MASTER DE SCIENCES ET TECHNOLOGIE
MENTION MATHÉMATIQUES FONDAMENTALES

Mémoire de Master 2

Toric Varieties and Tropical Geometry

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Année académique 2017/18

Acknowledgements

First and foremost, I am deeply grateful to Professor Omid Amini, the director of this memoire, for his commitment, support, and patience throughout this period. In adittion, I would like to thank Professor Marco Maculan for the valuable help in both the academic and the administrative sides.

I also thank all the members of the Institut de mathématiques de Jussieu for giving me all the means to acquire a solid training, as well as the Fondation Sciences Mathématiques de Paris for giving me the oportunity and financial support to study here in Paris.

Finally, I particularly thank my colleagues, friends and family for their solidarity and encouragement.

Abstract

In the last decades, there has been unexpected interactions between Algebraic geometry and other fields of mathematics. In particular, there are interesting connection with *polyhedral geometry*, the area of mathematics concerning the study of polyhedra, polytopes and triangulations among others.

One may see the above connection specially in two areas of Algebraic geometry: *Toric varieties* and *Tropical geometry*. In the context of Toric varieties, one uses polyhedral objects to construct interesting and tractable geometrical structures. On the other hand, in Tropical geometry one tries to understand geometric structures by attaching to them some simplified polyhedral version.

The objective of this *Mémoire de Master* is to give an overview of these two areas of algebraic geometry and to show some applications of them.

In Chapter 1 the main theory of Toric varieties is developed. We begin studying affine toric varieties and its correspondence with polyhedral cones. Afterwards, by gluing upon this case one obtain the general correspondence between normal toric varieties and *fans* (certain collection of cones). We study sheaves over these algebraic varieties and construct a resolution of singularities for them, which preserves its natural structures.

In Chapter 2 we prove the characteristic 0 semi-stable reduction theorem for surjective morphisms $f : X \to C$, where C is a smooth curve. For this we need to introduce the general concept of Toroidal variety which we developed during the first two sections.

In Chapter 3 we introduce Tropical geometry. We start this by defining the tropical semiring $\mathbb{T}$ and introducing tropical hypersurfaces as zeros of polynomials on it. Then we move to general tropical varieties by defining them as the sets obtained as *tropicalization* of algebraic varieties. Then we prove that these sets have a structure of weighted polyhedral complex satisfying the balancing condition, this is the Structure Theorem for Tropical varieties.

Contents

Chapter 1

Toric Varieties

The purpose of this chapter is to give a brief overview of the basics on toric varieties, also known as toroidal embeddings. This will be used in Chapter 2 to deduce the semistable reduction theorem of Deligne-Mumford, and in Chapter 3 to introduce the ideas of Tropical Geometry. We mainly follow [KKMSD73] for this chapter.

Throughout our discussion we fixed an algebraically closed field k. A variety is an integral scheme of finite type over k.

1.1 Facts about Algebraic Tori

We denote by $\mathbb{G}_m$ the algebraic group over k whose k-points are k^*. An algebraic torus T is an algebraic group isomorphic to $\mathbb{G}_m^n = \underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{n \text{ times}}$ for some $n \geq 1$. Sometimes we write T_k^n to indicate the dimension and the field of definition of T.

As $T^1 = \mathbb{G}_m = \mathbb{A}^1 \setminus \{0\}$, using coordinates in $\mathbb{A}^1$ we see that $\Gamma(T^1) = k\big[x, x^{-1}\big]$ and more generally

$$\Gamma(T^n) = k\big[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}\big]$$

This is the ring of Laurent polynomials in n variables, i.e., expressions of the form $f = \sum_{\alpha \in I} c_\alpha x^\alpha$ for some finite subset I of $\mathbb{Z}^n$, but it can also be seen as the group algebra of the group $\mathbb{Z}$.

For a given algebraic group T we define its group of characters and its group of one-parameters subgroups respectively as

$$M = \mathrm{Hom}_{\text{alg. groups}}(T, \mathbb{G}_m) \text{ and } N = \mathrm{Hom}_{\text{alg. groups}}(\mathbb{G}_m, T)$$

In order to understand these groups we need the following result.

Proposition 1.1.1. *The only endomorphisms of $\mathbb{G}_m$ are the n-powers, i.e,*

$$Hom_{alg.groups}(\mathbb{G}_m, \mathbb{G}_m) = \{x \mapsto x^n\}_{n \in \mathbb{Z}}$$

.

Proof. It is easy to see that

$$\mathrm{Hom}_{\text{alg.groups}}(\mathbb{G}_m, \mathbb{G}_m) \supseteq \{x \mapsto x^n\}_{n \in \mathbb{Z}}$$

On the other hand, we have

$$\mathrm{Hom}_{\text{alg.groups}}(\mathbb{G}_m, \mathbb{G}_m) \subseteq \mathrm{Hom}_{\text{alg.var}}(\mathbb{G}_m, \mathbb{G}_m) \overset{\text{bij}}{=} \mathrm{Hom}_{k-\text{alg}}\big(k\big[x, x^{-1}\big], k\big[x, x^{-1}\big]\big) = k\big[x, x^{-1}\big]^\times = \bigcup_{n \in \mathbb{Z}} kx^n$$

So $\mathrm{Hom}_{\text{alg.groups}}(\mathbb{G}_m, \mathbb{G}_m) \subseteq \{x \mapsto cx^n\}_{n \in \mathbb{Z}, c \in k^*}$. Now, by linearly independence of characters we have that $\mathrm{Hom}_{\text{alg.groups}}(\mathbb{G}_m, \mathbb{G}_m)$ is a linearly independent subset. Thus $x \mapsto cx^n$ and $x \mapsto x^n$

cannot be simultaneously endomorphisms of $\mathbb{G}_m$. Hence only $x \mapsto x^n$ is an endomorphism. $\square$

As Hom commutes with finite direct sums in both entries, we get that $M \simeq N \simeq \mathbb{Z}^n$ so M and N are lattices. More concretely we have

$$\begin{aligned}
\mathrm{Hom}_{\text{alg. groups}}(T^n, \mathbb{G}_m) &= \{(x_1, \ldots, x_n) \mapsto x_1^{a_1} \cdots x_n^{a_n} \mid (a_1, \ldots, a_n) \in \mathbb{Z}^n\} \\
\mathrm{Hom}_{\text{alg. groups}}(\mathbb{G}_m, T^n) &= \{x \to (x^{a_1}, \ldots, x^{a_n}) \mid (a_1, \ldots, a_n) \in \mathbb{Z}^n\}
\end{aligned}$$

From now on we will consider M, N just as lattices. For $r \in M$ and $a \in N$ we will denote by χ^r and λ_a the corresponding homomorphisms.

Using the previous notation we get $\Gamma(T) = k[\{\chi^r\}_{r \in M}]$. In other words, the coordinate ring of a torus is the group algebra of its character lattice M.

Moreover we have a canonical perfect pairing $M \times N \to \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$ giving by composition. In concrete terms this map attaches to each pair (r, a) the integer $\langle r, a \rangle$ defined by

$$\chi^r \circ \lambda_a(t) = t^{\langle r, a \rangle} \quad \forall t \in k^\times$$

We use this pairing to identify M as the dual of N.

1.2 Affine Toric Varieties

Definition 1.2.1. A toric variety is an algebraic variety X containing a torus T as an open subvariety together with an extension of the action $T \times T \to T$ to an action $T \times X \to X$ of the torus in the variety. A toric morphism between two toric varieties X and X' containing a copy of the same torus T is a morphism $f : X \to X'$ of algebraic varieties that is equivariant (i.e, $f(\alpha x) = \alpha f(x)$ $\forall \alpha \in T, x \in X$) and fix the torus (i.e, $f(1) = 1$). //

Notice that since the variety X is irreducible (recall it is an integral scheme) the torus T is dense in X.

Example 1.2.2. $\mathbb{P}^n$, $\mathbb{A}^n$ and the nodal cubic $V(x^3 - y^2) \subseteq \mathbb{A}^2$ are examples of toric varieties where the torus T is $\{(x_0 : \cdots : x_n) \mid x_i \neq 0\ \forall i\}$, $(k^*)^n$ and $\{(t^2, t^3) \mid t \in k^*\}$ respectively. //

The first objective is to understand affine toric varieties in terms of their coordinate rings. We start by translating the action of a torus on an arbitrary affine variety.

Proposition 1.2.3. *Given a variety $X = \operatorname{Spec} B$, the data given by an action of T on X is equivalent to a graduation of B of type M. Under this correspondence equivariant morphisms f translate to graded morphism of algebras φ.*

Proof. Given the action, we define $B_r = \{f \in B \mid f_r(\alpha x) = \chi^r(\alpha) f(x),\ \forall \alpha \in T, \forall x \in X\}$ for the graduation and given the graduation we define the action as the map induced by the morphism $B \to k[M] \otimes B$ given by $f \mapsto \sum \chi^r \otimes f_r$.

Now if $f : X \to X'$ is an equivariant morphism we have $f(\alpha \cdot x) = \alpha \cdot f(x)$ $\forall \alpha \in T$, so for $h \in B_r$ homogeneous we have

$$\begin{aligned}
\varphi(h)(\alpha x) &= h(f(\alpha x)) \\
&= h(\alpha f(x)) \\
&= \chi^r(\alpha) h(f(x)) \\
&= \chi^r(\alpha) \varphi(h)(x)
\end{aligned}$$

And hence φ is a homogeneous morphism. Similarly if φ is homogeneous from $\varphi(h)(\alpha x) = \chi^r(\alpha)\varphi(h)(x)$ we get $h(f(\alpha x)) = h(\alpha f(x))$ whenever $h \in B_r$ and then by linearity for every $h \in B$, hence $f(\alpha x) = \alpha f(x)$. $\square$

Now let's use this to describe all affine toric varieties in terms of subgroups of the group of characters M:

If $T \hookrightarrow X$ is a toric variety with $X = \operatorname{Spec} B$ we have a map between k-algebras $\phi : B \to k[M]$. By the proposition above ϕ is a morphism of graded algebras and, as the inclusion of the torus is dominant, ϕ is also injective. Therefore we can consider B as a subgraded algebra of $k[M] = \bigoplus_{r\in M} \chi^r k$. Then it must be of the form $k[S]$ for some finitely generated sub-semigroup $S \subseteq M$ and also as the function field of T is equal to the function field of X we must have $\text{Frac } k[S] = \text{Frac } k[M]$, or in other words, S generate M as a group.

Conversely take a finitely generated sub-semigroup $S \subseteq M$ such that S generate M as a group. Then $k[S] \subseteq k[M]$ and $k[M]$ is a localization of $k[S]$ with respect to finitely many elements. Hence $T = \operatorname{Spec} k[M]$ is an open subset of the affine variety $X = \operatorname{Spec} k[S]$ and as the inclusion $k[S] \hookrightarrow k[M]$ is a graded morphism, X is a toric variety.

Writing down what we have got:

Theorem 1.2.4. *There is a correspondence between finitely generated sub-semigroups S of M that generate M as a group and isomorphism classes of affine toric varieties of T. In concrete terms it is given by $S \mapsto \operatorname{Spec} k[S]$. Moreover, this induce a order reversing correspondence between toric morphisms and inclusions of semigroups.*

Proof. Only the morphism correspondence is new.

If $f : X_1 \to X_2$ is a toric morphism between $X = \operatorname{Spec} k[S_1]$ and $X' = \operatorname{Spec} k[S_2]$ it induce a graded morphism between algebras $\varphi : k[S_2] \to k[S_1]$ that is injective (because f is dominant) and that extend to the identity in $k[M]$ (because $f|_T = \mathrm{Id}_T$) so φ is the inclusion $S_2 \subseteq S_1$.

In the other hand if we have $S_2 \subseteq S_1$ then $k[S_2] \subseteq k[S_1]$ and as this inclusion is a graded morphism we get a toric morphism $X_1 \to X_2$ that fix pointwise the torus because the map between k-algebras extend to the identity on $k[M]$. □

Remark 1.2.5. We notice here for future references that (k-valued) points of the variety $X = \operatorname{Spec} k[S]$ correspond to morphisms of varieties $\operatorname{Spec}(k) \to X$ and hence to k-algebra morphisms $k[S] \to k$ or equivalently, semigroup morphisms $S \to k$.

In concrete terms, the semigroup morphism $\gamma : S \to k$ corresponding to the point $x \in X$ is given by $\gamma(r) = \chi^r(x)\ \forall r \in S$. //

From now on we will be mainly interested in normal toric varieties. In this context the correspondence seen in the above section can be adapted in a more combinatorial way. For this we introduce the following.

Definition 1.2.6. A sub-semigroup $S \subseteq M$ is called *saturated* if for every integer $n \geq 1$ and every $r \in M$ the condition $nr \in S$ implies $r \in S$. //

We can translate this concept to the algebro-geometric side

Proposition 1.2.7. *The bijection in Theorem 5 makes a correspondence between saturated semigroups and normal affine toric varieties.*

Proof.: Suppose $\operatorname{Spec} k[S]$ is normal. Then $k[S]$ is integrally closed. Therefore, if $nr \in S$ for some $n \in \mathbb{N}, r \in S$ we have that χ^r satisfy the integral equation $x^n - \chi^{rn} = 0$ and hence $\chi^r \in k[S]$, i.e $r \in S$.

For the converse we suppose that S is satured and we need to prove that $k[S]$ is integrally closed. Since $k[S] \subseteq k[M]$ and the later is integrally closed and has the same fraction field, we just need to prove that if $f \in k[M]$ is integral over $k[S]$ then belongs to $k[S]$. Actually, as $k[S]$ is a graded algebra it is enough to work with f homogeneous (see [ZS60], p.158) so we have an element χ^r satisfying the equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0$$

with coefficients $a_i \in k[S]$. Each a_i is a sum of characters but we can throw out all the ones whose degree do not add up, i.e, $nr = r(n-i) + \deg\, a_i$. Then taking i with $a_i \neq 0$ we have $\deg\, a_i = ri \in S$ so by saturation $r \in S$ so $\chi^r \in k[S]$. □

Now we are going to visualize this saturated semigroups in a more combinatorial way as polyhedral cones sitting inside $M_{\mathbb{R}} = M \otimes \mathbb{R}$. For the notion of polyhedral cone we refer the reader to the appendix.

Lemma 1.2.8. *There is a correspondence between strictly convex, rational polyhedral cones σ in $N_{\mathbb{R}}$ and saturated finitely generated sub-semigroups of M given by $\sigma \mapsto \sigma^\vee \cap M$.*

Proof. Given a strictly convex, rational polyhedral cones σ in $N_{\mathbb{R}}$. The semigroup $\sigma^\vee \cap M$ is saturated (intersection of saturated sets) and generates M as a group, because $\sigma^\vee$ is not contained in any hyperplane of $M_{\mathbb{R}}$. Hence, we just have to prove that it is finitely generated. For this, as $\sigma^\vee$ is rational, we can take $x_1, \dots, x_r \in M$ such that $\text{cone}(x_1, \dots, x_r) = \sigma^\vee$. Now taking the bounded subset

$$K = \left\{ \sum_{i=1}^{r} \delta_i x_i \,\middle|\, 0 \leq \delta_i < M_{\mathbb{R}} \right\}$$

We have that $(K \cap M) \cup \{x_1, \dots, x_n\}$ is a finite set of generators for $\sigma^\vee \cap M$. Indeed, for any $x \in \sigma^\vee \cap M$ there are $\lambda_i \in \mathbb{Z}_{\geq 0}$ such that $x - \sum_i \lambda_i x_i \in K \cap M$.

Conversely, let $S \subseteq M$ be a saturated finitely generated sub-semigroup of M. If $x_1, \dots, x_n$ are generators for S then the cone σ, whose dual is given by $\{\sum_i x_i \lambda_i \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$, is a convex rational polyhedral cone in $N_{\mathbb{R}}$ not contained in any hyperplane and whose lattice points are S. ☐

Using this we can state the theorem that classifies affine normal toric varieties.

Theorem 1.2.9. *The correspondence $\sigma \mapsto \text{Spec}(\sigma^\vee \cap M)$ gives a bijection between the set of strictly convex rational polyhedral cones in $N_{\mathbb{R}}$ and the set of affine normal toric variaties containing T.*

Proof. This is a direct concatenation of Proposition 1.2.7 and Lemma 1.2.8 above. ☐

We will use the notation $S_\sigma := \sigma^\vee \cap M$ for the semigroup associated to the cone and $U_\sigma := \text{Spec}\, k[S_\sigma]$ for the toric variety associated.

Example 1.2.10. In Example 1.2.2 we saw the three toric variaties. Namely, $\mathbb{P}^n$, $\mathbb{A}^n$ and $V(x^3 - y^2) \subset \mathbb{A}^2$. As the projective space is not an affine variety and the nodal cubic is not a normal variety only the affine space is an example of a toric normal variety. Under the correspondence described above we see that $U_\sigma = \mathbb{A}^n$ where $\sigma = \text{cone}(e_1, \dots e_n) \subset N_{\mathbb{R}}$. In fact we have $\sigma^\vee = \text{cone}(e_1^*, \dots, e_n^*)$ so $S_\sigma = \sigma^\vee \cap M_{\mathbb{R}} \cong \mathbb{N}^n$ and then $U_\sigma = \text{Spec}\, k[S_\sigma] = \text{Spec}\, k[x_1, \dots x_n] = \mathbb{A}^n$ //

Now recall that even though a smooth variety is always normal, the inverse is not true in general. Hence there exists the possibility that the variety U_σ obtained with the construction above gives a nonsmooth variety. Fortunately there is a criterion to determine exactly when an affine toric variety is smooth in term of the associated cone.

Theorem 1.2.11. *U_σ is non singular $\iff$ σ is an smooth cone as defined in appendix A.*

Proof. Suppose σ is smooth. Then there is a basis $\{e_1, \dots, e_k, e_{k+1}, \dots, e_n\}$ of the lattice N such that $\sigma = \text{cone}(e_1, \dots, e_k) \subseteq N_{\mathbb{R}}$. Hence

$$\sigma^\vee = \text{cone}(e_1^*, \dots, e_k^*, e_{k+1}^{*\pm 1}, \dots, e_n^{*\pm 1}) \text{ so } U_\sigma = \text{Spec}\left(k[e_1^*, \dots, e_n^*]_{e_{k+1}^* \cdots e_n^*}\right) = k^n \times (k^*)^{n-r}$$

and this is a non singular variety.

