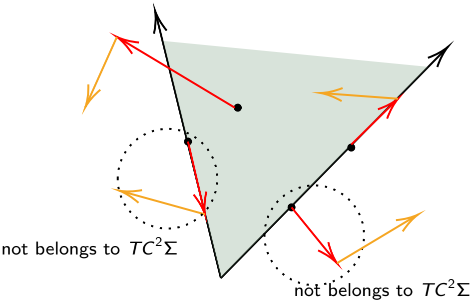
3. $x + \varepsilon_1 w_1 + \varepsilon_2 w_2 \in \Sigma$ for $\varepsilon_1 > 0$ small and $\varepsilon_2 > 0$ small w.r.t ε_2.
4. ....



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$$D_{w_1,w_2}F(x) = \lim_{h\to 0}\frac{D_{w_1+hw_2}F(x)-D_{w_2}F(x)}{h}$$

$$\vdots$$

$$D_{w_1,\dots,w_k}F(x) = \lim_{h\to 0}\frac{D_{w_1,\dots,w_{k-1}+hw_k}F(x)-D_{w_1,\dots,w_{k-1}}F(x)}{h}$$

Duality Theorem

Theorem 2, Duality Theorem (Amini - I '21)

There is an isomorphism of bundles over $\mathscr{M}(D) \simeq \Sigma(D)$

$$\begin{array}{ccc} \mathscr{M}^k(D) & \xrightarrow{\simeq} & T\mathcal{C}^{k-1}\,\Sigma(D) \\ \downarrow & & \downarrow \\ \mathscr{M}(D) & \xrightarrow{\simeq} & \Sigma(D) \end{array}$$

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which is a monomial valuation with respect to the weights

$$(x;\underline{w})^T = (\alpha_1, \ldots, \alpha_k) \in (\mathbb{R}^k)_{\geq_{\mathrm{lex}} 0}.$$

Topology on $TC^{k-1}\Sigma(D)$

If we come back to the original problem, the map

$$\begin{aligned} \Delta_\bullet : TC^{k-1}\Sigma(D) &\longrightarrow \mathrm{BC}(\mathbb{R}^k) \\ (x;\underline{w}) &\longmapsto \Delta_{\nu_{(x;\underline{w})}} \end{aligned}$$

is not continuous with the euclidean topology. This is understandable because the maps $f \longmapsto \nu_{(x;\underline{w})}(f)$ are not continuous.

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We introduce the tropical topology as the weakest topology that makes all the maps

$$\begin{aligned}T\mathcal{C}^{k-1}\,\Sigma(D) &\longrightarrow \mathbb{R}^k\\ (x,\underline{w}) &\longmapsto \nu_{(x;\underline{w})}(f)\end{aligned}$$

continuous, when $\mathbb{R}^k$ is endowed with its euclidean topology.

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- It is not locally compact nor second countable. It is separable and first countable.
- A set is dense in the tropical topology if and only if it is dense in the euclidean topology.
- The local basis of neighborhoods of an irrational point is the same in the tropical and euclidean topology.

Variation of Okounkov Bodies

Theorem 3 I. Amini

The map

$$\begin{aligned}\Delta\colon \mathring{T\mathcal{C}}^{d-1}\Sigma(D) &\longrightarrow \mathrm{BC}(\mathbb{R}^d)\\ \nu &\longmapsto \Delta_\nu := \overline{\bigcup_{n\geq 0}\left\{\frac{\nu(f)}{n} \mid f\in H(nE)\setminus\{0\}\right\}}\end{aligned}$$

which attaches to each valuation its corresponding Newton-Okounkov body is continuous with respect to the tropical topology.

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$$\Delta_{(1:s),(0,1)} = \begin{cases} \text{conv-hull}\big((0,0),(1,0),(s,1)\big) & \text{if } 1 \le s < 2 \\ \text{conv-hull}\big((0,0),(2,0),(s/2,1/2)\big) & \text{if } 2 \le s < 5 \\ \text{conv-hull}\big((0,0),(5/2,0),(2s/5,2/5)\big) & \text{if } 5 \le s < 6 + \frac{1}{4} \end{cases}$$

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Our theorem explains why this map is continuous on irrational points, and why the discontinuities are only from one side. All this follows from the embedding of the line $[1, 6 + \frac{1}{4})$ on $\mathring{T\mathcal{C}}^{d-1}\Sigma(D)$ and noticing that the induced topology is generated by intervals of the form $[a, b)$ with $a, b \in \mathbb{Q}$.