Higher rank tropical geometry and the variation of the demand

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Joint work with Jaime Tobar

The University of Texas at Austin

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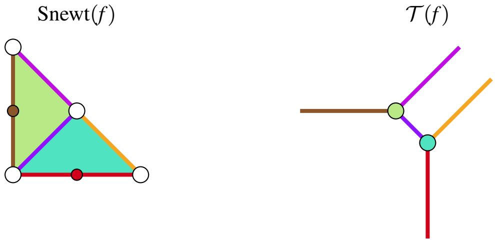
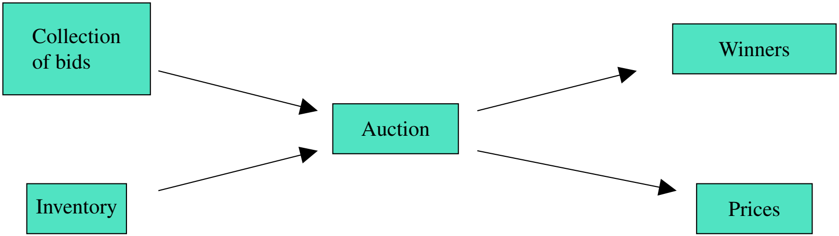
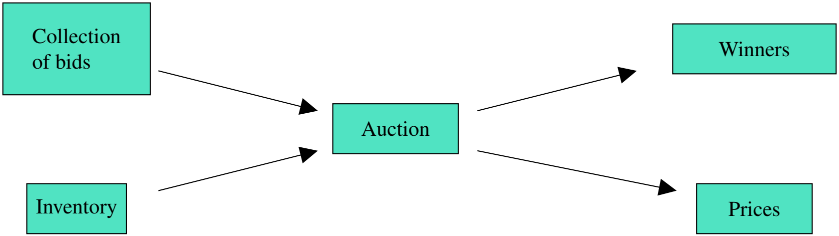



Collection of bids
Inventory
Auction
Winners
Prices

Introduction



Slogan: The input data in the diagram above is tropical geometry data in a natural way. Therefore, the auction process should depend on tropical geometry as well.

Elizabeth Baldwin y Paul Klemperer (2019). *Understanding preferences:"demand types", and the existence of equilibrium with indivisibilities.* Econometrica, 87(3), 867-932.

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The function f_u is an instance of an n-th variable *tropical polynomial*.

Tropical Geometry

The tropical semiring is

$$\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$$

with the *tropical sum* $\oplus$ and *tropical multiplication* $\odot$ given by

$$a \oplus b := \max(a, b) \quad , \quad a \odot b := a + b. \quad \forall a, b \in \mathbb{T}.$$

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A tropical Laurent polynomial is an expression of the form

$$f_u = \bigoplus_{a \in A} u_a \odot x^a$$

for some $A \subseteq \mathbb{Z}^n$. Written in a different form

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A tropical zero of f_u is an element $x \in \mathbb{R}^n$ for which f_u achieves the maximum at least twice. The set of all tropical zeros is called a tropical hypersurface.

Tropical Hypersurfaces

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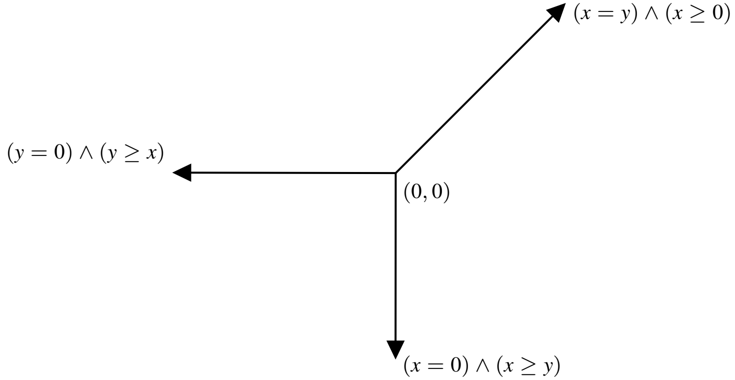
$$\{(x = y) \wedge (x \geq 0)\} \cup \{(x = 0) \wedge (x \geq y)\} \cup \{(y = 0) \wedge (y \geq x)\}.$$

which gives the tropical hypersurface $\mathcal{T}(p_1)$



Tropical Hypersurfaces

Using **polymake**:

$$p_2(x,y) = 2 \oplus 4x \oplus 3x^2 \oplus 5x^3 \oplus 7x^4 \oplus 6x^5 \oplus 8x^6 \oplus 10x^7 \oplus 8x^8 \oplus 9y$$
$$\oplus 10y^2 \oplus 8y^3 \oplus 7y^4 \oplus 5y^5 \oplus 6y^6 \oplus 3y^7 \oplus 4y^8 \oplus 15x^4y^4$$