For the other implication let σ be a cone such that U_σ is smooth. We are going to reduce the problem to the case $\dim(\sigma) = n$. If $\dim(\sigma) = r < n$ let N_1 be the smallest vector subspace of N containing σ. Then N/N_1 is a torsion free abelian group. Hence, there exists a vector subspace $N_2 \subseteq N$ such that $N_1 \oplus N_2 = N$. This decomposition induces a decomposition $M_1 \oplus M_2 = M$. Now, since we can see σ as a cone of N or of N_1, we can compute the two semigroups $S_{\sigma,N}$ and S_{σ,N_1}. These semigroups satisfy $S_{\sigma,N} = S_{\sigma,N_1} \oplus M_2$ and therefore we have $k[S_{\sigma,N}] = k[S_{\sigma,N_1}] \otimes k[M_2]$ from where $U_{\sigma,N} = U_{\sigma,N_1} \times (k^*)^{n-r}$. As we are assuming $U_{\sigma,N}$ smooth we can deduce that U_{σ,N_1} is smooth and therefore we can suppose σ is a cone of maximal dimension.

Now assuming σ has maximal dimension de dual $\sigma^\vee \subseteq M$ is strictly convex and hence no element of $k[S_\sigma]$ is invertible. Taking the maximal ideal I generated by $\{\chi^\alpha \mid \alpha \in S_\sigma \setminus \{0\}\}$ we see that it corresponde with a point $p \in U_\sigma$ (in fact it is the unique fixed point of the action), hence by the non singular assumption the ring R_I is a graded regular local ring of dimension n so there are $\chi^{\alpha_1}, \dots, \chi^{\alpha_n}$ such that $IB_I = (\chi^{\alpha_1}, \dots, \chi^{\alpha_n})B_I$ or in other words, for all $\beta \in S$

$$\chi^\beta = \sum_{i=1}^{n} a_i \chi^{\alpha_i} \quad \text{where} \quad a_i \in R_I$$

clearing denominators we have

$$u\chi^\beta = \sum_{i=1}^{n} b_i \chi^{\alpha_i} \quad \text{where} \quad u \in R \setminus I \text{ and } b_i \in R$$

And hence looking at the degree β part of this equation we get $\beta = \gamma + \alpha_i$ for some $\gamma \in S$ and $1 \leq i \leq n$. After doing this again for γ we get $\beta = \gamma' + \alpha_j + \alpha_i$ for some $\gamma \in S$ and $1 \leq i, j \leq n$ and continuing the process at some point it must end because S does not have invertible elements. Hence the $\alpha_1, \ldots \alpha_n$ generate S as a semigroup and then they form a basis of M over $\mathbb{Z}$. Then passing to the dual is not difficult to see that $\sigma \cap N$ is also generated by a basis of the lattice of N and so σ is an smooth cone. □

Now, in a similar vein as in Theorem 1.2.4. the correspondence between affine normal toric varieties and cones induce a correspondence between inclusions of cones and toric morphisms.

Theorem 1.2.12. *There exist a morphism of toric varieties $f : U_{\sigma_1} \to U_{\sigma_2}$ if and only if $\sigma_1 \subseteq \sigma_2$ and in such a case, f is uniquely defined. Moreover, if σ_1 is a face of σ_2 then f is an open immersion.*

Proof. We have $\sigma_1 \subseteq \sigma_2 \iff S_{\sigma_2} \subseteq S_{\sigma_1}$ hence the first part follows directly by Theorem 1.2.4.

Now if σ_1 is a face of σ_2 there is an $\alpha \in S_{\sigma_2}$ such that $\alpha \geq 0$ on σ_2 and $\sigma_1 = \{x \in \sigma_2 \mid \alpha(x) = 0\}$. Notice that $-\alpha \in S_{\sigma_1}$ and for every $\beta \in S_{\sigma_1}$ there is an $n \geq 0$ such that $\beta + n\alpha \geq 0$ in σ_2. Hence

$$\begin{aligned} \chi^\beta \in \Gamma(\mathcal{O}_{U_{\sigma_1}}) &\iff \beta \geq 0 \text{ on } \sigma_1 \\ &\iff \text{ there is an } n \geq 0 \text{ such that } \beta + n\alpha \geq 0 \text{ on } \sigma_2 \\ &\iff \chi^\beta \in \Gamma(\mathcal{O}_{U_{\sigma_2}})\left[\frac{1}{\chi^\alpha}\right] \end{aligned}$$

So the inclusion $\Gamma(\mathcal{O}_{U_{\sigma_2}}) \hookrightarrow \Gamma(\mathcal{O}_{U_{\sigma_1}})$ is a localization map and hence $U_{\sigma_1} \to U_{\sigma_2}$ is an open immersion. □

1.3 Orbit-Cone Correspondence

In this section we are going to state and prove the Orbit-Cone Correspondence and for this we need to study the action of T in U_σ in more detail.

We start by introducing distinguished points.

Proposition 1.3.1. *Fix a strictly convex rational cone σ.*

1. *If $a \in N$ we have*
$$a \in \sigma \iff \lambda_a(0) := \lim_{t \to 0} \lambda_a(t) \text{ exists in } U_\sigma$$

2. *Let $a_1, a_2 \in \sigma \cap N$, then*
$$\lambda_{a_1}(0) = \lambda_{a_1}(0) \iff a_1 \text{ and } a_2 \text{ lie in the interior of the same face of } \sigma$$

3. *The point of the form $\lambda_a(0)$ for some a in the interior of the face τ of σ is given in terms of the semigroup morphism $\gamma_\tau : S_\sigma \to k$ by*

$$\gamma_\tau(r) = \begin{cases} 1 & \text{if } r \in S_\sigma \cap \tau^\perp \\ 0 & \text{if not} \end{cases}$$

Proof.

1. $$\begin{aligned} \lim_{t \to 0} \lambda_a(t) \text{ exists in } U_\sigma &\iff \lim_{t \to 0} \chi^r(\lambda_a(t)) = \lim_{t \to 0} t^{\langle r, a \rangle} \text{ exists in } \mathbb{A}^1 \text{ for every } r \in S_\sigma \\ &\iff \langle r, a \rangle \geq 0 \text{ for every } r \in S_\sigma \\ &\iff a \in ((\sigma^\vee)^\vee = \sigma \end{aligned}$$

2. Let $\{r_1, \ldots, r_n\}$ be a set of generators for S_σ, then $\{\chi^{r_1}, \ldots, \chi^{r_n}\}$ are coordinates for U_σ. Now as

$$\chi^{r_i}(\lambda_a(0)) = \begin{cases} 1 & \text{if } \langle r_i, a\rangle = 0 \\ 0 & \text{if } \langle r_i, a\rangle > 0 \end{cases}$$

we have that $\lambda_{a_1}(0) = \lambda_{a_2}(0)$ exactly when $\langle r_i, a_1\rangle = 0 \iff \langle r_i, a_2\rangle = 0$ and this happen exactly when a_1 and a_2 belong to the interior of the same face.

3. Fix a face $\tau \subseteq \sigma$ and $a \in \operatorname{int}(\tau)$. For $r \in S_\sigma$ we have

$$\begin{aligned} \gamma_\tau(r) = 1 &\iff r \in S_\sigma \cap \tau^\perp \\ &\iff \langle r, a\rangle = 0 \\ &\iff \chi^r(\lambda_a(0)) = 1 \end{aligned}$$

Hence the point corresponding to γ_τ is exactly $\lambda_a(0)$.

□

Definition 1.3.2. The points γ_τ associated to faces $\tau \subseteq \sigma$ by the proposition above are called the distinguished points of U_σ. //

Let's denote by $\mathcal{O}(\tau)$ the orbit of the distinguished point of τ. The next proposition tell us how to understand this sets in terms of semigroup morphisms.

Lemma 1.3.3.

1. *For any cone σ we have*

$$\begin{aligned} \mathcal{O}(\sigma) &\cong \{\gamma : S_\sigma \to k \mid \gamma(r) \neq 0 \iff r \in \sigma^\perp \cap M\} \\ &\cong Hom_{\mathbb{Z}}(\sigma^\perp \cap M, k^*) \end{aligned}$$

2. *If τ is a face of σ then $\overline{\mathcal{O}(\tau)} = V(I) \subseteq U_\sigma$ where*

$$I = \bigoplus_{\substack{r \geq 0 \text{ on } \sigma \\ r > 0 \text{ on } Int\ \tau}} k\chi^r \subseteq \Gamma(U_\sigma)$$

Proof.

1. For simplicity let's call $\mathcal{O}' = \{\gamma : S_\sigma \to k \mid \gamma(r) \neq 0 \iff r \in \sigma^\perp \cap N\}$ momentaneusly. For $r \in S_\sigma$ we have

$$\gamma_\sigma(r) \neq 0 \iff \langle r, a\rangle = 0 \text{ for } a \in \operatorname{int}(\sigma) \iff \langle r, a\rangle = 0\ \forall a \in \sigma \iff r \in \sigma^\perp \cap N$$

Hence $\gamma_\sigma \in \mathcal{O}'$. Also $\mathcal{O}'$ is invariant under the action of the torus T, because if $\gamma \in \mathcal{O}'$ then $t \cdot \gamma : r \mapsto \chi^r(t)\gamma(r)$ is also in $\mathcal{O}'$.

Next, if $\gamma \in \mathcal{O}'$ we have $\gamma(r) = 0\ \forall r \notin \sigma^\perp \cap N$ and hence the map

$$\begin{aligned} \mathcal{O}' &\to \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, k^*) \\ \gamma &\mapsto \gamma\,|_{\sigma^\perp \cap M}\ . \end{aligned}$$

is injective and well defined (the restriction is a group homomorphism). In the other hand, taking $\psi \in \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, k^*)$ we can extend it to a semigroup homomorphism on S_σ by defining $\psi(r) = 0$ if $r \notin \sigma^\perp \cap M$ and then the map above is a bijection.

Finally denote by N_σ the sublattice of N generated by $\sigma \cap N$ and $N(\sigma)$ the quotient N/N_σ. The perfect pairing over $M \times N$ induce a perfect pairing over $\sigma^\perp \cap M \times N(\sigma)$ so we can identify $\operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, k^*)$ with a torus $T_{N(\sigma)} = N(\sigma) \otimes_{\mathbb{Z}} k^*$. Then the surjection $N \to N(\sigma)$ induce another surjection $N \otimes_{\mathbb{Z}} k^* \to N(\sigma) \otimes_{\mathbb{Z}} k^*$ and this translate to the map

$$T_N \to T_{N(\sigma)} = \operatorname{Hom}_{\mathbb{Z}}(\sigma^\perp \cap M, k^*) = \mathcal{O}'$$

who is no other than the action of T_N on $\mathcal{O}'$. Hence T_N acts transitively in $\mathcal{O}'$ so $\mathcal{O}' = \mathcal{O}(\sigma)$.

2. In part 1. we saw that $\mathcal{O}(\sigma) \cong Hom_{\mathbb{Z}}(\tau^\perp \cap M, k^*)$ but as any group morphism from $\tau^\perp \cap M$ is also determined by its restriction to $\sigma^\vee \cap \tau^\perp \cap M$ we have that $\mathcal{O}(\sigma) \cong \mathrm{Mor}(\sigma^\vee \cap \tau^\perp \cap M, k^*)$ where Mor stands for semigroup morphisms.

   Now for $f = \sum_r a_r \chi^r \in \Gamma(U_\sigma)$ we have

$$\begin{aligned} f \in I(\overline{\mathcal{O}(\tau)}) &\iff f(x) = 0 \quad \forall x \in \mathcal{O}(\tau) \;\cong\; \mathrm{Mor}(\sigma^\vee \cap \tau^\perp \cap M, k^*) \\ &\iff \sum_r a_r \gamma(r) = 0 \quad \forall \gamma \in \mathrm{Mor}(\sigma^\vee \cap \tau^\perp \cap M, k^*) \\ &\iff a_r = 0 \quad \forall r \in \sigma^\vee \cap \tau^\perp \cap M \\ &\iff a_r = 0 \quad \forall r \in M \text{ s.t } r \geq 0 \text{ on } \sigma \text{ and } r = 0 \text{ on Int } \tau \\ &\iff f \in \bigoplus_{\substack{r \geq 0 \text{ on } \sigma \\ r > 0 \text{ on Int } \tau}} k\chi^r \end{aligned}$$

□

Now we are ready to proof the Orbit-cone correspondence

Theorem 1.3.4. *Fix a strictly convex rational cone σ.*

1. *There is a bijection $\tau \mapsto \mathcal{O}(\tau)$ between the faces of σ and the orbits of the action. This bijection send τ to the orbit $\mathcal{O}(\tau)$ generated by the point x_τ.*
2. $\dim \tau + \dim \mathcal{O}(\tau) = n$
3. *Under the bijection above*

$$\tau_1 \subseteq \tau_2 \iff \mathcal{O}(\tau_2) \subseteq \overline{\mathcal{O}(\tau_1)}$$

Proof.

1. Let $\mathcal{O}$ be a orbit and τ the minimal face in σ such that $\mathcal{O} \subseteq U_\tau$, it exist because $\mathcal{O} \subseteq U_{\tau_1}$ and $\mathcal{O} \subseteq U_{\tau_2}$ implies $\mathcal{O} \subseteq U_{\tau_1} \cap U_{\tau_2} = U_{\tau_1 \cap \tau_2}$. We claim that $\mathcal{O} = \mathcal{O}(\tau)$ from where the bijection follows.

   To prove the claim is enough to fix $\gamma \in \mathcal{O}$ and show that $\gamma \in \mathcal{O}(\tau)$. For this consider $\gamma^{-1}(k^*) \subseteq S_\tau$, as $\gamma(r+s) = \gamma(r)\gamma(s)$ we have $r + s \in \gamma^{-1}(k^*)$ implies $r, s \in \gamma^{-1}(k^*)$ moreover as $\gamma^{-1}(k^*)_{\mathbb{R}}$ is a convex subset subset of $M_{\mathbb{R}}$ we have that it is a face of $\tau^\vee$. Hence, there is a face ς of τ such that $\gamma^{-1}(k^*)_{\mathbb{R}} = \tau^\vee \cap \varsigma^\perp$, so

$$\gamma^{-1}(k^*) = \tau^\vee \cap \varsigma^\perp \cap M$$

   From this we see that $\gamma(r) = 0 \ \forall r \notin \varsigma^\perp \cap M$, so γ can be identified with a semigroup morphism $\bar{\gamma} : \varsigma^\vee \cap M \to k^*$ and then γ correspond to a point in X_ς therefore $\mathcal{O} \subset X_\varsigma$ and from minimality $\varsigma = \tau$. Then as $\gamma(r) = 0 \iff r \in \tau^\perp$ we get $\gamma \in \mathcal{O}(\tau)$ and we conclude $\mathcal{O} = \mathcal{O}(\tau)$.

2. As we saw in Lemma 1.3.3, if N_σ is the smallest sublattice containing $\sigma \cap N$, then $\mathcal{O}(\tau) = \mathrm{Hom}(\sigma^\perp \cap M, k^*)$ can be identified with N/N_σ. Hence

$$\dim(\mathcal{O}(\tau)) = \mathrm{rank}(N/N_\sigma) = \mathrm{rank}(N) - \mathrm{rank}(N_\sigma) = n - \dim(\sigma)$$

3. By Lemma 1.3.3 we have

$$\begin{aligned} \mathcal{O}(\tau_2) \subseteq \overline{\mathcal{O}(\tau_1)} &\iff \bigoplus_{\substack{r \geq 0 \text{ on } \sigma \\ r > 0 \text{ on Int } \tau_2}} k\chi^r \subseteq \bigoplus_{\substack{r \geq 0 \text{ on } \sigma \\ r > 0 \text{ on Int } \tau_1}} k\chi^r \\ &\iff \sigma^\vee \cap \tau_2^\perp \cap M \subseteq \sigma^\vee \cap \tau_1^\perp \cap M \\ &\iff \tau_1 \subseteq \tau_2 \end{aligned}$$

□

Corollary 1.3.5. *As in any set with an action of a group we have a partition given by its orbit. In this case this give us the following disjoint union*

$$U_\sigma = \bigcup_{\tau < \sigma} \mathcal{O}(\tau)$$

1.4 Classification of General Toric Varieties

Now we are going to extend our classification of normal affine toric varieties to arbitary normal affine toric varieties. For this we need the following result due to Sumihiro to reduced the problem to the affine case.

Theorem 1.4.1. (Sumihiro) *Let T be a torus acting in a normal variety X, then every point of X admits an open invariant affine neighborhood.*

Proof. See [KKMSD73]. □

The combinatorial objects that will encode our toric varieties are called fans. These are collections of strictly convex rational polyhedral cones satisfying properties resembling the ones for simplicial complexes (Definition A.6).

By Theorem 1.2.11 a fan Σ give us a directed system of varieties that we can glue (take a colimit) to form a new variety that we denote $\mathbb{P}_\Sigma$.

We can see that $\mathbb{P}_\Sigma$ is normal as it has a cover $\{U_{\sigma_\alpha}\}_{\alpha\in\Lambda}$ by affine normal toric varieties. It is also separable as $U_{\sigma_\alpha} \cap U_{\sigma_\beta} = U_{\sigma_\alpha\cap\sigma_\beta}$ and the algebra $\Gamma(\mathcal{O}_{U_{\sigma_\alpha\cap\sigma_\beta}})$ is generated by the two subalgebras $\Gamma(\mathcal{O}_{U_{\sigma_\alpha}})$ and $\Gamma(\mathcal{O}_{U_{\sigma_\beta}})$. Notice that if Σ is the fan generated by a single cone σ, then $\mathbb{P}_\Sigma = U_\sigma$, so this construction generalize the one we saw in the previous chapters.

Now we can classify normal toric varieties in terms of fans and extend the theorems we saw for affine normal toric varieties in the previous sections.

Theorem 1.4.2.

1. *The correspondence $\Sigma \mapsto \mathbb{P}_\Sigma$ defines a bijection between fans over $N_\mathbb{R}$ and isomorphism classes of normal toric varieties containing the torus T.*

2. *The correspondence $\Sigma \ni \sigma \mapsto U_\sigma$ defines a bijection between the cones in the fan and the invariant affine open subsets of $\mathbb{P}_\Sigma$.*

3. **(Orbit-cone correspondence)** *The map which associates to each $\sigma \in \Sigma$ the unique closed orbit of U_σ denoted again by $\mathcal{O}(\sigma)$ gives a correspondence between the cones of Σ and the orbits of $\mathbb{P}_\Sigma$. We have also $\sigma_1 \subset \sigma_2 \iff \mathcal{O}(\sigma_2) \subseteq \overline{\mathcal{O}(\sigma_1)}$*

4. *There is a toric morphism $\mathbb{P}_{\Sigma_1} \to \mathbb{P}_{\Sigma_2}$ iff for all $\sigma_1 \in \Sigma_1$ there is a $\sigma_2 \in \Sigma_2$ such that $\sigma_1 \subseteq \sigma_2$.*

Proof.

1. and 2. Let X be a normal toric variety. By Sumihiro's theorem and quasi-compactness we have a finite cover of X by maximal affine invariant open sets. Then by Theorem 1.2.9 each of this invariant sets is given by a toric variety of the form U_σ for some cone $\sigma \subseteq N_\mathbb{R}$. Then the fan Σ generated from this cones induce exactly the variety X with which started. Also, by maximality any other invariant affine open subset U_τ of X must be contained in one open set U_σ from the ones we started with. Hence τ is a face of σ and then $\tau \in \Sigma$.

3. Let $\mathcal{O}$ be a orbit of $\mathbb{P}_\Sigma$ and define $U = \{x \in \mathbb{P}_\Sigma \mid \mathcal{O} \subseteq \overline{\mathcal{O}(x)}\}$ where $\mathcal{O}(x)$ is the orbit of the point x. By Sumihiro's theorem there is an open affine invariant subset containing U. Hence we can use the affine version of the Orbit-cone correspondence to see that U is an invariant affine open subset, hence of the form U_σ for some $\sigma \in \Sigma$ and then $\mathcal{O} = \mathcal{O}(\sigma)$.

4. If there is $\sigma_1 \in \Sigma_1$ and $\sigma_2 \in \Sigma_2$ such that $\sigma_1 \subset \sigma_2$ then by Theorem 1.2.12 we have a unique toric map $U_{\sigma_1} \to U_{\sigma_2}$. Gluing this maps we get a toric morphism $\mathbb{P}_{\Sigma_1} \to \mathbb{P}_{\Sigma_2}$.

   In the other hand, take a toric morphism $f : \mathbb{P}_{\Sigma_1} \to \mathbb{P}_{\Sigma_2}$ and $\sigma_1 \in \Sigma_1$. As f is equivariant $f(\mathcal{O}(\sigma_1))$ is also an orbit of $\mathbb{P}_{\Sigma_2}$ and then is of the form $\mathcal{O}(\sigma_2)$ for some $\sigma_2 \in \Sigma_2$. If τ_1 is a face of σ_1 we have $\mathcal{O}(\tau_1) \subseteq \overline{\mathcal{O}(\sigma_1)}$ and hence $f(\mathcal{O}(\tau_1)) \subseteq f(\overline{\mathcal{O}(\sigma_1)}) \subseteq \overline{\mathcal{O}(\sigma_2)}$ so $f(\mathcal{O}(\tau_1)) = \mathcal{O}(\tau_2)$ for some face τ_2 of σ_2. From this using Corollary 1.3.5 we get

$$f(U_{\sigma_1}) = f(\cup_{\tau\subset\sigma}\mathcal{O}(\tau)) \subseteq \cup_{\tau\subseteq\sigma_2}\mathcal{O}(\tau) = U_{\sigma_2}$$

   as we wanted.

□

Relaxing the affine condition allow us to have proper and projective toric variaties. Luckily we have good criteria to identify this properties in terms of the associated fans, to state them we need the definition of complete fan and polytopal fan. We refer the reader to [?] for this.

Now we can state each criterion

Theorem 1.4.3. *A toric variety $\mathbb{P}_\Sigma$ is*

1. *proper iff Σ is a complete fan*
2. *projective iff Σ is a polytopal fan*

Proof. The proof of 1. is contained in [Ful93] section 2.4. and 2. is proved at [Ewa96] section VII.3. □

1.5 Equivariant Sheaves of Fractional Ideals Over Toric Varieties

Fix a fan Σ on $N_\mathbb{R}$ and call by $X = \mathbb{P}_\Sigma$ its associated toric variety.

In this section we will classify certain coherent sheaves over X well behaved with respect to the action of the torus. This objects will be understood in terms of certain piecewise linear concave function attached to them and its understanding will give us an interpretation for the Picard group and the equivariant part of the class group of the variety.

If we denote by $i : T \hookrightarrow X$ the inclusion morphism then $i_*(\mathcal{O}_T)$ is a quasi-coherent $\mathcal{O}_X$-submodule of the sheaf $\mathcal{K}_X$ of rational functions over X and it is equipped with an action of T given by

$$(\alpha \cdot f)(x) = f(\alpha^{-1}x) \ \ \forall \alpha \in T, \ \forall f \in i_*(\mathcal{O}_T)(U_\sigma), \ \forall \sigma \in \Sigma$$

The sheaves we are going to classify are the coherent sheaves $\mathcal{F}$ contained on $i_*(\mathcal{O}_T)$ such that we can restrict the action above to them. We will call these sheaves *equivariant sheaves of fractional ideals.*

Notice that for such an $\mathcal{F}$ we have $\mathcal{F}(T) \subseteq \Gamma(\mathcal{O}_T)$. This allow us to look at the variety $V(\mathcal{F}(T))$ which from the T-invariance is either $\varnothing$ or T, then by the coherent hypothesis $\mathcal{F}|_T$ is either 0 (from where $\mathcal{F} = 0$) or $\mathcal{O}_T$. This implies that all the interesting behavior of $\mathcal{F}$ occurs outside T. To understand this behavior we introduce the *order function of $\mathcal{F}$*

$$\operatorname{Ord} \mathcal{F} : |\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \to \mathbb{R}.$$

It is defined as follows:

First, we define it on lattice points $a \in N_\mathbb{R} \cap |\Sigma|$. For this notice that, because of Proposition 1.3.1, a is attached to a function

$$\lambda_a : \operatorname{Spec} k[x] \to X$$

so we can define

$$\operatorname{Ord}(\mathcal{F})(a) := \operatorname{Ord}_0(\lambda_a^* \mathcal{F})$$

Let's understand this definition. Taking $\sigma \in \Sigma$ such that $a \in \sigma$ we can restrict the codomain of λ_a to U_σ.

Now $U_\sigma = \operatorname{Spec} R_\sigma$ where $R_\sigma = \bigoplus_{\substack{r \in M \\ r \geq 0 \text{ on } \sigma}} k\chi^r$ and as $\mathcal{F}$ is coherent $\mathcal{F}(U_\sigma)$ is a finitely generated R_σ-submodule of $\mathcal{F}(T) = \Gamma(\mathcal{O}_T)$, say

$$J_\sigma := \sum_{i=1}^{N} R_\sigma \chi^{r_i} \subseteq \bigoplus_{r \in M} k\chi^r.$$

From this, as in affine varieties the pullback of a coherent $\mathcal{O}_X$-module is the $\mathcal{O}_X$-module induced by the extension of coordinates of its global sections, we have

$$\lambda_a^*(\mathcal{F}) = \lambda_a^*(\mathcal{F}|_{U_\sigma}) = (J_\sigma \otimes_{R_\sigma} k[x])^\sim = \left(\sum_{i=1}^{N} k[X]X^{\langle r_i, a\rangle}\right)^\sim$$

To see this last isomorphism notice that under the isomorphism $R_\sigma \otimes_{R_\sigma} k[x] \cong k[x]$ the set of generators $\{\chi^{r_i} \otimes p \mid p \in k[x], i = 1, \dots, N\}$ of $J_\sigma \otimes_{R_\sigma} k[x]$ is send to $\{x^{\langle r_i, a\rangle} p \mid p \in k[x], i = 1, \dots, N\}$ as

$$\chi^{r_i} \otimes p = 1 \otimes \lambda_a(\chi^{r_i})p = 1 \otimes x^{\langle r_i, a\rangle} p.$$

Then $\operatorname{Ord}\mathcal{F}(a) = \min_i \langle r_i, a\rangle$.

This allow us to extend $\operatorname{Ord}\mathcal{F}$ to all σ by the formula above, i.e,

$$\operatorname{Ord}\mathcal{F}(x) = \min_i \langle r_i, x\rangle \ \ \forall\, x \in \sigma$$

and by doing the same for the other cones in Σ we can defined $\operatorname{Ord}\mathcal{F}$ for all $|\Sigma|$.

With this construction we can state all the results of this section.

Theorem 1.5.1. *Let Σ be a fan, $X = \mathbb{P}_\Sigma$ its induced toric variety and $\mathcal{F}$ a equivariant coherent subsheaf of $i_*(\mathcal{O}_T)$. Then*

I. *The function $\operatorname{Ord}\mathcal{F}$ satisfies the following properties*

   (a) $\operatorname{Ord}\mathcal{F}(\lambda x) = \lambda \cdot \operatorname{Ord}\mathcal{F}(x), \quad \forall \lambda \in \mathbb{R}^+$

   (b) *$\operatorname{Ord}\mathcal{F}$ is continuous and piecewise-linear.*

   (c) $\operatorname{Ord}\mathcal{F}(N \cap |\Sigma|) \subseteq \mathbb{Z}$

   (d) *$\operatorname{Ord}\mathcal{F}$ is concave on each σ*

   A function satisfying all this conditions will be called an ***order function****. Conversely, any function satisfying this conditions is of the form $x \mapsto \min_i \langle r_i, x\rangle$ for some $r_i \in M$.*

II. *Take an order function $f : |\Sigma| \to \mathbb{R}$. For all $\sigma \in \Sigma$ we put*

$$(J_f)_\sigma = \bigoplus_{\substack{r \in M \\ r \geq f \text{ on } \sigma}} k\chi^r$$

   Then $(J_f)_\sigma$ is a T-invariant $\Gamma(\mathcal{O}_{U_\sigma})$-module and the sheaves $(J_f)_\sigma^\sim$ can be glued together to obtain a coherent sheaf $\mathcal{F}_f$ of equivariant fractional ideals contained in $i_*(\mathcal{O}_T)$.

   Moreover, the sheaf $\mathcal{F}_f$ is complete as a sheaf of fractional ideals. This means that each $(J_f)_\sigma$ above is a complete fractional ideal in the following precise sense:

   Given a domain A with quotient field K and $J \subseteq K$ a fractional ideal we say that J is complete if for all $z \in K$ satisfying an equation of the form

$$z^q + a_1 z^{q-1} + \cdots + a_q = 0 \quad \text{with } a_i \in J^i \tag{$*$}$$

   we have $z \in J$

III. *The constructions above satisfy the following properties.*

   (a) $\operatorname{Ord}\mathcal{F}_f = f$

   (b) *$\mathcal{F}_{\operatorname{Ord}\mathcal{F}}$ is the completion of $\mathcal{F}$*

   (c) *The maps $\mathcal{F} \mapsto \operatorname{Ord}\mathcal{F}$ and $f \mapsto \mathcal{F}_f$ define a bijection between the set of coherent sheaves $\mathcal{F}$ of equivariant complete fractional ideals over X and the set of order functions $f : |\Sigma| \to \mathbb{R}$.*

   (d) $\mathcal{F} \subseteq \mathcal{F}_f \iff \operatorname{Ord}\mathcal{F} \geq f$

   (e) $\operatorname{Ord}\mathcal{F}_1\mathcal{F}_2 = \operatorname{Ord}\mathcal{F}_1 + \operatorname{Ord}\mathcal{F}_2$

(f) $\mathcal{F}|_{U_\sigma} = \mathcal{O}_{U_\sigma} \iff Ord\mathcal{F} \equiv 0$ *on* σ

(g) There is a toric isomorphism of $\mathcal{O}_X$*-modules between* $\mathcal{F}_{f_1}$ *and* $\mathcal{F}_{f_2}$ *if and only if* $f_1 - f_2$ *is linear on* $|\Sigma|$.

IV. *Given a cone* σ *we denote by* $sk^1\,\sigma$ *the set of all primitive integral vectors inside its one dimensional faces and for a fan* Σ *we define* $sk^1\,\Sigma = \bigcup_{\sigma\in\Sigma} sk^1\,\sigma$. *For a function* $h : sk^1\,\sigma \to \mathbb{Z}$ *we define its convex interpolation as the least order function* $\tilde{h} : \sigma \to \mathbb{R}$ *majorizing* h *on* σ. *In other words*

$$\tilde{h}(x) = \min_{r\geq h \text{ over } sk^1\,\sigma} \langle r, x\rangle$$

where the minimum goes over all integral linear functional $r \in M$ *that majorizes* h *over* $sk^1\,\sigma$. *For a function* $h : sk^1\,\Sigma \to \mathbb{Z}$ *we define its convex interpolation as the function* $\tilde{h} : |\Sigma| \to \mathbb{R}$ *given as the convex interpolation of* h *restricted to each cone. In particular it is an order function.*

With this notation we have the following properties:

(a) $\mathcal{F}^{-1} = \mathcal{F}_g$ *where* g *is the convex interpolation of* $-Ord\mathcal{F}$ *on* $sk^1\,\Sigma$

(b) $(\mathcal{F}^{-1})^{-1} = \mathcal{F}$ *if and only if* $\mathcal{F}$ *is complete and* $Ord\,\mathcal{F}$ *is the linear interpolation of an integral function over* $sk^1\,\Sigma$. *Moreover, there exists a bijective correspondence between the set of* T*-invariant Weil divisors and the set of integral functions on* $sk^1\,\Sigma$.

(c) The following are equivalent:

i) $\mathcal{F}$ *is invertible*

ii) $\mathcal{F}\mathcal{F}^{-1} = \mathcal{O}_X$

iii) $Ord\mathcal{F}$ *is linear on each* σ.

(d) $\Omega^n_X \cong \mathcal{F}_k$ *where* k *is the convex interpolation of the constant function with values* -1 *on* $sk^1\,\Sigma$.

Proof.

I. All this properties follow directly from the formula defining the order function. For the converse let f be an order function. As it is piecewise linear there are finitely many $r_i \in M_{\mathbb{R}}$ such that for each $x \in |\Sigma|$ there is an i with $f(x) = \langle r_i, x\rangle$. Then as f is concave $f(x) = \min_i\langle r_i, x\rangle$ and as $f(N \cap |\Sigma|) \subseteq \mathbb{Z}$ we can take each r_i in M.

II. As $(J_f)_\sigma$ is closed under multiplication by elements of R_σ we see that it is an R_σ-module. Also if $h = \sum_r a_r\chi^r \in (J_f)_\sigma$ and $\alpha \in T$ we have $\alpha \cdot h = \sum_r a_r\alpha^{-r}\chi^r \in (J_f)_\sigma$ so $(J_f)_\sigma$ is T-invariant R_σ-module. Next to prove that these modules can be glued together is enough to prove that

$$(J_f)^{\sim}_\sigma|_{U_\tau} = (J_f)^{\sim}_\tau$$

For this notice that, as U_τ is a principal open subset for $g = \prod_{\substack{r\in M\cap\sigma^\vee \\ r=0 \text{ on } \tau}} \chi^r$, we have

$$\Gamma(U_\tau, (J_f)^{\sim}_\sigma) = \left(\bigoplus_{\substack{r\in M \\ r\geq f \text{ on } \sigma}} k\chi^r\right)_g = \bigoplus_{\substack{r\in M \\ r\geq f \text{ on } \tau}} k\chi^r = \Gamma(U_\tau, (J_f)_\tau)$$

Now to see that $\mathcal{F}_f$ is complete we need to prove that each $(J_f)_\sigma$ is complete. For this notice that if $f(x) = \min_{i=1,\dots,s}\{\langle r_i, x\rangle\}$ for all $x \in \sigma$ then

$$\mathcal{R} := \sum_{i=1}^{s} R_\sigma\chi^{r_i} = \bigoplus_{\substack{r\in M \\ r\geq r_i \text{ on } \sigma \\ \text{for some } i}} k\chi^r \subseteq \bigoplus_{\substack{r\in M \\ r\geq f \text{ on } \sigma}} k\chi^r = (J_f)_\sigma$$

and we will prove that $(J_f)_\sigma$ is exactly the completion of $\mathcal{R}$ and thus complete.

We start by proving that there is an equation over $\mathcal{R}$ of the form $(*)$ for each $\chi^r \in (J_f)_\sigma$, this will prove that $(J_f)_\sigma$ is contained in the completion of $\mathcal{R}$. From $\chi^r \in (J_f)_\sigma$ we have $r \geq f$ on σ and if $\sigma^\vee = \text{cone}(v_1, \ldots, v_n)$ we have $\langle x, v_j \rangle \geq 0$ for all $1 \leq i \leq n$ exactly when $x \in \sigma$, hence

$$\min\{\langle x, r_i - r\rangle, \langle x, v_j\rangle\}_{i,j} \leq 0 \ \ \forall x \in N_\mathbb{R}$$

Now define $h(x) = -\min\{\langle x, r_i - r\rangle, \langle x, v_j\rangle\}_{i,j} = \max\{\langle x, r - r_i\rangle, \langle x, -v_j\rangle\}_{i,j}$. This is a convex function that attains its minimum at 0. Hence, 0 belongs to the subgradient $\partial h(0)$, but this subgradient is also given by $\text{conv}(\bigcup_{i,j} \partial\langle \cdot, r - r_i\rangle \cup \partial\langle \cdot, -v_j\rangle) = \text{conv}\{r - r_i, -v_j\}_{i,j}$ and so there are integers $n_i, m_j \geq 0$ not all zero such that

$$\sum_i n_i(r - r_i) + \sum_j m_j v_j = 0$$

and then $\sum_i n_i r_i + \sum_j m_j v_j = \left(\sum_i n_i\right) r$, so for $N = \sum n_i$ we have

$$(\chi^r)^N = \chi^{\langle \sum_j m_j v_j\rangle} \prod_i (\chi^{r_i})^{n_i}$$

and this is a monic equation for χ^r over $\mathcal{R}$ of the desired form.