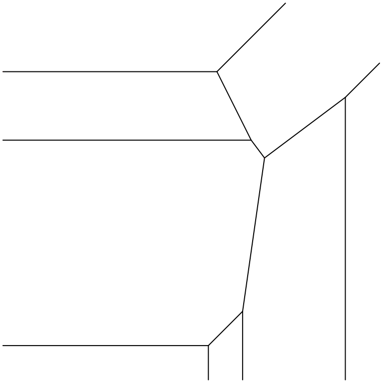
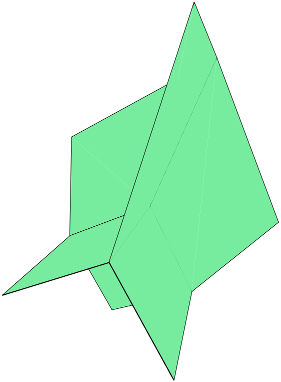


Figure 1: $\mathcal{T}(p_2)$

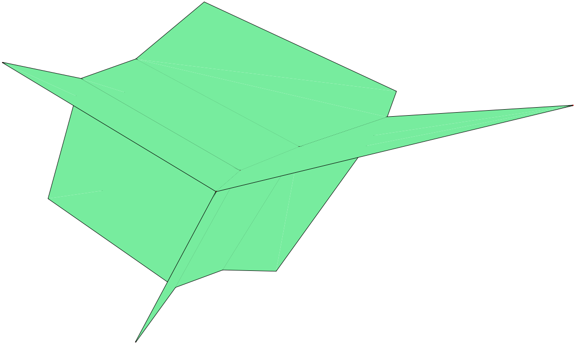
Tropical Hypersurfaces

$$p_3(x,y,z) = 1 \oplus 1x \oplus 1y \oplus 1z$$

$$p_4(x,y,z) = 6 \oplus 5x \oplus 4y \oplus 3z \oplus 3x^2 \oplus 2y^2 \oplus 1z^2$$



(a) $\mathcal{T}(p_3)$



(b) $\mathcal{T}(p_4)$

Back to Auctions

Given a market with n different types of goods and a set $A \subseteq \mathbb{Z}^n$ of bundles of goods. An agent has a valuation $u : A \to \mathbb{R}$ which defines a utility function

$$f_u(p) = \max_{x \in A} \left(u_j(x) - \langle x, p \rangle \right).$$

- The *demand* of the agent for a given price p is the set $D_u(p) = \operatorname{argmax}_{x \in A}(f_u(p))$.

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2. Surprisingly, the demands of the agent also fit into a polyhedral complex: The *regular subdivision* of the Newton polytope of f_u.
3. Moreover, the demand complex is dual to the tropical hypersurface.

Regular subdivision

$$p(x,y) = 1 \oplus 1x \oplus 1x^2 \oplus 1x^3 \oplus 1y \oplus 2yx \oplus 2yx^2 \oplus 1yx^3 \oplus 1y^2 \oplus$$
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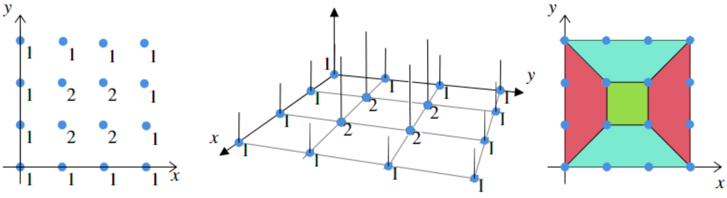

(a) (b) (c)

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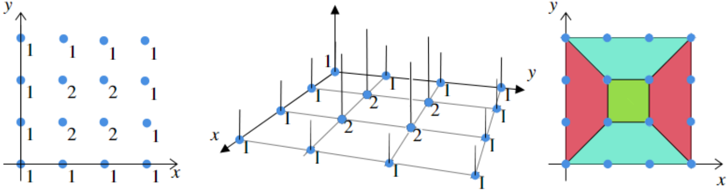

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(b)