In the other hand, if χ^r is in the completion of $\mathcal{R}$ then

$$\chi^{rm} + a_1\chi^{r(m-1)} + \cdots + a_{m-1}\chi^r + a_m = 0$$

for some $a_i \in \mathcal{R}^i$. Looking only at the degree rm part of the equation we can assume that each non zero a_i is a character, taking some N with $a_N \neq 0$ we have $a_N = \chi^c \cdot \chi^{\sum n_j r_j}$ with $\sum n_j = N$ and $c \geq 0$ as a function over σ, then

$$\begin{aligned}
rm = \deg(a_N) + r(m - N) &\iff r = c/N + \sum \frac{n_j r_j}{N} \\
&\implies r \geq \frac{n_j r_j}{N} \text{ over } \sigma \\
&\implies \text{ there is some } j \text{ such that } r \geq r_j \text{ over } \sigma \\
&\iff r \geq f \text{ over } \sigma \\
&\iff \chi^r \in (J_f)_\sigma
\end{aligned}$$

Hence $(J_f)_\sigma$ is the completion of $\mathcal{R}$ and in particular is complete.

III. (a) As $\mathcal{F}$ is coherent we have

$$\Gamma(U_\sigma, \mathcal{F}_f) = \bigoplus_{\substack{r \in M \\ r \geq f \text{ on } \sigma}} k\chi^r = \sum_{j=1}^m R_\sigma \chi^{s_j}$$

for some $s_j \in M$ from where $\text{Ord}\,\mathcal{F}_f = \min_j\langle s_j, x\rangle = f$.

(b) If $\Gamma(U_\sigma, \mathcal{F}) = \sum_i R_\sigma\chi^{r_i}$ we have $\text{Ord}\,\mathcal{F} = \min_i\langle r_i, x\rangle$ on σ and then as it was shown in the proof of part II. the sheaf $\mathcal{F}_{\text{Ord}\,\mathcal{F}}$ is the completion of $\sum_i R_\sigma\chi^{r_i} = \mathcal{F}$.

(c) This follows formally from parts (a) and (b).

(d) $\mathcal{F} \subseteq \mathcal{F}_f \iff \mathcal{F}|_{U_\sigma} \subseteq \mathcal{F}_f|_{U_\sigma}$ for every $\sigma \in \Sigma$. As these sheaves are coherent, if $\Gamma(U_\sigma, \mathcal{F}) = \sum_i R_\sigma\chi^{r_i}$ we have

$$\mathcal{F}|_{U_\sigma} \subseteq \mathcal{F}_f|_{U_\sigma} \iff \sum_i R_\sigma\chi^{r_i} \subseteq \bigoplus_{\substack{r \in M \\ r \geq f \text{ on } \sigma}} k\chi^r \iff r_i \geq f \ \ \forall i \iff \text{Ord}\ \mathcal{F} \geq f$$

(e) Suppose $\Gamma(U_\sigma, \mathcal{F}_1) = \sum_i R_\sigma\chi^{r_i}$ and $\Gamma(U_\sigma, \mathcal{F}_2) = \sum_j R_\sigma\chi^{s_i}$, then $\Gamma(U_\sigma, \mathcal{F}_1\mathcal{F}_2) = \sum_{ij} R_\sigma\chi^{r_i + s_j}$ so

$$\text{Ord}(\mathcal{F}_1\mathcal{F}_2)(x) = \min_{ij}\langle r_i + s_j, x\rangle = \min_i\langle r_i, x\rangle + \min_j\langle s_j, x\rangle = \text{Ord}\ \mathcal{F}_1(x) + \text{Ord}\ \mathcal{F}_2(x)$$

(f) This follows from $\Gamma(U_\sigma, \mathcal{O}_{U_\sigma}) = R_\sigma \chi^0$ so Ord $\mathcal{F}(x) = \langle 0, x\rangle = 0$

(g) Suppose $f_2 - f_1$ is linear, say $f_2 - f_1 = s \in M$. Then for any cone $\sigma \in \Sigma$ we have the isomorphism

$$\begin{aligned} \varphi\colon \Gamma(\mathcal{F}_{f_1}, U_\sigma) &\to \Gamma(\mathcal{F}_{f_2}, U_\sigma) \\ \chi^r &\mapsto \chi^{r+s}. \end{aligned}$$

And this glue to an isomorphism of $\mathcal{O}_X$-modules.

In the other hand, if there is a toric isomorphism of $\mathcal{O}_X$-modules $\varphi : \mathcal{F}_{f_1} \to \mathcal{F}_{f_2}$ then for each $\sigma \in \Sigma$ we have a toric isomorphism

$$\varphi_\sigma : \Gamma(\mathcal{F}_{f_1}, U_\sigma) \to \Gamma(\mathcal{F}_{f_2}, U_\sigma).$$

The T-equivariance implies that $\varphi_\sigma(\chi^r) = a_r \chi^s$ for some a_r and after renormalization we can assume that $a_r = 1$. Now we have an induced bijection between the exponents

$$\overline{\varphi_\sigma} : \{r \in M \mid r \geq f_1 \text{ over } \sigma\} \to \{r \in M \mid r \geq f_2 \text{ over } \sigma\}$$

And from φ_σ being a R_σ-module morphism we get

$$\overline{\varphi_\sigma}(r + r') = \overline{\varphi_\sigma}(r) + r' \text{ for all } r' \in M \text{ with } r' \geq 0 \text{ on } \sigma$$

Now suppose for every $r, r' \in M$ there is a $r^* \in M$ such that $r^* \geq r, r'$ on σ. Then

$$\overline{\varphi_\sigma}(r) + r^* - r = \overline{\varphi_\sigma}(r + r^* - r) = \overline{\varphi_\sigma}(r^*) = \overline{\varphi_\sigma}(r' + r^* - r') = \overline{\varphi_\sigma}(r') + r^* - r'$$

and so $\overline{\varphi_\sigma}(r) - r = s$ is independent of r. Then φ_σ is given by $\chi^r \mapsto \chi^{r+s}$ and $f_1 - f_2 = s$ over σ, varying sigma one get $f_1 - f_2 = s$ over $|\Sigma|$.

To see that an upper bound r^* exists for every r, r', notice that $r^* \geq r$ on σ iff $r^* - r \in \sigma^\vee$ and so the result follows from the fact that $\sigma^\vee$ has non empty interior (because σ is strictly convex) and then after a translation you can put both $-r, -r'$ inside of it.

IV. (a) Fix σ in Σ. For $r \in M$ we have

$$\begin{aligned} \chi^r \in \Gamma(U_\sigma, \mathcal{F}^{-1}) &\iff \chi^r \cdot \Gamma(U_\sigma, \mathcal{F}^{-1}) \subseteq \Gamma(X_\sigma, \mathcal{O}_{X_\sigma}) \\ &\iff r + \{s \in M \mid s \geq \text{Ord } \mathcal{F} \text{ on } \sigma\} \subseteq \{s \in M \mid s \geq 0 \text{ on } \sigma\} \\ &\iff r \geq -\text{Ord } \mathcal{F} \text{ on } \sigma \\ &\iff r \geq g \\ &\iff r \in \Gamma(U_\sigma, \mathcal{F}_g) \end{aligned}$$

where g is the convex interpolation of $-\text{Ord } \mathcal{F}$ on $\text{sk}^1 \sigma$.

(b) By the part above we have $(\mathcal{F}^{-1})^{-1} = \mathcal{F}_f$ with f the convex interpolation of the restriction of Ord $\mathcal{F}$ to $\text{sk}^1 \Sigma$. Then by III (c) this sheaf is equal to $\mathcal{F}$ iff the sheaf is complete and Ord $F = f$.

Now let D be a Weil divisor on X. As X is a normal integral variety it correspond to the coherent sheaf

$$\mathcal{O}_X(D)(U) = \{f \in \mathcal{K}_X \mid (f) + D \geq 0\}$$

and we have $\mathcal{O}_X(D)\mathcal{O}_X(D') = \mathcal{O}_X(D + D')$. Hence, $\mathcal{O}_X(D)^{-1} = \mathcal{O}_X(-D)$ from where $(\mathcal{O}_X(D)^{-1})^{-1} = \mathcal{O}_X(D)$ and so $\mathcal{O}_X(D)$ is complete and its order function correspond to an integral function on $\text{sk}^1 \Sigma$.

In the other hand, given an integral function over $\text{sk}^1 \Sigma$ we can take its convex interpolation $f : |\Sigma| \to \mathbb{R}$ and it correspond to a $\mathcal{F}_f$ such that $(\mathcal{F}_f^{-1})^{-1} = \mathcal{F}_f$. This property translate to $\mathcal{F}_f$ being reflexive and as it is also of rank 1 (as it is a subsheaf of $\mathcal{K}_X$) it must be of the form $\mathcal{O}_X(D)$ for some T-invariant Weil divisor D over X (see [Sch], Proposition 3.7).

(c) (iii $\implies$ i and ii) Suppose Ord $\mathcal{F}$ is linear on σ, say Ord $\mathcal{F}(x) = \langle r, x\rangle$ $\forall x \in \sigma$. Then in the finitely generated R_σ-module inducing $\mathcal{F}|_{U_\sigma}$

$$J_\sigma = \sum_{i=1}^{N} R_\sigma \chi^{r_i} \subset \Gamma(\mathcal{O}_T)$$

we would have Ord $\mathcal{F} = \min_i \langle r_i, x\rangle = \langle r, x\rangle$ from where there is an i such that $\langle r_i, x\rangle = \langle r, x\rangle$ on σ. For that i we have $J_\sigma = R_\sigma\chi^{r_i}$ so $\mathcal{F}|_{U_\sigma} = \chi^{r_i}\mathcal{O}_{U_\sigma}$ and hence $\mathcal{F}$ is an invertible sheaf.

In the other hand by IV (a) we have Ord $\mathcal{F}^{-1} = -$Ord $\mathcal{F}$ and so by III (e) Ord$(\mathcal{F}\mathcal{F}^{-1}) = \mathcal{F} + \mathcal{F}^{-1} = 0$ from where $\mathcal{F}\mathcal{F}^{-1} = \mathcal{O}_X$ using III (f).

(ii $\implies$ i) Suppose $\mathcal{F}\mathcal{F}^{-1} = \mathcal{O}_X$. Then as all this sheafs are coherent we have for each σ

$$\mathcal{F}(U_\sigma)\mathcal{F}(U_\sigma)^{-1} = \Gamma(U_\sigma, \mathcal{O}_X)$$

From which $1 \in \mathcal{F}(U_\sigma)\mathcal{F}(U_\sigma)^{-1}$, i.e, $1 = \sum_j g_j g'_j$ with $g_j \in \mathcal{F}(U_\sigma)$ and $g'_j \in \mathcal{F}(U_\sigma)^{-1}$. Then for every $h \in \mathcal{F}$ we have

$$h = h\cdot 1 = \sum_j g_j h g'_j \in g\mathcal{F}(U_\sigma)\mathcal{F}(U_\sigma)^{-1} = g\Gamma(U_\sigma, \mathcal{O}_X)$$

so $\mathcal{F}|_{U_\sigma} \leq g\mathcal{O}_X|_{U_\sigma}$. As the other implication is trivial we conclude $\mathcal{F}|_{U_\sigma} \cong \mathcal{O}_X|_{U_\sigma}$. As this isomorphism is equivariant it gives an equality $\mathcal{F}|_{U_\sigma} = \chi^r\mathcal{O}_X|_{U_\sigma}$ from which Ord $\mathcal{F}(x) = \langle r, x\rangle$ over σ.

(i $\implies$ iii) By Proposition 4.2.2 on [CLS11] we have that Pic(X) is trivial for every affine toric variety X. Now if $\mathcal{F}$ is an invertible sheaf over X, for each $\sigma \in \Sigma$ we have that $\mathcal{F}|_{U_\sigma}$ is trivial and hence it is isomorphic to $\mathcal{F}|_{U_\sigma}$. By III (g) this happens iff Ord F is linear over σ.

(d) The only differential of degree n over the torus $T = \text{Spec } k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ up to constant is given by

$$w = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{x_n}{x_n}$$

This differential form has pole of order 1 on each orbit of codimension 1. Hence for every one parameter subgroup $\lambda_a : \text{Spec } k[x, x^{-1}] \to X$ with $a \in N$ primitive we have $\text{Ord}_0(\lambda_a^*(w)) = -1$ and the result follows.

□

1.6 Equivariant Resolution of Singularities of Toric Varieties

Now we use our understanding on equivariant sheaves of fractional ideals from the last sections to resolve the singularities of a toric variety in a way compatible with the action of the torus.

For this we let $\mathcal{F}$ an equivariant sheaf of fractional ideals as in last section. The blow up of X with respect to these fractional ideal is defined as the morphism

$$\tilde{X} = \underline{\text{Proj}}_X\left(\bigoplus_{k=0}^{n} \mathcal{F}^k\right) \xrightarrow{f} X$$

and it satisfy that $f^*(\mathcal{F})$ is an invertible sheaf over Y. Moreover this is the minimal variety satisfying this: Every morphism $Y \to X$ such that the pullback of $\mathcal{F}$ along it is an invertible sheaf factorize uniquely along $\tilde{X} \to X$.

Now recall that this variety $\tilde{X}$ is not normal in general (it is for example when $\mathcal{F}$ defines a reduced closed subscheme of X, but this is not necessarily the case). For that reason we will work with the normalization of $\tilde{X}$ and we will call this variety $\mathbf{B}_\mathcal{F}(X)$.

Using the universal property of the normalization we conclude that $\mathbf{B}_{\mathcal{F}}(X)$ satisfy the following universal property: Any birational map $Y \xrightarrow{f} X$ from a normal variety Y to X such that induced sheaf of fractional ideals $f^*\mathcal{F}$ is invertible must factorize uniquely along $\mathbf{B}_{\mathcal{F}}(X)$.

Using this construction we present the following result.

Theorem 1.6.1. *Fix a fan Σ over $N_{\mathbb{R}}$, let $X = \mathbb{P}_\Sigma$ and $\mathcal{F}$ an equivariant sheaf of fractional ideals over X. Then $B_{\mathbb{F}}(X)$ defined above, as an allowable embedding of T, is described by the fan of $\Sigma_{\mathcal{F}}$ over $N_{\mathbb{R}}$ obtained by subdividing all the $\sigma \in \Sigma$ into the biggest polyhedra on which Ord $\mathcal{F}$ is linear.*

Proof. From the explicit construction of $f\colon \mathbf{B}_{\mathcal{F}}(X) \to X$ we conclude that $f^{-1}(T) = T \subseteq \mathbf{B}_{\mathcal{F}}(X)$ and from this $\mathbf{B}_{\mathcal{F}}(X)$ is a normal toric variety. Let Σ' denotes its associated fan, we need to prove $\Sigma' = \Sigma_{\mathcal{F}}$. For this denote $\tilde{X}$ the toric variety attached to the fan $\Sigma_{\mathcal{F}}$. By theorem 1.4.2 there is a toric morphism $g : \tilde{X} \to X$ and it is constructed in the following way.

For any $\tau \in \Sigma_{\mathcal{F}}$ there is a $\sigma \in \Sigma$ such that $\tau \subseteq \sigma$. Then $U_\tau \to U_\sigma$ is given by the inclusion of rings $R_\sigma \subseteq R_\tau$. Then if $\mathcal{F}|_{U_\sigma} = \tilde{J}_\sigma$ is generated by $J_\sigma = \bigoplus_i R_\sigma \chi_i^r$ we have that $J_\sigma \otimes_{R_\sigma} R_\tau$ is generated by $\chi^{r_i} \otimes 1$ as an R_τ-module and so we conclude

$$\mathrm{Ord}_X\ \mathcal{F} = \min\langle r_i, x\rangle = \mathrm{Ord}_{\tilde{X}} f^*(\mathcal{F}) \quad \forall x \in \tau$$

Hence by theorem 1.5.1 part IV (c) we have that $f^*(\mathcal{F})$ is an invertible sheaf. Then by the universal property of $\mathbf{B}_{\mathcal{F}}(X)$ discused above there is a morphism $\mathbf{B}_{\mathcal{F}}(X) \to \tilde{X}$ and from Theorem 1.4.2 again we conclude that Σ' is a subdivision of $\Sigma_{\mathcal{F}}$ from which $\Sigma' = \Sigma_{\mathcal{F}}$. □

This theorem together with 1.2.11 translate the problem of finding a resolution of singularities for X into a combinatorial problem. We will use this approach to prove the following.

Theorem 1.6.2. *Let $X = \mathbb{P}_\Sigma$ be a toric variety. Then there exists an equivariant sheaf of ideals $\mathcal{F} \subseteq \mathcal{O}_X$ such that $B_{\mathcal{F}}(X)$ is non singular.*

Using Theorem 1.2.11 and Theorem 1.6.1 above in order to prove this we need to find an order function $f : \mathcal{F} \to \mathbb{R}$ such that each cone in Σ_f is smooth. Also, notice that the condition $f(|\Sigma| \cap N) \subseteq \mathbb{Z}$ in the definition of order function can be relaxed to $f(|\Sigma| \cap N) \subseteq \mathbb{Q}$ since then a suitable multiple of f will be an order function and Σ_f doesn't get affected. Now the objective of this section is to construct an example of such a function.

Let us start defining for each simplicial cone, i.e, a cone of the form $\sigma = \mathrm{cone}(x_1, \ldots, x_n) \subset N_{\mathbb{R}}$ with $x_1 \ldots, x_n$ linearly independent, its multiplicity as the rank of the subgroup $\sum_i \mathbb{Z}x_i$ inside the lattice $N \cap \sum_i \mathbb{R}x_i$. This multiplicity is denoted by $\mathrm{mult}(\sigma)$ and σ is smooth iff $\mathrm{mult}(\sigma) = 1$.

The idea of the construction is to proceed by induction in the number of cones inside Σ.

We will use the following lemma for the inductive step.

Lemma 1.6.3. *Let σ be a polyhedral cone and f_0 a piecewie linear concave rational function defined over its boundary $\partial\sigma$. Let $x_0 \in N \cap int(\sigma)$.*

The function $f : \sigma \to \mathbb{R}$ defined by

$$f(\alpha x_0 + \beta y) = \alpha C + \beta f_0(y), \ \ y \in \partial\sigma, \alpha, \beta \geq 0$$

where $C \in \mathbb{Q}^+$.

Then, if C is large enough, f is concave and the associated polyhedra of f are of the form $cone(\tau \cup \{x_0\})$ for $\tau \in \Sigma_{f_0}$.

Using this result suppose we have a function $f_0 : \bigcup_{\sigma \in \Sigma \setminus \{\sigma_0\}} \to \mathbb{R}$ satisfying all properties we need over this domain. Then we can use the lemma above to extend f_0 to a function f over $|\Sigma|$ such that the cones added to Σ_{f_0} are all simplicial, let F denote the set of all this extensions.

We define the multiplicity of f as $\mathrm{mult(f)} := \max_{\tau \in \Sigma_f}(\mathrm{mult})(\tau)$, so the objective is to show there is a $f \in F$ with $\mathrm{mult}(f) = 1$.

We do this by, given an $f \in F$, constructing another function $f' \in F$ with fewer cones $\tau \in \Sigma_{f'}$ such that $\mathrm{mult}(f') = \mathrm{mult}(\tau)$. The following easy results assure that we can always do this.

Lemma 1.6.4. *Let f be a convex piecewise linear rational positive function on a polyhedron σ. Let $x_0 \in N \cap \Sigma$ and let us consider the set*

$$\overline{\Sigma} = \{\tau \in \Sigma_f \mid x_0 \notin \tau\} \cup \{cone(\tau \cup \{x_0\}) \mid x_0 \notin \tau, \tau \in \Sigma_f \text{ and } \tau \text{ is the face of some } \tau' \text{ s.t } x_0 \in \tau'\}$$

Then $\overline{\Sigma}$ is a fan with support σ. If for $\varepsilon \in \mathbb{Q}_{>0}$ we define $f_{x_0,\varepsilon}$ on σ by

$$f_{x_0,\varepsilon} = \begin{cases} f(x) & \text{if } x \in \tau \text{ and } x_0 \notin \tau, \tau \in \overline{\Sigma} \\ f(x) + \varepsilon g(x) & \text{if } x \in \tau \text{ and } x_0 \in \tau, \tau \in \overline{\Sigma} \end{cases}$$

where g is a linear function such that $g(x_0) = 1$ and $g|_{\tau_i} \cong 0$. Then for ε small enough $f_{x_0,\varepsilon}$ is still concave and its associated fan is $\overline{\Sigma}$.

Lemma 1.6.5.

1. *If τ_1 and τ_2 are two simplicial cones in $N_{\mathbb{R}}$ such that τ_1 is a face of τ_2, then $mult(\tau_1) \mid mult(\tau_2)$.*
2. *Let $\tau = \langle x_1, \ldots, x_N \rangle$ be a simplical cone such that the x_i are primitive and let $x = \alpha_1 x_1 + \cdots + \alpha_l x_l, 0 < a_i < 1$, $l \leq N$ be a primitive lattice point. For $1 \leq i \leq l$ let $\tau_i = cone(x_1, \ldots, \hat{x}_i, \ldots, x_N\rangle)$, then*

$$mult(cone)(\tau_i \cup \{x\}) = \alpha_i \cdot mult(\tau)$$

So we can conclude the induction and with this the proof of Theorem 1.6.2.

1.7 Cohomology and convexity

If X is a toric variety containing a torus T, we will denote by Σ_X its associated fan and the support of Σ_X by $|X|$. Further given a toric morphism $f : X \to Y$ we represent by $|f| : |X| \to |Y|$ the associated function between the fans.

Let $\mathcal{J}$ be a complete equivariant sheaf of fractional ideals on a toric variety X. Set $j = \text{Ord } \mathcal{J}$. Define the sheaf of (absolutely regular) differentials with coefficients in $\mathcal{J}$, written $\mathcal{K}_j$, by the formula:

$$\mathcal{K}_j(U_\sigma) = \left(\bigoplus_{\substack{r \in M \\ \langle r,x \rangle > j(x) \text{ on } \sigma \setminus \{0\}}} k\chi^r \right)^{\sim}$$

For all $\sigma \in \Sigma_X$. Clearly, $\mathcal{K}_j$ is a complete equivariant sheaf of fractional ideals.

If $\mathcal{J} = \mathcal{O}_X$, then $j = 0$ and $\mathcal{K}_0 = \hat{\hat{\Omega}}_X$ because by Theorem 1.5.1 $\hat{\hat{\Omega}}_X = \mathscr{F}_\delta$ where δ is the convex interpolation of 1 on $\text{sk}^1(\Sigma_X)$. Also, let $\mu : \overline{X} \to X$ be a toric morphism. Let $\overline{j} = j \circ |\mu|$. Then, $\mu^*(s)$ is a section of $\mathcal{K}_{\overline{j}}$ for every section s of $\mathcal{K}_j$. Furthermore, if μ is proper, we have $\mu_* \mathcal{K}_{\overline{j}}$ is exactly $\mathcal{K}_j$.

Let $\mathcal{J}$ be a equivariant sheaf of fractional ideals on a toric variety X. As $\mathcal{J} = \bigoplus_\chi \mathcal{J}^\chi$, we have that $H^i(X, \mathcal{J}) = \bigoplus_\chi H^i(X, \mathcal{J}^\chi)$. Obviously, $H^i(X, \mathcal{J}^\chi) = H^i(X, \mathcal{J})^\chi$. Therefore, the cohomology $H^i(X, \mathcal{J})$ can be computed by

Theorem 1.7.1. *For any character χ of T, we have*

$$H^i(X, \mathcal{J})^\chi \simeq H^i_A(|X|, k)$$

Where the right hand side is the relative cohomology with respect to a closed subset A of $|X|$ that is either:

a) *$A = |U|$ where U is the largest T-invariant open in X such that $\chi \in \mathcal{J}(U)$*

b) *in case $\mathcal{J}$ is complete.*

$$A = \{x \in |X| \mid \langle \chi, x \rangle \geq Ord\ \mathcal{J}(x)\}.$$

Furthermore, if j =Ord($\mathcal{J}$), the cohomology of the differentials $\mathcal{K}_j$ is given by the formula

$$H^i(X, \mathcal{K}_j)^\chi \simeq H^i_{B^*}(|X|, k)$$

Where $B = \{x \in |X| : \langle \chi, x \rangle \geq j(x)\}$, $B^* = B - \bigcap($*open unit ball in* $N_\mathbb{R})$

Proof.
I) Assume X is affine. Then by Serre's Vanishing Theorem and by an explicit computation we have

$$H^i(X, \mathcal{J}) = 0 \text{ for } i > 0 \text{ and}$$

$$H^0(X, \mathcal{J})^\chi = \begin{cases} k\chi, & \text{if } \chi \in \mathcal{J}(X), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, both of the opens $|X| - A$ are convex. $|X| - A$ is empty if and only if $\chi \in \mathcal{J}(X)$. The long exact sequence

$$0 \to H^0_A(|X|, k) \to H^0(|X|, k) \to H^0(|X| - A, k) \to H^1_A(|X|, k) \to \cdots$$

has zero at the dots because $|X|$ and $|X| - A$ are conctractible. therefore, $H^i_A(|X|, k) = 0$ for $i > 0$ and $H^0_A(|X|, k) = \begin{cases} k, & \text{if } A = |X|, \\ 0 & \text{otherwise.} \end{cases}$

This completes the affine case of the first isomorphism.

II) Let $\{U_i\}$ be a covering of X by T-invariant open affines. Let σ_i be the simplex $|U_i|$. Intersections of U_i's are open affines and these correspond to intersections of the σ_i's. By part I, $H^i_{\bigcap(\sigma_j \cap A)}(\bigcap \sigma_j, k) = 0$ if $i > 0$. Hence by Leray's theorem,

$$H^i_A(|X|, k) \simeq \hat{H}^i_A(\{\sigma_j\}, k)$$

Also Serre's theorem says that

$$H^i(X, \mathcal{J})^\chi \simeq \hat{H}^i(\{U_j\}, \mathcal{J})^\chi$$

Again by part I, the two Čech complexes

$$\hat{C}^i_A(\{\sigma_j\}, k) \text{ and } \hat{C}^i(\{U_j\}, \mathcal{J})^\chi$$

Are isomorphic. Thus

$$\hat{H}^j_A(\{\sigma_i\}, k) \simeq \hat{H}^j(\{U_i\}, \mathcal{J})^\chi$$

This completes the proof of the first isomorphism. The proof of the second isomorphism is formally the same. □

Let $\phi : X \to Y$ be a proper toric morphism. Let $\mathcal{J} = \mathfrak{F}_f$ be a complete sheaf on Y for some order function $f : |Y| \to \mathbb{R}^+$ and denote $g = f \circ |\phi|$. Then we have a ϕ-homomorphism $\mathfrak{F}_f \to \mathfrak{F}_g$.

Corollary 1.7.2. *In the above situation,*

a) $H^i(X, \mathfrak{F}_g) \to H^i(Y, \mathfrak{F}_f)$ *is an isomorphism for all* i.

b) $H^i(X, \mathcal{F}_g) \to H^i(Y, \mathcal{K}_f)$ *is an isomorphism for all* i

c) $R^i\phi_*\mathfrak{F}_g = 0$ *for all* $i > 0$ *and* $\phi_*\mathfrak{F}_g \simeq \mathfrak{F}_f$

d) $R^i\phi_*\mathcal{K}_g = 0$ *for all* $i > 0$ *and* $\phi_*\mathcal{K}_g \simeq \mathcal{K}_f$.

Proof. $|\phi| : |X| \to |Y|$ is a homeomorphism. The second A and the B in the theorem for X and Y are equivalent under $|\phi|$. The theorem says that the cohomology groups are topological invariants of X, A and B. This proves a) and b). Statements c) and d) follow formally from a) and b) when Y is affine. □

Corollary 1.7.3. *Let X be a toric variety and $\mathcal{J}$ be a complete equivariant sheaf of fractional ideals. Set f =Ord $\mathcal{J}$. Assume $|X|$ is convex and f is convex. Then, $H^i(X, \mathcal{J})$ is zero for all $i > 0$.*

Proof. For a given character χ, let A be given by part b) of Theorem 1.7.1. By assumption, $|X|$ and $|X| - A$ are contractible is $|X| \neq A$. The conclusion follows from the long exact sequence of the relative cohomology used in the proof of the theorem. □

First, let's do a numerical corollary of the theorem.

Corollary 1.7.4. *Let $\mathcal{J} = \mathfrak{F}_f$ be a complete equivariant sheaf of ideals on an affines toric variety. Assume that the support of $\mathcal{O}_X/\mathcal{J}$ is the zero-dimensional orbit. Then, the function* dim $\Gamma(X, \mathcal{O}_X/\mathfrak{F}_{nf})$ *is a polynomial for all integers $n \neq 0$. the degree of this polynomial is the dimension of X.*

Proof. Let k be the dimension of X. Let Y be the normalization of X after $\mathcal{J}$ is blown up. The morphism $\mu : Y \to X$ is proper. We have $\mu^*\mathfrak{F}_{nf} \simeq (\mu^*\mathfrak{F}_f)^{\otimes n}$ as $\mu^*\mathfrak{F}_f \equiv \mathcal{Q}$ is an invertible ideal. Let F be the closed subscheme of Y defined by $\mathcal{Q}$. Set $K = \mathcal{Q}/\mathcal{Q}^2$,which is an ample invertible sheaf of F. As F is projective of dimension $k - 1$,

$$\chi(F, K^{\otimes n}) = \sum (-1)^i h^i(F, K^{\otimes n})$$

is a polynomial for all n and has degree $k - 1$ by Serre's form of Hilbert's theorem.

Claim $\chi(F, K^{\otimes}) = \dim \Gamma(X, \mathfrak{F}_{nf}/\mathfrak{F}_{(n+1)f})$ for $n \geq 0$. This claim implies the corollary. For,

$$\dim \Gamma(X, \mathcal{O}_X/\mathfrak{F}_{nf}) = \sum_{n>i\geq 0} \dim \Gamma(X, \mathfrak{F}_{if}/\mathfrak{F}_{(i+1)f}) = \sum_{n>i\geq 0} \chi(F, K^{\otimes i})$$

To prove the claim, consider the direct images by μ of the exact sequence

$$0 \to \mathcal{Q}^{\otimes n+1} \to \mathcal{Q}^{\otimes n} \to K^{\otimes n} \to 0$$

By the first corollary, $\mathcal{Q}^{\otimes n}$'s have no higher direct images. Hence $K^{\otimes n}$ has none. Furthermore, the sequence

$$0 \to \underbrace{\mu_*(\mathcal{Q}^{\otimes n+1})}_{=\mathfrak{F}_{(n+1)f}} \to \underbrace{\mu_*(\mathcal{Q}^{\otimes n})}_{=\mathfrak{F}_{nf}} \to \mu_*(K^{\otimes n}) \to 0$$

is exact. Thus, by Leray's spectral sequence,

$$H^i(F, K^{\otimes n}) \simeq H^i(X, \mathfrak{F}_{nf}/\mathfrak{F}_{(n+1)f})$$

These last group are zero if $i > 0$. Therefore,

$$\chi(F, K^{\otimes n}) = h^0(F, K^{\otimes n}) = \dim \Gamma(X, \mathfrak{F}_{nf}/\mathfrak{F}_{(n+1)f})$$

□

Before studying the singularities of toric varieties further, let's consider some of the geometric meaning of the convexity condition in corollary 1.7.2

Lemma 1.7.5. *Let X be a toric variety.*

a) $|X|$ is convex if and only if X is proper over an affine.

b) A complete T-sheaf of fractional ideals on X, $\mathcal{J} = \mathfrak{F}_f$,is generated by its global sections if and only if $f = Inf_{\chi \geq f on |X|} \chi$ In this case f is convex.

Proof. a) let $\lambda : \mathbb{G}_m \to T$ be a 1-P.S. of T. Assume that X is proper over an affine. Then, $\lim_{t\to 0} \lambda(t)$ exists in X if and only if, for all $f \in \Gamma(X, \mathcal{O}_X)$, $f \circ \lambda$ is regular at zero.

Therefore $\lambda \in |X| \cap N$ if and only if $< \alpha, \lambda >\geq 0$ for all $\alpha \in |X|^\vee \cap M$. Therefore $|X| = |X|^{\vee\vee}$, hence $|X|$ is convex. To prove the converse, let

$$Y = Speck[\ldots, \chi^\alpha, \ldots]_{\alpha \in M \cap |X|^\vee}$$

and note that there is a caonical map $f : X \to Y$ It follows easily that f is proper using the valuative criterion.

b) Obviously □

Theorem 1.7.6. *Let X be a toric variety and $\mathcal{L} = \mathfrak{F}_f$ be an invertible equivariant sheaf. Then, $\mathcal{L}$ is ample if and only if f is strictly concave in the sense that for each $\sigma \in \Sigma_X$ there exists a character χ and positive integer n, for which*

a) $\chi \geq nf$ *on* $|X|$

b) $\sigma_\alpha = \{x \in |X| : \langle \chi, x \rangle = nf(x)\}$

Proof. $\mathcal{L}$ is ample if and only if there is an $n > 0$ and characters $\{\chi_\beta\}$ such that a) the χ_β's are sections of $\mathcal{L}^{\otimes(n_\beta)}$, form an affine cover of x. But then any invariant affine U must be contained in one of the U_β's hence there will also be a section $\chi' \circ \chi_\beta^m$ of $\mathcal{L}^{\otimes n'}$ such that U is the open set where this generates $\mathcal{L}^{\otimes n'}$. Since the subsets $|U|$, and σ in $|X|$ are the same, the theorem is a direct translation of this criteion. □

1.8 Generalities on rational Resolutions

Let $f : X \to S$ be a proper morphism between smooth varieties. Denote the dimensions of X and S by x and s. Let $\mathcal{L}$ be an invertible sheaf on X. A special case of the duality theorem for the proper morphism f is

Theorem 1.8.1. *Assume $R^i f_*(\omega_X^X \otimes \mathcal{L}^{\otimes -1}) = 0$ for $i > 0$. Then there are natural isomorphisms*

$$R^i f_* \mathcal{L} \simeq \mathcal{E}xt^{s-x+i}_{\mathcal{O}_S}(f_*(\Omega_X^X \otimes \mathcal{L}^{\otimes -1}), \Omega_S^S)$$

Recall that a coherent sheaf F on S is called Cohen-Macaulay of pure dimension k if and only if $\mathcal{E}\mathrm{xt}^{s-j}_{\mathcal{O}_S}(F, \Omega_S^S)$ is zero unless $j = k \equiv$dimension of the support of F. This fact together with the theorem immediately implies

Corollary 1.8.2. *Assume $R^i f_*(\omega_X^X \otimes \mathcal{L}^{\otimes -1})$ and $R^i f_* \mathcal{L}$ are zero for $i > 0$. Then $f_*(\omega_X^X \otimes \mathcal{L}^{\otimes -1})$ and $f_* \mathcal{L}$ are Cohen-Macaulay of pure dimension x. In fact, $\mathcal{E}xt^{s-j}_{\mathcal{O}_S}(-, \Omega_S^S)$ interchanges them.*

Let's call one of these shaves the Ext-dual of the other.