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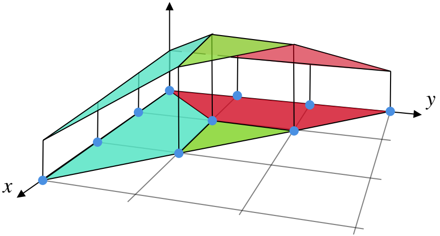
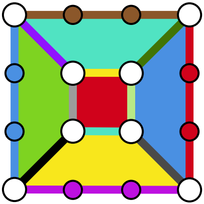
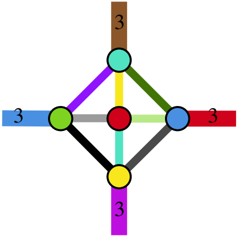


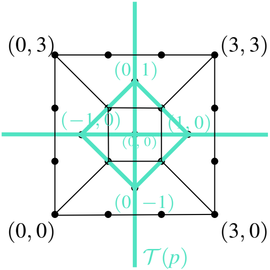
Hypersurface Duality



Newt(p)



$\mathcal{T}(p)$



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Other big advantage of the tropical geometry setting is that it handles easily multiple agents.

Definition

Given a family of agents J, their **aggregate demand** at a price $p \in \mathbb{R}^n$ is the Mikowski sum

$$D_{u_J}(p) := \sum_{j \in J} D_{u_j}(p).$$

The aggregate demand coincides to the demand of a fictional *aggregate agent*. The utility function of this aggregate agent will be obtained as a product of the polynomials

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Remark: Notice that an auction will be successful exactly if a competitive equilibrium exists.

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Theorem (Hemicontinuity Theorem)

Given $A \subseteq Z^n$, the map

$$\begin{aligned} D : \mathrm{Val}(A) \times \mathbb{R}^n &\to \mathcal{P}(\mathbb{Z}^n) \\ (u, p) &\mapsto D_u(p) \end{aligned}$$

satisfy that for each $u \in \mathrm{Val}(A)$ and $p \in \mathbb{R}^n$ there exists an open neighborhood of $V \subseteq \mathrm{Val}(A) \times \mathbb{R}^n$ of (u, p) such that $\forall (u', p') \in V$

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Do we have a way to understand how is the change exactly?

Tropical Geometry of Higher Rank

The *tropical semiring of rank k* $\mathbb{T}_k = (\mathbb{R}^k \cup \{-\infty\}, \oplus, \odot)$ is the semiring over $\mathbb{R}^k$ in which $\odot$ is the addition and $\oplus$ the lexicographic order

$$(a^{(1)}, \ldots, a^{(k)}) \prec (b^{(1)}, \ldots, b^{(k)}) \iff a_i < b_i \text{ for the minimum } i \text{ such that } a_i \neq b_i$$

Elements of $a \in \mathbb{T}_k$ should be thought as

$$a^{(1)} + \varepsilon a^{(2)} + \ldots + \varepsilon^{(k-1)} a^{(k)}$$

where ε is a very small but positive element.

We can introduce tropical polynomials $f_u = \bigoplus_{a \in A} u_a \odot x^a$ and tropical hypersurfaces $\mathcal{T}(f_u) \subseteq \mathbb{T}_k^n$ in the same way as we did before.

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How do we visualize $\mathcal{T}(f_u)$?

Iterated Fibrations

There are natural projections maps

$$\begin{aligned} \pi_r \colon \mathbb{T}_k &\longrightarrow \mathbb{T}_r \\ a &\longmapsto a^{[r]} := (a^{(1)}, \dots, a^{(r)}) \end{aligned}$$

This projection maps extend to maps elements in $\mathbb{T}_k^n$ and to polynomials.
Then, for any Laurent polynomial f we have

$$\mathcal{T}(f^{[r]}) = \mathcal{T}(f)^{[r]}.$$

Which gives us a sequence of projections

$$\mathcal{T}(f^{[r]}) \xrightarrow{\pi_{k-1}} \mathcal{T}(f^{[r-1]}) \xrightarrow{\pi_{k-2}} \dots \xrightarrow{\pi_1} \mathcal{T}(f^{[1]})$$

The base of this fibration is a tropical hypersurface of rank 1, and all the fibers of points are tropical hypersurfaces of rank 1. Moreover, the hypersurface duality generalize to this context.