Let $g : X \to Y$ be a proper birational morphism with X smooth. Call g a resolution (of the singularities) of Y. Define such a morphism g a rational resolution if

a) Y es normal; i.e., $\mathcal{O}_Y \to g_* \mathcal{O}_X$ is an isomorphism,

b) $R^i g_* \mathcal{O}_X$ is zero for all $i > 0$,

c) $R^i g_* \omega_X^X$ is zero for all $i > 0$

Remark 1.8.3. Condition c) is always satisfied in characteristic zero by a generalization of Kodaira's vanishing theorem. //

Consider the two more conditions on the singularities of Y

d) $\mathcal{O}_Y$ is Cohen-Macaulay,

e) The natural homomorphism $g_* \Omega_X^X \to \omega_Y$ is a isomorphism where ω_Y is the sheaf on Y, which is isomorphic to $\mathcal{E}\mathrm{xt}^{s-x}_{\mathcal{O}_S}(\mathcal{O}_Y, \mathcal{O}_S)$ if Y were embedded in a smooth S. Also, if Y es a normal variety, ω_Y is isomorphic to the double dual of the highest differentials, Ω_Y^Y

Proposition 1.8.4. *Assuming the above conditions c), we have that a) and b) are equivalent to d) and e).*

Proof. The problem is local on Y. So, assume Y is embedded of a smooth variety S by i. Set $f = i \circ g$. As condition c) is verified, the theorem gives us isomorphisms

$$i_* R^i g_* \mathcal{O}_X \simeq \mathcal{E}\mathrm{xt}^{s-x-i}_{\mathcal{O}_S}(f_* \Omega_X^X, \Omega_S^S) \qquad (*)$$

Thus, b) is true if and only if $f_* \Omega_X$ or $g_* \Omega_X$ is Cohen-Macaulay. If a) and b), we have that $\mathcal{O}_Y \simeq g_* \mathcal{O}_X$ is the Ext-dual of the Cohen-Macaulay sheaf $f_* \Omega_X^X$. Thus, $\mathcal{O}_Y$ is Cohen-Macaulay and the homomorphism in e) is the Ext-dual of the isomorphism $\mathcal{O}_Y \to g_* \mathcal{O}_X$. Hence, a) + b) $\Longrightarrow$ d) + e)

Assume d) and e) are verified. Then, $\mathcal{O}_Y$ is Cohen-Macaulay with dual Cohen-Macaulay sheaf ω_Y by d). By e), we have that $g_*\Omega^X_X$ is Cohen-Macaulay. Equation $(*)$ implies condition b) and that $g_*\mathcal{O}_X$ is the Ext-dual of $g_*\mathcal{O}_X$. Further,the Ext-dual of the homomorphism $\mathcal{O}_Y \to g_*O_X$ is an isomorphism. Hence, d) + e) $\implies$ a) + b).

□

The forward implication gives an easy way to check Cohen-Macaulay-ness and to find Ext-duals. In characteristic zero, when Y is normal, Grauert and Riemenscheider have given an intrinsic characterization of the sheaf $g_*\Omega^X_X$ which is independent of the resolution g. In this case, the conditions d) and e) essentially deal only with Y. One says that Y has rational singularities if these conditions d) and e) are verified.

Returning to toric variety, we can apply the above theory to prove

Theorem 1.8.5. *a) Any toric variety Y is Cohen-Macaulay.*

b) ω_Y is $\hat{\hat{\Omega}}_Y$.

c) Any proper T-invariant resolution of the singularities of Y is rational.

Proof. By Theorem 1.6.2, there is a proper toric morphism $f : X \to Y$ with X smooth. By Corollary 1 of Theorem 1.7.1, the conditions b) and c) in the definition of rational resolutions are verified. Thus, the third statement in the theorem follows because we are dealing only with normal toric varieties. The last general theorem shows that the first two statement follows from the third. □

Chapter 2

The Semistable Reduction Theorem

We have 2 goals in this chapter: the first is to generalize the concept of toric variety of Chapter I to the concept of *toroidal variety.* This are embeddings of non-singular varieties U in normal varieties X such that formally at each $x \in X \setminus U$ the pair (X, U) is isomorphic to $\left(\overline{T}, T\right)$ for some torus T and equivariant $T \subseteq \overline{T}$. This allows us to apply an analysis similar to that of Chapter I to a much greater range of examples. In particular we want then to apply the theory to prove the famous:

Theorem 2.0.1 (Semi-stable reduction theorem)**.** *Assume char$(k) = 0$. Let C be a non-singular curve, $0 \in C$ a point and $f : X \longrightarrow C$ a morphism of a variety X onto C such that*

$$\textit{res } f : X \setminus f^{-1}(0) \longrightarrow C \setminus \{0\}$$

is smooth. Then there is a finite morphism $\pi : C' \longrightarrow C$ with C' non-singular, $\pi^{-1}(0) = \{0'\}$ and a proper morphism p as follows

$$\begin{array}{ccccc} & & & \overset{p}{\swarrow} & X' \\ X & \longleftarrow & X \times_C C' & & \\ \downarrow f & & \downarrow & \swarrow f' & \\ C & \overset{\pi}{\longleftarrow} & C' & & \end{array}$$

such that

1. *p is an isomorphism over $C' \setminus \{0'\}$.*
2. *p is projective; in fact p is obtained by blowing up a sheaf of ideals $\mathscr{I}$ with $\mathscr{I}|_{C' \setminus \{O'\}} \cong \mathcal{O}_{X \times_C C'}|_{C' \setminus \{0'\}}$.*
3. *X' is non-singular, and the fibre $f'^{-1}(0')$ is reduced, with non-singular components crossing normally.*

As we will see, everything here is an easy consequence of Hironaka's resolution theorems except for "reduced".

2.1 Toroidal Varieties

Definition 2.1.1. Let X be a normal variety of dimension n, U a smooth Zariski open set of X. We say that $U \subseteq X$ is a toroidal variety if for every closed point x in X there exists an n-dimensional torus T, an affine toric variety X_σ containing T, a point t in X_σ and an isomorphism of k-local algebras

$$\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{X_\sigma,t}$$

such that the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal of $X \setminus U$ corresponds under this isomorphism to the ideal in $\hat{\mathcal{O}}_{X_\sigma,t}$ generated by the ideal of $X_\sigma \setminus T$. //

By restricting X_σ if necessary, we can assume that the orbit of t is closed in X_σ: such an affine toric variety plus a formal isomorphism as above will be called a local model at x.

Notice that it follows formally from this definition and from the fact that $X_\sigma \setminus T$ is purely 1-codimensional, that $X \setminus U$ is also 1-codimensional (see [Gro67] Theorem 7.1.3): we shall write

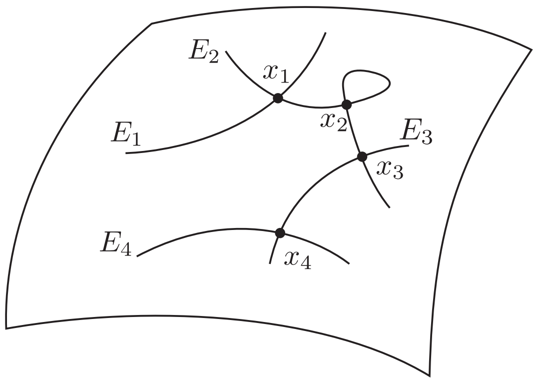
$$X \setminus U = \bigcup_{i \in I} E_i,$$

where the E_i's are irreducible subvarieties of dimension $n-1$.

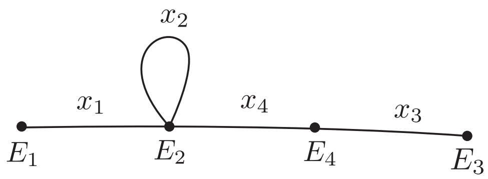
To get an idea of this definition, let us assume for a moment that dim $X = 2$: in that case the E_i's are curves and let x be a point in $\bigcup\limits_{i \in I} E_i$. Then:

either a. x is a non-singular point of $\bigcup\limits_{i \in I} E_i$: i.e., x is in only one E_i and is non-singular on it. Then in our local model t is in a dimension 1 orbit, X_σ is non-singular at t, hence X is non-singular at x,

or b. x is singular on $\bigcup\limits_{i \in I} E_i$: then $\{t\}$ is a dimension 0 orbit, $(X_\sigma \setminus T)_{\text{red}}$ has an ordinary double point at t, hence x either (b_1) belongs to two components E_{i_1} and E_{i_2} which are non-singular at x and meet transversaly, or (b_2) belongs to one E_i with an ordinary double point at x. Moreover the singularity of X at x is quite elementary: i.e., formally $X \cong \mathbb{A}^2/(\text{cyclic gp.})$ which the two branches of $\bigcup\limits_{i \in I} E_i$ being the image of the two coordinate axes.



We can associate to this situation a graph, the *dual graph* of the configuration, by attaching a vertex to each E_i, an edge between two vertices corresponding to a singularity of type (b_1) and a loop through a vertex corresponding to a singularity of type (b_2)



We define also a *stratification* [1] on X as follows:

1. the set U
2. the connected components of the sets $E_i \setminus \left\{\text{double points of } \bigcup_{i \in I} E_i\right\}$

[1] a *stratification* on a variety X is a finite set $S_1, S_2, \ldots, S_n$ of locally closed subsets, called the strata, such that every point of X is in exactly one stratum and such that the closure of a stratum is a union of strata.

If Y is a stratum, we define

$$\text{Star } Y = \bigcup_{\substack{\text{Strata } Z \\ \text{s.t } Y \subseteq \overline{Z}}} Z = X \setminus \bigcup_{\substack{\text{Strata } Z \\ \text{s.t } \overline{Z} \cap Y = \emptyset}} Z$$

3. the double points of $\bigcup_{i \in I} E_i$

Now we can go back to our general situation described in the Def. 1: we shall restrict ourselves to the case where E_i are normal varieties and we shall say in this case that $U \subseteq X$ is a toroidal variety without self-intersection.

Proposition 2.1.2. *Let $U \subseteq X$ be a toroidal variety without self-intersection and $X \backslash U = \bigcup_{i \in I} E_i$. Then for any subset $J \subseteq I$,*

- $\bigcap\limits_{j \in J} E_j$ *is normal*
- $\left(\bigcap\limits_{j \in J} E_j\right) \setminus \left(\bigcup\limits_{i \notin J} E_i\right)$ *is non-singular.*

The components of the sets $\left(\bigcap\limits_{j \in J} E_j\right) \setminus \left(\bigcup\limits_{i \notin J} E_i\right)$ define a stratification of X; moreover the components of $\bigcap\limits_{i \in J} E_i$ are the closure of the strata. Finally, if $x \in X$ and (X_σ, t) is a local model at x, then the closures of $\overline{Z}$ of the strata Z, with $x \in \overline{Z}$, correspond formally to the closures of the orbits in X_σ; in particular if $x \in$ Stratum Y, then Y corresponds formally to $\mathcal{O}(t)$ itself.

Proof. Let $O_{\nu'}, \nu \in I'$, be the orbits of codimension 1 in X_σ, corresponding to the vertices of σ by the Orbit-Cone Correspondence. Then the sets $\overline{\mathcal{O}_\nu}$ are normal and if we denote $\widehat{\overline{\mathcal{O}_\nu}}$ the inverse image of $\overline{\mathcal{O}_\nu}$ in Spec $\widehat{\mathcal{O}}_{X_\sigma,t}$, then $\widehat{\overline{\mathcal{O}_\nu}}$ is still reduced, irreducible, and normal; hence are the analytics branches of $X_\sigma \setminus T$.

On the other, let $I_X = \{i \in I \mid x \in E_i\}$. Then since the E_i are assumed normal, $\widehat{E_i}, i \in I$, are the analytic branches of $X \setminus U$ at x. Thus, the formal isomorphism

$$\operatorname{Spec} \widehat{\mathcal{O}}_{X,x} \xrightarrow{\approx} \operatorname{Spec} \widehat{\mathcal{O}}_{X_\sigma,t},$$

induces an isomorphism of I_X and I' under which the $\widehat{E_i}$ correspond to the $\widehat{\mathcal{O}_\nu}$. Now, if J is any subset of I_X and J corresponds to $J' \subseteq I'$, let τ be the smallest face of σ containing the vertices $\nu, \nu' \in J'$. Then it follows from the Orbit-Cone Correspondence that

$$\overline{\mathcal{O}^\tau} = \bigcap_{\nu \in J'} \overline{\mathcal{O}_\nu}.$$

Since we know $\overline{\mathcal{O}^\tau}$ is normal, it follows that $\widehat{\overline{\mathcal{O}^\tau}}$ is normal and corresponds formally to $\bigcap_{i \in J} \widehat{E_i}$. Therefore $\bigcap_{i \in J} E_i$ is normal at x. Since x is arbitrary, $\bigcap_{i \in J} E_i$ is normal. Moreover, if we take $J = I_X$ and $J' = I'$, then

$$\mathcal{O}(t) = \bigcap_{\nu \in I'} \overline{\mathcal{O}_\nu}$$

and $\mathcal{O}(t)$ is non-singular. It follows then that $\bigcap_{i \in I_X} E_i$ is non-singular at x. Since x is an arbitrary point of $\left(\bigcap_{i \in I_X} E_i\right) \setminus \left(\bigcup_{i \notin I_X} E_i\right)$, it folows that this set is non-singular.

Call the components of the set $\left(\bigcap_{i \in J} E_i\right) \setminus \left(\bigcup_{i \notin J} E_i\right)$ strata. Obviously X is the disjoint union of these sets and their closures are just tje components of the sets $\bigcap_{i \in J} E_i$ and we have checked that these do correspond formally to the closure orbits. It remains to check the axiom of the frontier: if Y, Z are strata, $x \in Y \cap \overline{Z}$, then $Y \subseteq \overline{Z}$. But it suffices to prove that

$$I(Y)\widehat{\mathcal{O}}_{X,x} \supseteq I(\overline{Z})\widehat{\mathcal{O}}_{X,x}. \tag{2.1}$$

In our local model, Y corresponds to $\mathcal{O}(t)$ and $\overline{Z}$ to $\overline{\mathcal{O}^\tau}$, some face τ of σ. Since $\mathcal{O}(t) \subseteq \overline{\mathcal{O}^\tau}$, it follows that

$$I(\mathcal{O}(t))\widehat{\mathcal{O}}_{X_\sigma,t} \supseteq I(\overline{\mathcal{O}^\tau})\widehat{\mathcal{O}}_{X_\sigma,t}$$

which proves (2.1) □

Remark 2.1.3. If we do not assume that $U \subseteq X$ is without self-intersection, it is still possible to define a stratification such that for $x \in X$, the stratum containing x is formally isomorphic to the orbit of t as follows:

If x is r_i-fold of E_i, $i \in I_x, 1 \le r_i < \infty$ then the stratum containing x is the connected component through x of the subset of $\left(\bigcap_{j\in J} E_j\right) \setminus \left(\bigcup_{i\notin J} E_i\right)$ where the multiplicity along each E_i is equal to r_i.

//

Definition 2.1.4. Let Y be a stratum.

$$\begin{aligned} M^Y &= \text{Group of Cartier divisors on Star}(Y)\text{, supported on Star}(Y) \setminus U \\ M^Y_{\mathbb{R}} &= M^Y \otimes \mathbb{R} \\ N^Y &= \text{Hom}\big(M^Y, \mathbb{Z}\big) \\ N^Y_{\mathbb{R}} &= N^Y \otimes \mathbb{R} = \text{Hom}\big(M^Y_{\mathbb{R}}, \mathbb{R}\big) \\ M^Y_+ &= \text{sub-semigroup of } M^Y \text{ of effective divisors} \\ \sigma^Y &= \big\{x \in N^Y_{\mathbb{R}} \mid \langle D, x\rangle \ge 0 \quad \forall D \in M^Y_+\big\} \subseteq N^Y_{\mathbb{R}}. \end{aligned}$$

//

If $\text{Star}(Y) \setminus U = \bigcup_{i\in J} E_i$, then M^Y is a subgroup of the free abelian group of Weil divisors $\sum_{i\in J} n_i E_i$ on Star(Y). Thus M^Y and hence N^Y are finitely generated free abelian groups, and $M^Y_{\mathbb{R}}, N^Y_{\mathbb{R}}$ are dual finite-dimensional real vector spaces. Moreover M^Y_+ is clearly the intersection of M^Y with the convex rational polyhedral cone in $M^Y_{\mathbb{R}}$ and σ^Y is the dual cone in $N^Y_{\mathbb{R}}$. We can relate the M, N and σ's to our models via:

Lemma 2.1.5. *Let x be a closed point of Y and (T, X_σ, t) a local model at x. Let E_i correspond to the codimension 1 orbit $\mathcal{O}_i \subseteq X_\sigma$, and hence to the face $\mathbb{R}^+ \cdot v_i \subseteq \sigma$. Then the following are equivalent:*

1. *$\sum n_i E_i$ is a Cartier divisor on Star(Y)*
2. *$\sum n_i E_i$ has a local equation at x*
3. *$\sum n_i \overline{\mathcal{O}_i}$ has a local equation at t*
4. *$\sum n_i \mathcal{O}_i$ is a Cartier divisor on X_σ*
5. *$\sum n_i \overline{\mathcal{O}_i} = (\mathfrak{X}^r)$ for some $r \in M(T)$.*

Proof. The equivalence $(ii) \iff (iii)$ follows because an ideal I in a Noetherian local ring $\mathcal{O}$ is principal if and only if $I \cdot \widehat{\mathcal{O}}$ is principal.

Now $(v) \implies (iv) \implies (iii)$ and $(i) \implies (ii)$ are obvious. Next start with any Weil divisor $D = \sum n_i \overline{\mathcal{O}_i}$. Then $\{y \in X_\sigma \mid D \text{ has a local equation at } y\}$ is an open T-invariant set V. If $t \in V$, it follows that V contains all orbits $\mathcal{O}$ such that $t \in \overline{\mathcal{O}}$, i.e., $V = X_\sigma$. Thus $(iii) \implies (iv)$. $(iv) \implies (v)$ [See Theorem 1.4.3]. Finally to see $(ii) \implies (i)$, it suffices to show, for all closed points $x' \in Y$,

$$\sum n_i E_i \text{ has a local equation at } x' \iff \sum n_i E_i \text{ has a local equation at } \eta_Y$$

where η_Y is the generic point of Y. Again $(\implies)$ is obvious. To prove $(\impliedby)$, we may as well rename $x' = x$, and use the local model (T, X_σ, t). Let η_0 a generic point of $\mathcal{O}(t) \subseteq X_\sigma$. Then localizing $\widehat{\mathcal{O}}_{X,x} \xrightarrow{\approx} \widehat{\mathcal{O}}_{X\sigma,t}$, with respect to the prime ideals of Y, $\mathcal{O}(t)$ resp., one checks that

$\sum n_i E_i$ has a local equation at η_Y if and only if $\sum n_i \overline{\mathcal{O}_i}$ has a local equation at η_0. By the same argument as above, $\sum n_i \overline{\mathcal{O}_i}$ has a local equation at η_0 implies $\sum n_i \overline{\mathcal{O}_i}$ is a Cartier divisor on X_σ, hence $\sum n_i E_i$ has a local equation at x. □

Corollary 2.1.6. *There are canonical isomorphisms:*

1. $M^Y \cong M(T)/\{r \mid r \equiv 0 \text{ on } \sigma\}$

2. $N^Y_{\mathbb{R}} \cong \mathit{Vect.span}_{N(T)_{\mathbb{R}}}(\sigma)$

3. $\sigma^Y \cong \sigma$.

Proof. Define the map

$$\begin{array}{rcl} M(T) & \longrightarrow & M^Y \\ x & \longmapsto & \sum n_i E_i \qquad \text{if } (\mathfrak{X}^r) = \sum n_i \overline{\mathcal{O}_i}. \end{array}$$

By Lemma 2.1.5 it is surjective and the kernel is just $\{r \mid (\mathfrak{X}^r) = 0 \text{ or } \mathfrak{X}^r \in \Gamma(\mathcal{O}^*_{X_\sigma})\} = \{r \mid r \equiv 0 \text{ on } \sigma\}$. This implies (i) and (ii). From the obvious fact that $\sum n_i E_i$ is effective if and only if $\sum n_i \overline{\mathcal{O}_i}$ is effective, it follows that in this map, M^Y_+ is the image of $\check{\sigma} \cap M(T)$, hence in the map of (ii), σ^Y corresponds to σ. □

Corollary 2.1.7.

1. *If Z is a stratum in Star(Y), then there exists a positive Cartier divisor D on Star(Y) such that Star$(Z) =$ Star$(Y) \setminus$ Supp(D).*

2. *If D is a positive Cartier divisor on Star(Y), then Star(Y) – Supp D = Star(Z) for some stratum $Z \subseteq$ Star(Y)*

Proof.

1. Let $\overline{Z}$ correspond formally to $\overline{\mathcal{O}^\tau}$ via the isomorphism

$$\widehat{\mathcal{O}}_{X,x} \xrightarrow{\approx} \widehat{\mathcal{O}}_{X\sigma,t}.$$

   There exists $r \in M(T)$ such that $r \geq 0$ on σ, and $\tau = \{\sigma \cap x \mid r(x) = 0\}$. Then $(X_\sigma)_{\mathfrak{X}^r} \cong X_\tau$, i.e., it is the open subset consisting of orbits $\mathcal{O}^{\tau'}$ for all faces τ' of τ. Let $(\mathfrak{X}^r) = \sum n_i \overline{\mathcal{O}_i}$ and let $D = \sum n_i E_i$ be the corresponding divisor in Star(Y). Star$(Y) \setminus$ Supp(D) is a union of various strata Z', and if $\overline{Z'}$ corresponds formally to $\overline{\mathcal{O}^{\tau'}}$, then

$$\begin{aligned} Z' \subseteq \text{Star}(Y) \setminus \text{Supp } D &\Longleftrightarrow \overline{Z'} \not\subseteq \text{Supp } D \\ &\Longleftrightarrow \overline{\mathcal{O}^{\tau'}} \not\subseteq \text{Supp}\,(\mathfrak{X}^r) \\ &\Longleftrightarrow \overline{\mathcal{O}^{\tau'}} \supseteq \overline{\mathcal{O}^\tau} \\ &\Longleftrightarrow \overline{Z'} \supseteq Z \\ &\Longleftrightarrow Z' \subseteq \text{Star}(Z). \end{aligned}$$

2. If $D = \sum n_i E_i$, let $\sum n_i \overline{\mathcal{O}_i} = (\mathfrak{X}^r)$. Since $n_i \geq 0$, $\mathfrak{X}^r \in \Gamma(\mathcal{O}_{X_\sigma})$ and $r \geq 0$ on σ. Let

$$\tau = \sigma \cap \{x \mid r(x) = 0\},$$

   and let $\overline{\mathcal{O}^\tau}$ correspond formally to $\overline{Z}$ for some stratum $Z \subseteq$ Star(Y). Then the same argument as in the previous item, shows that Star$(Y) \setminus$ Supp D = Star(Z).

□

Now we saw in Proposition 2.1.2 that there is a bijection between the strata in Star(Y) and orbits in X_σ. This now induces further bijections:

$$\begin{aligned}\{\text{Strata in Star}(Y)\} &\cong \{\text{Orbits in } X_\sigma\}\\ &\cong \{\text{Faces of } \sigma\}\\ &\cong \left\{\text{Faces of } \sigma^Y\right\}\end{aligned}$$

We can identify this bijection intrinsically on X without use of local models as follows:

1. If $Z_i = E_i \setminus \left(\bigcup_{j\neq i} E_j\right)$ is a codimension 1 stratum in Star(Y), and E_i corresponds formally to $\overline{\mathcal{O}_i} \subseteq X_\sigma$, then $\overline{\mathcal{O}_i}$ corresponds to the ray $\mathbb{R}\cdot v_i \subseteq N(T)$ where

$$\begin{aligned} v_i : M(T) &\longrightarrow \mathbb{Z}\\ r &\longmapsto \left(\text{order of vanishing of } \mathfrak{X}^r \text{ on } \overline{\mathcal{O}_i}\right).\end{aligned}$$

   Therefore the above bijection takes Z_i to the linear function:

$$e_i : \left(\sum_{i\in J} n_i E_i\right) \longmapsto n_i.$$

   Note that by definition

$$M^Y_+ = \left\{x \in M^Y \mid e_i(x) \geq 0 \quad \forall i\right\},$$

   hence σ^Y is the span of the vectors e_i's.

2. If $Z \subseteq \text{Star}(Y)$ is any stratum, then

$$Z = \left[\bigcap_{i\in K} E_i - \bigcup_{i\notin K} E_i\right] \cap \text{Star}(Y)$$

   where $K = \{i \mid Z \subseteq E_i\}$. Then if Z corresponds to τ, for all i,

$$Z \subseteq E_i \iff \tau \supseteq \mathbb{R}^+ \cdot e_i.$$

   Therefore τ is the face of σ spanned by $\{e_i\}_{i\in K}$.

Definition 2.1.8.

$$\text{R.S}^U(X) \doteq \{\lambda : \operatorname{Spec} k[[t]] \longmapsto X \mid \lambda \text{ is a } k\text{-morphism and } \lambda(\eta) \in U \text{ for } \eta \text{ the generic point}\}$$

//

R.S. is short for "Riemann Surface" as used, more or less, by Zariski (in, for example, [ZS60] page 110). Notice that we have a pairing:

$$\begin{aligned}\langle\cdot,\cdot\rangle : \text{R.S}^U(\text{Star}(Y)) \times M^Y &\longrightarrow \mathbb{Z}\\ (\lambda, D) &\longmapsto \langle\lambda, D\rangle \doteq \text{Ord}_0\left(\lambda^{-1}D\right),\end{aligned}$$

where Ord_O is the order of vanishing at the closed point 0 of a divisor. This pairing dualizes to a map:

$$\text{Ord} : \text{R.S}^U(\text{Star}(Y)) \longrightarrow N^Y$$

and since $\langle \lambda, D \rangle \geq 0$ if D is effective, Ord factors through σ^Y:

$$\text{Ord} : \text{R.S}^U(\text{Star}(Y)) \longrightarrow \sigma^Y \cap N^Y$$

This is the non-linear analog of the interpretation of $N(T)$ as the 1-P.S.'s of T and of $\sigma \cap N(T)$ as the 1-P.S.'s which extend to morphisms $\lambda : \mathbb{A}^1_k \longrightarrow X_\sigma$. We will see below that, in fact, $\sigma^Y \cap N^Y$ correspond to the image of Ord. For the moment, notice that:

$$\left(\text{Int } \sigma^Y\right) \cap N^Y \subseteq \text{Image(Ord)} \tag{2.2}$$

In fact, taking a closed point $x \in Y$ and a local model (T, X_σ, t), we may assume that t is the distinguished point in its orbit in the sense of 1.3.2, i.e., for each $a \in (\text{Int}(\sigma)) \cap N(T)$, $\lambda_a(0) = t$. Hence λ_a induces $\widehat{\lambda_a}$:

$$\begin{array}{ccc}
\text{Spec}\, k[t] = \mathbb{A}^1_k & \xrightarrow{\lambda_a} & X_\sigma \\
\uparrow & & \uparrow \\
\text{Spec}\, k[[t]] & \overset{\widehat{\lambda_a}}{\dashrightarrow} & \text{Spec}\left(\widehat{\mathcal{O}}_{X_\sigma,t}\right) \\
 & & \wr\Vert \\
 & & \text{Spec}\, \widehat{\mathcal{O}}_{X,x} \\
 & & \downarrow{\scriptstyle p} \\
 & & \text{Star}\, Y
\end{array}$$

hence $p \cdot \widehat{\lambda}_a \in \text{R.S.}^U(\text{Star}(Y))$ and it is immediate that in the isomorphism of Corollary of Lemma 2.1.5, $a = \text{Ord}\left(p \cdot \hat{\lambda}_a\right)$.

"Ord" also gives us a direct relation between the strata in $\text{Star}(Y)$ and the faces of σ^Y:

Lemma 2.1.9. *Let $\lambda \in R.S.^U(Star(Y))$. If $Z \subseteq Star(Y)$ is a stratum and r is the corresponding face of σ^Y as in Proposition 2.1.2, then:*

$$\lambda(0) \quad \Longleftrightarrow \quad \text{Ord } \lambda \in \mathit{Int}\ \tau$$

Proof. Suppose $K = \{i \mid Z \subseteq E_i\}$, so that $\tau = \text{cone}(\{e_i\}_{i \in K})$. Then $\text{Ord } \lambda \in \text{Int } \tau$ is equivalent to:

$$\forall D \in M^Y \text{ such that } D \geq 0 \text{ on } \sigma^Y : \quad \langle D, \text{Ord } \lambda \rangle = 0 \Longleftrightarrow D \equiv 0 \text{ on } \tau$$

Such a D comes from a positive Cartier divisor $D = \sum n_i E_i$ on $\text{Star}(Y)$ and:

1. $\langle D, \text{Ord } \lambda \rangle = 0 \iff \lambda(o) \notin \text{Supp } D$

2. $$\begin{aligned} D \equiv 0 \text{ on } \tau &\Longleftrightarrow \langle D, e_i \rangle = 0 \quad \forall i \in K \\ &\Longleftrightarrow n_i = 0 \quad \forall i \in K \\ &\Longleftrightarrow \text{Supp } D \cap Z = \emptyset \end{aligned}$$

Thus $\text{Ord } \lambda \in \text{Int } \tau$ is equivalent to:

$$\forall \text{ positive Cartier divisor } D \text{ on } \text{Star}(Y): \quad \lambda(0) \notin \text{Supp } D \Longleftrightarrow Z \cap \text{Supp } D = \emptyset.$$

By the second Corollary to 2.1.5, this means $\lambda(0) \in Z$. □

The final step is to glue together all the polyhedral cones σ^Y into one big conical polyhedral complex. We must first compare σ^Y and σ^Z when Z is a stratum in $\text{Star}(Y)$: There are canonical maps $\alpha^{Y,Z}$ and $\beta^{Y,Z}$ (which we abbreviate to α and β):

1. $M^Y \xrightarrow{\alpha} M^Z$, the restriction of divisors from $\operatorname{Star} Y$ to $\operatorname{Star} Z$. Hence,

2.
$$M^Y_{\mathbb{R}} \xrightarrow{\alpha} M^Z_{\mathbb{R}}$$
$$N^Y \xleftarrow{\beta} N^Z$$
$$N^Y_{\mathbb{R}} \xleftarrow{\beta} N^Z_{\mathbb{R}}$$
and β is the dual of α.

3. $M^Y_+ \xrightarrow{\alpha} M^Z_+$ (since restriction of positive divisor is positive). Thus,

4. $\sigma^Y \xleftarrow{\beta} \sigma^Z$.

where β is the dual of α.

Lemma 2.1.10. *$M^Y \longrightarrow M^Z$ is surjective.*

Proof. Let $\{E_i\}_{i\in T}$ be the components of $(\operatorname{Star}(Y)) \setminus U$. Let $\{E_i\}_{i\in K}$ be those $E_i \subseteq \operatorname{Star}(Y) \setminus \operatorname{Star}(Z)$. If D is a Cartier divisor on $\operatorname{Star} Z \setminus U$, we can write

$$D = \sum_{i\in J\setminus K} n_i E_i$$

and extend it by the same formula to a Weil divisor on $\operatorname{Star}(Y)$. We seek

$$D' = D + \sum_{i\in K} m_i E_i$$

which is a Cartier divisor on $\operatorname{Star}(Y)$. If $x \in Y$, it suffices by Lemma 2.1.5 that D' have a local equation at x. Let (X_σ, t) be a local model at x and let

$$D^* = \sum_{i\in J\setminus K} n_i \overline{\mathcal{O}_i}$$

be the corresponding Weil divisor. If Z corresponds to the face τ of σ, then by the argument used in Lemma 2.1.5,

$$D \text{ Cartier divisor in } \operatorname{Star}(Z) \implies D^* \text{ Cartier divisor in } X_\tau.$$

But then again by Lemma 2.1.5, $D*|_{X_\tau} = (\mathfrak{X}^r)|_{X_\tau}$ for some $r \in M(T)$. Then let $D^{*'} = (\mathfrak{X}^r)$: this is a Cartier divisor on X_σ extending $D^*|_{X_\tau}$. Then D' can be taken as the corresponding divisor on X. □

Corollary 2.1.11. *$N^Z_{\mathbb{R}} \longrightarrow N^Y_{\mathbb{R}}$ is injective and considering this as an inclusion: $N^Z = N^Z_{\mathbb{R}} \cap N^Y$.*

Corollary 2.1.12. *If Z corresponds to the face τ of σ^Y, then the inclusion $N^Z_{\mathbb{R}} \longrightarrow N^Y_{\mathbb{R}}$ maps σ^Z isomorphically onto τ.*

Proof. Using the notation of the lemma, let $e_i \in N^Y$ be the map $\sum n_j E_j \mapsto n_i$, $i \in J$. Then if $i \in J \setminus K$, $e_i(\sum n_j E_j)$ depends only on the restriction of $\sum n_j E_j$ to $\operatorname{Star}(Z)$, hence these e_i are in N^Z. Now we know that since $\operatorname{Star}(Z) - U = \bigcup_{i\in J\setminus K} E_i$, σ^Z is spanned by $\{e_i\}_{i\in J\setminus K}$. On the other hand, $M^Y_{\mathbb{R}}$, we know that τ is spanned by $\{e_i\}_{i\in J\setminus K}$. Therefore σ^Z goes onto τ in the inclusion $M^Z_{\mathbb{R}} \longrightarrow M^Y_{\mathbb{R}}$. □

Corollary 2.1.13. *The diagram*

$$\begin{array}{ccc} R.S.^U(\operatorname{Star} Z) & \xrightarrow{\text{Ord}} & \sigma^Z \\ \cap\!\!| & & \big\downarrow \beta \\ R.S.^U(\operatorname{Star} Y) & \xrightarrow{\text{Ord}} & \sigma^Y \end{array}$$

commutes, hence

$$\text{Ord} : R.S.^{U}(Star(Y)) \longrightarrow \sigma^{Y} \cap N^{Y}$$

is surjective.

Proof. The commutativity is immediate from the definitions and then surjectivity of Ord follows from the fact that for all Z, $\text{Im}\Big(\text{Ord}^{(Z)}\Big) \supseteq \Big(\text{Int } \sigma^{Z}\Big) \cap N^{Z}$, hence by the first Corollary to Lemma 2.1.5

$$\text{Im}\Big(\text{Ord}^{(Y)}\Big) \supseteq (\text{Interior of face } \tau \text{ corresponding to } Z) \cap N^{Y}.$$

□

Next two general definitions on the kind of objects we are seeking to define

Definition 2.1.14. A conical (compact resp.) plyhedral complex Δ is a topological space $|\Delta|$ plus a finite family of closed subsets $\sigma_\alpha \subseteq |\Delta|$ called its cells plus a finite-dimensional real vector space V_α of real-valued continuous functions on σ_α such that

(Conical case) via a basis $f_1, \ldots, f_{n_\alpha}$ of V_α, we get a homeomorphism

$$\phi_\alpha : \sigma_\alpha \xrightarrow{\approx} \sigma'_\alpha \subseteq \mathbb{R}^{n_\alpha}$$

where σ'_α is a canonical convex polyhedra in $\mathbb{R}^{n_\alpha}$, not contained in a hyperplane,

(Compact case)

1. $V_\alpha \supseteq \mathbb{R}$, the constant functions, and via a basis $1, f_1, \ldots, f_{n_\alpha}$ of V_α, we get a homeomorphism:
$$\phi_\alpha : \sigma_\alpha \xrightarrow{\approx} \sigma'_\alpha \subseteq \mathbb{R}^{n_\alpha}$$
where σ'_α is a canonical convex polyhedra in $\mathbb{R}^{n_\alpha}$, not contained in a hyperplane,
2. $\phi_\alpha^{-1}(\text{faces of } \sigma'_\alpha) = \text{other } \sigma'_\beta s$, which we call the faces of σ_α; we call $\phi_\alpha^{-1}(\text{Int } \sigma'_\alpha)$ the interior of σ_α
3. $|\Delta| =$
4. If $\sigma_\beta =$ a face of σ_α, then $\text{res}_{\sigma_\beta} V_\alpha = V_\beta$

//

We can "explain´´ the idea of V_α as follows: we want σ_α to be a conical or compact polyhedron in an actual real vector space, but unique only up to linear transformations, or affine transformations in the 2 cases.