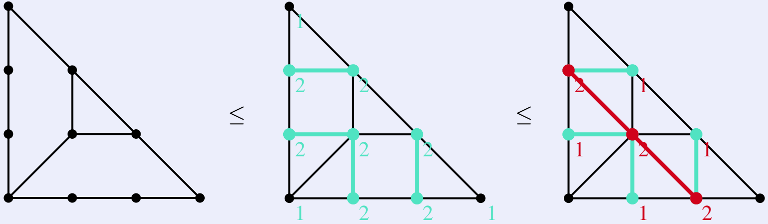
Consider the set

$$A = \{(0,0),\ (1,0),\ (2,0),\ (3,0),\ (0,1),\ (0,2),(0,3),(1,2),(2,1),(1,1)\}$$

In this case, the layered regular subdivision induced by the map $u : A \to \mathbb{T}_3$

$$\begin{aligned} u(0,0) &= 1 + 1\varepsilon + 1\varepsilon^2 \\ u(1,0) &= 1 + 2\varepsilon + 1\varepsilon^2 \\ u(2,0) &= 1 + 2\varepsilon + 2\varepsilon^2 \\ u(3,0) &= 1 + 1\varepsilon + 1\varepsilon^2 \end{aligned} \qquad \begin{aligned} u(0,1) &= 1 + 2\varepsilon + 1\varepsilon^2 \\ u(0,2) &= 1 + 2\varepsilon + 2\varepsilon^2 \\ u(0,3) &= 1 + 1\varepsilon + 1\varepsilon^2 \\ u(1,2) &= 2 + 2\varepsilon + 1\varepsilon^2 \end{aligned} \qquad \begin{aligned} u(2,1) &= 2 + 2\varepsilon + 1\varepsilon^2 \\ u(1,1) &= 2 + 2\varepsilon + 2\varepsilon^2. \end{aligned}$$

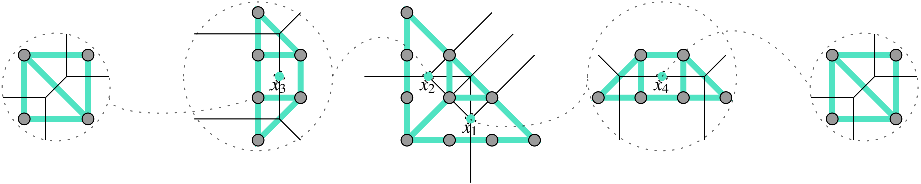
Will be the following:



Higher Rank Hypersurface Duality

Theorem (Hypersurface Duality)

Given a higher rank tropical polynomial and its corresponding layered fibration of its Newton polytope, there is a way to read from this subdivisions the combinatorial structure of the iterated fibration in its tropical hypersurface.



$\mathcal{T}_{x_2,x_3}\left(f_u^{[3]}\right)$ $\mathcal{T}_{x_2}\left(f_u^{[2]}\right)$ $\mathcal{T}\left(f_u^{[1]}\right)$ $\mathcal{T}_{x_1}\left(f_u^{[2]}\right)$ $\mathcal{T}_{x_1,x_4}\left(f_u^{[3]}\right)$

Part of the usefulness of this framework is that it mixes two perspectives.

1. On one hand, the elements of $\mathbb{T}_k$ are rigid, and this allow us to draw the diagrams that generalize the ideas from $\mathbb{T}_1$.
2. On the other hand, given an element

$$x^{(1)} + \varepsilon x^{(2)} + ... + \varepsilon^{(k-1)} x^{(k)} = x \in \mathbb{T}_k$$

   we can replace ε by a concrete small real number, giving rise to a perturbation of the element $x^{(1)}$. More generally, “finitelly generated” objects $X/\mathbb{T}_k$ should give rise to perturbations X_ε in this way.

As working with perturbations is generally a difficult thing (What is the perturbation of the demand $D_u(p)$ as u changes?), the formal point of view of working directly in $\mathbb{T}_k$ simplify the study.

Perturbation of the Demand

Theorem (Demand)

Consider a map $u : A \subseteq \mathbb{Z}^n \to \mathbb{R}_k$. Then, for $\delta > 0$ a small real number, the demand

$$D_{u^{(1)}+\delta u^{(2)}+\cdots+\delta^{k-1}u^{(k)}}(p^{(1)} + \delta p^{(2)} + \cdots + \delta^{k-1} p^{(k)}).$$

coincides with the corresponding cell in the layered subdivision dual to the cell in the tropical hypersurface containing $p = p^{(1)} + \varepsilon p^{(2)} + \cdots + \varepsilon^{k-1} p^{(k)}$

Theorem (Perturbation of Competitive Equilibria)

Consider a family of agents, each with a valuations which has been perturbed by functions $\{u^j : A \subseteq \mathbb{Z}^n \to \mathbb{D}_k\}_{j \in J}$. This family posses a competitive equilibrium for $x \in A$ for each $\delta > 0$ small iff the corresponding valuations have *formally* a competitive equilibrium over $\mathbb{T}_k$.