Definition 2.1.15. An integral structure on a conical (compact resp.) polyhedral complex is a set of finitely generated abelian groups $L_\alpha \subseteq V_\alpha$ such that:

0. **(Compact case only)** $L_\alpha \supseteq n\mathbb{Z}$, the constant function with values in $n\mathbb{Z}$, for some n,
1. $L_\alpha \otimes \mathbb{R} \xrightarrow{\approx} V_\alpha$,
2. If σ_β is a face of σ_α, then $\text{res}_{\sigma_\beta} L_\alpha = L_\beta$.

//

Remark 2.1.16. If Δ is a compact polyhedral complex, we can "expand´´ Δ to a conical polyhedral complex Δ' canonically:

$$|\Delta'| = {|\Delta| \times [0,\infty)}\big/{\sim}$$

where $\sim$ identifies $|\Delta| \times \{0\}$ to one point. Afterwards, setting

$$|\sigma'_\alpha| = {|\sigma_\alpha| \times [0,\infty)}\big/{\sim}$$

$$V'_\alpha = \{\text{functions on } \sigma'_\alpha \text{ of form } (x,t) \mapsto t \cdot f(x),\ f \in V_\alpha\}.$$

where $\sim$ identifies $\sigma_\alpha \times \{0\}$ to one point.

Given a conical polyhedral complex Δ, if f is a continuous function on $|\Delta|$ such that

- $\operatorname{res}_{\sigma_\alpha} f \in V_\alpha$, for all α,
- $f(x) \geq 0$ for all x with equality only if $x = \text{apex}$

then we get a compact polyhedral complex Δ_0:

$$\begin{aligned} |\Delta_0| &= \{x \in |\Delta| \mid f(x) = 1\} \\ (\sigma_\alpha)_0 &= \sigma_\alpha \cap |\Delta_0| \\ (V_\alpha)_0 &= \operatorname{res}_{(\sigma_\alpha)_0} V_\alpha. \end{aligned}$$

//

Our final point is simply:

To every toroidal variety without self-intersection $U \subseteq X$, we can associate a conical polyheral complex with integral structure $\Delta = \left(|\Delta|, \sigma^Y, M^Y\right)$ whose cells are in $1-1$ correspondence with the strata of X.

This is now quite simple to define. We set

$$|\Delta| \doteq \left. \coprod_{\text{Strata } Y} \sigma^Y \middle/ \sim \right.$$

where the relation $\sim$ is generated by isomorphisms

$$\beta^{Y,Z} : \sigma^Z \xrightarrow{\approx} \text{face of } \sigma^Y$$

whenever $Z \subseteq \operatorname{Star} Y$.
Explicitly, the equivalence relation is:

$$x_1 \in \sigma^{Y_1} \sim x_2 \in \sigma^{Y_2} \quad \Longleftrightarrow \quad \left\{ \begin{array}{c} \text{the faces } \tau_i \text{ of } \sigma^{Y_i} \text{ containing } x_i \\ \text{correspond to the same stratum} \\ Z \subseteq \operatorname{Star} Y_1 \cap \operatorname{Star} Y_2 \text{ and } x_1, x_2 \\ \text{correspond to the same point of } \sigma^Z \end{array} \right\}$$

so that $|\Delta| = \left\{\text{disjoint union of Int } \sigma^Y\right\}$ and the identification is carried out like this: for every Y_1, Y_2, $\operatorname{Star}(Y_1) \cap \operatorname{Star}(Y_2)$ is an open set in $\operatorname{Star}(Y_1)$ and a union of strata; hence it is the set of strata corresponding to the set of faces of a closed subpolyhedron $\sigma^{Y_1,Y_2} \subseteq \sigma^{Y_1}$. Define

$$\begin{array}{ccc} \sigma^{Y_1,Y_2} & \xrightarrow[\approx]{h^{Y_1,Y_2}} & \sigma^{Y_2,Y_1} \\ \cap\!\!| & & \cap\!\!| \\ \sigma^{Y_1} & & \sigma^{Y_2} \end{array}$$

by requiring that for all $Z \subseteq \operatorname{Star}(Y_1) \cap \operatorname{Star}(Y_2)$, h^{Y_1,Y_2} equals $\beta^{Y_2,Z} \circ \left(\beta^{Y_1,Z}\right)^{-1}$:

$$\begin{array}{ccc} & \sigma^Z & \\ \swarrow^{\beta^{Y_1,Z}}_{\approx} & & \searrow^{\beta^{Y_2,Z}}_{\approx} \\ (\text{Face corresponding to } Z) & \dashrightarrow & (\text{Face corresponding to } Z) \\ \cap\!\!| & & \cap\!\!| \\ \sigma^{Y_1} & & \sigma^{Y_2} \end{array}$$

on the face corresponding to Z. For this to be possible, there is a compatibility condition to check whenever $W \subseteq \operatorname{Star} Z$, $Z \subseteq \operatorname{Star} Y$, i.e., that the diagram

$$
\begin{array}{ccccc}
 & & \sigma^Z & & \\
 & \overset{\beta^{Z,W}}{\nearrow} & & \overset{\beta^{Y,Z}}{\searrow} & \\
\sigma^W & & \xrightarrow{\beta^{Y,W}} & & \sigma^Y
\end{array}
$$

is commutative. This is immediate.

Remark 2.1.17. It also follows immediately that all the "Ord" maps patch together to one big "Ord" map:

$$
\begin{array}{ccc}
\mathrm{R.S.}^U(\mathrm{Star}\, Y) & \xrightarrow{\mathrm{Ord}} & \sigma^Y \\
\cap\!\!\!\mid & & \big\downarrow \\
\mathrm{R.S.}^U(X) & \xrightarrow{\mathrm{Ord}} & |\Delta|.
\end{array}
$$

//

2.2 Theorems on Toroidal Varieties

We should like to study to what extent a toroidal variety is determined by Δ. More precisely, one cannot, of course, expect that Δ determines $U \subset X$; rather if you fix one toroidal variety $U \subset X$, it turns out that polyhedral subdivisions of the associated Δ determine canonically new toroidal variety $U \subset Z$ dominating $U \subset X$:

$$
\begin{array}{ccc}
 & & Z \\
 & \nearrow & \big\downarrow \text{birational morphism} \\
U & & \\
 & \searrow & \\
 & & X
\end{array}
$$

we work up to this in a sequence of theorems that we number analogously to those in chapter I. For the whole of this section, fix a particular toroidal variety $U \subset X$ without self-intersection. The following idea is due to Hironaka:

Definition 2.2.1. A birrational morphism f :

$$
\begin{array}{ccc}
 & & Z \\
 & \nearrow & \big\downarrow f \\
U & & \\
 & \searrow & \\
 & & X
\end{array}
$$

//

is called canonical if for all $x_1, x_2 \in X$ in the same stratum Y and all

$$\alpha : \widehat{\mathcal{O}}_{X,x_1} \xrightarrow{\approx} \widehat{\mathcal{O}}_{X,x_2}$$

which preserve the strata, i.e., if $Y \subset \bar{Y}^*$ for some stratum $\bar{Y}^*$, then α takes the ideal of $\bar{Y}^*$ at x_1 to the ideal of $\bar{Y}^*$ at x_2, α lifts:

$$
\begin{array}{ccc}
Z \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x_1} & \xleftarrow{\approx} & Z \times_X \operatorname{Spec} \widehat{\mathcal{O}}_{X,x_1} \\
\big\downarrow & & \big\downarrow \\
\operatorname{Spec} \widehat{\mathcal{O}}_{X,x_1} & \xleftarrow{\operatorname{Spec} \alpha} & \operatorname{Spec} \widehat{\mathcal{O}}_{X,x_1}
\end{array}
$$

Now fix a stratum $Y \subset X$ and consider diagrams:

$$\begin{array}{ccc} & & Z \\ & \nearrow & \\ U & & \big\downarrow f \\ & \searrow & \\ & & \operatorname{Star} Y \end{array} \qquad (*)$$

f affine and canonical

Z normal

Theorem 2.2.2. *There is a 1-1 correspondence between the set of diagrams* $(*)$ *and the set of rational polyhedral cones* $\tau \subset \sigma^Y$ *given by* $\tau \mapsto Z_\tau = \mathfrak{U}_\tau$ *where* $\mathfrak{U}_\tau =$ *subsheaf* $\sum_{D \in \tau^\vee \cap M^Y} \mathcal{O}_{Star\ Y}(-D)$ *of* $\underline{\mathbb{R}(X)}$. *For all such diagrams,* $U \subseteq Z$ *is another toroidal variety without self-intersection, and with a unique closed stratum* $\tilde{Y}$. *moreover,* $\exists$ *unique linear isomorphism* γ *making the following commute:*

$$\begin{array}{ccccc} R.S.^U(Z) & \xrightarrow{ord} & \sigma^{\tilde{Y}} & & \\ & & & \overset{\gamma}{\underset{\simeq}{\searrow}} & \\ \cap & & & & \tau \\ & & & & \supset \\ R.S.^U(\operatorname{Star} Y) & \xrightarrow{ord} & \sigma^Y & & \end{array}$$

and if $\lambda \in R.S.^U(Star\ Y)$, *then*

$$\lambda \in R.S.^U(Z) \iff Ord\ \lambda \in \tau$$

Proof. To make Z_τ more explicit, note that we can construct it as follows: Let $D_1, ..., D_N$ be a basis of the semi-group $\tau^\vee \cap M^Y$. Then if $V \subset \operatorname{Star} Y$ is any open set where each D_i has a local equation δ_i, then

$$\mathfrak{U}_\tau|_V = \mathcal{O}_V[\delta_1, ..., \delta_N]$$

In particular, $\mathfrak{U}_\tau$ is quasi-coherent and Z_τ is well-defined. It is evident that $f : Z_\tau \to \operatorname{Star} Y$ is affine and canonical.

In particular, $\mathfrak{U}_\tau$ is a quasi-coherent and Z_τ is well-defined. It is evident that $f : Z_\tau \to \operatorname{Star} Y$ is affine and canonical. On the other hand, abbreviating σ^Y by σ, we can use the theory of chapter 1, to construct affine toric varieties containing the n-dimensional torus T:

$$\begin{array}{ccc} & & X_\tau \\ & \nearrow & \\ T & & \big\downarrow g \\ & \searrow & \\ & & X_g \end{array}$$

for any closed point $x \in \operatorname{Star} Y$, one can choose a suitable orbit in X_σ, a closed point t in this orbit and a local model $\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_{X_\sigma,t}$. In this isomorphism, δ_i corresponds to $u_i\chi^{r_i}$ for some unit u_i, and $r_i \in M(T)$. Then $r_1, ..., r_N$ generate the corresponding semi-group $\tau^\vee \cap M(T)$ in $M(T)$, and $X_\tau = \underline{\operatorname{Spec}}\mathcal{O}_{X_\sigma}[\chi^{r_1}, ..., \chi^{r_N}]$. Therefore our formal isomorphism lifts to ϕ:

$$\begin{array}{ccccccc} & & Z_\tau \times_X \operatorname{Spec} \hat{\mathcal{O}}_{X,x} & \xrightarrow[\phi]{\approx} & X_\tau \tau_{X_\sigma} \operatorname{Spec} \hat{\mathcal{O}}_{X_g,t} & & \\ & \swarrow^{p} & & & & \searrow^{q} & \\ Z_\tau & & \big\downarrow & & \big\downarrow & & X_\tau \\ \big\downarrow f & & & & & & \big\downarrow g \\ X & \longleftarrow & \operatorname{Spec} \hat{\mathcal{O}}_{X,x} & \xrightarrow{\approx} & \operatorname{Spec} \hat{\mathcal{O}}_{X_g,t} & \longrightarrow & X_g \end{array}$$

Passing to completions at a point $x' \in Z_\tau$, this gives us simultaneously local models at all points of Z_τ over x, hence proves that Z_τ is normal and that $U \subset Z_\tau$ is a toroidal variety. Moreover since in the morphism $X_\tau \to X_\sigma$, each orbit of X_τ is smooth over some orbit of X_g, it follows that every stratum of Z_τ is smooth over some stratum of $\mathrm{Star}\, Y$.

We may make a simplifying reduction to the case $\tau \cap \mathrm{Int}\sigma \neq \phi$ at this point. Because if σ' is the smallest face of σ containing τ and σ' corresponds to the stratum $Y' \subset X$, then it follows from the existence of ϕ and the fact that $\mathrm{Im}(X_\tau \to X_\sigma) \subset \mathrm{Star}\ \mathcal{O}_{\sigma'}$. To go further we need: □

Lemma 2.2.3. *For all orbits $\mathcal{O} \subset X_\tau$, $q^{-1}(\bar{\mathcal{O}})$ is irreducible and normal.*

Proof. Note that q is a flat morphism with regular fibres, hence $\bar{\mathcal{O}}$ normal $\implies q^{-1}(\bar{\mathcal{O}})$. If $\mathcal{O}_O =$ image of $\mathcal{O}$ in X_σ, note that

a) $\bar{\mathcal{O}}_0$ is normal

b) Since the stabilizer of a point of $\mathcal{O}_0$ is connected, $\mathcal{O} \simeq \mathcal{O}_0 \times T$ for some torus T', hence $\mathbb{R}(\mathcal{O}_O)$ is algebraically closed in $\mathbb{R}(\mathcal{O}_0)$

Irreducibility now follows from: □

Lemma 2.2.4. *Let $f : X \to Y$ be a morphism of varieties, $Y_1 = f(\bar{X})$ and let $y \in Y_1$. If*

a) Y_1 is normal at Y

b) $\mathbb{R}(Y_1)$ is algebraically closed in $\mathbb{R}(X)$, then:

$$X \times_Y \mathit{Spec}\ \hat{\mathcal{O}}_{y,Y}$$

is irreducible.

Proof. Since $X \times_Y \mathrm{Spec}\ \hat{\mathcal{O}}_{y,Y}$ is flat over X, all its generic points lie over the generic point of x. Thus it is enough to show that $\mathbb{R}(X) \otimes_{\mathcal{O}_{y,Y}} \hat{\mathcal{O}}_{y,Y}$ is an integral domain. But

$$\mathbb{R}(X) \otimes_{\mathcal{O}_{y,Y}} \hat{\mathcal{O}}_{y,Y} \simeq \mathbb{R}(X) \otimes_{\mathbb{R}(Y_1)} [\mathbb{R}(Y_1) \otimes_{\mathcal{O}_{y,Y_1}} \hat{\mathcal{O}}_{y,Y_1}]$$

by (a), $\hat{\mathcal{O}}_{y,Y}$ is a domain and $\mathbb{R}(Y_1) \otimes_{\mathcal{O}_{y,Y_1}} \hat{\mathcal{O}}_{y,Y_1}$ is part of its quotient field, which is separable over $\mathbb{R}(Y_1)$. Therefore by (b), the whole thing is still a domain. □

Now assume that we have chosen a point $x \in Y$, so that the corresponding point t is in the closed orbit of X_σ. In this case:

a) $\forall$ strata $W \subset Z_\tau$, $\eta_W \in$ Image p

b) $\forall$ orbits $\mathcal{O} \subset X_\tau$, $\eta_{\mathcal{O}} \in$ Image q

because the image of η_W in X (resp. of $\eta_{\mathcal{O}}$ in X_σ) equals the generic point of some stratum (resp. some orbits) which lies in Spec $\mathcal{O}_{x,X}$ (resp. Spec $\mathcal{O}_{t,X_\sigma}$). Let

$$E_1, ..., E_N = \text{comp. of } Z_\tau - U$$

$$\bar{\mathcal{O}}_1, ..., \bar{\mathcal{O}}_M = \text{comp. of } X_\tau - T$$

It follows from lemma 2.1.5 that the sets:

1. of schemes $q^{-1}(\bar{\mathcal{O}}_i)$
2. of components of shemes $q^{-1}(\bar{\mathcal{O}}_i)$
3. of components of $X_\tau \times_{X_\sigma} \hat{\mathcal{O}}_t - q^{-1}(T)$
4. of components of $Z_\tau \times_X \hat{\mathcal{O}}_X - p^{-1}(U)$
5. of components of schemes $p^{-1}(E_i)$

are all equal or in 1-1 correspondence. We would like to show that the scheme $p^{-1}(E_i)$ are irreducible so that we can add to our list the set:

6. of schemes $p^{-1}(E_i)$

Suppose to the contrary $p^{-1}(E_{i_0})$ has ≥ 2 components which correspond say to $q^{-1}(\bar{\mathcal{O}}_{i_1})$ and $q^{-1}(\bar{\mathcal{O}}_{i_2})$. we can find a character chi^r of T with $r \geq 0$ on τ such that

a) $\chi^r \equiv 0$ on $\bar{\mathcal{O}}_{i_1}$

b) χ^r unit generically on $\bar{\mathcal{O}}_{i_2}$

But (χ^r) correspond formally to the Cartier divisor D on Z_τ supported on $Z_\tau - U$ and $r \geq 0$ on τ implies D effective and a) and b)imply:

a') $D \equiv 0$ on the branch of E_{i_0} corresponding to $\bar{\mathcal{O}}_{i_1}$

b') $D \not\equiv 0$ on the branch of E_{i_0} corresponding to $\bar{\mathcal{O}}_{i_2}$

But E_{i_0} is irreducible so D has a definite multiplicity along E_{i_0} which is either positive or zero, i.e., a' and b' are incompatible. Thus $p^{-1}(E_{i_0})$ is irreducible. This shows incidentally that for each i, $\phi(p^{-1}(E_i)) = q^{-1}(\bar{\mathcal{O}}_j)$ for some j, hence the E_i are normal. Therefore $U \subset Z_\tau$ is a toroidal variety without self-intersection. But more than that, we can show that Z_τ has a unique closed stratum. In fact, let $\mathcal{O}^*$ be the closed orbit in X_τ and let $t^* \in \mathcal{O}^*$ be a closed point lying over $t \in X_\sigma$. This gives us $(t^*, t) \in X_\tau \times_{X_\sigma} \text{Spec } \bar{\mathcal{O}}_{t,X_\sigma}$ and $X^* = p(\phi^{-1}(t^*, t)) \in Z_\tau$. Then I claim:

$$\forall \text{ strata } W \subset Z_\tau, x^* \in \bar{W} \text{ hence the stratum } \tilde{Y} \text{containing } x^* \text{ is the only closed stratum.}$$

In fact, suppose $\bar{W}$ is a component of $\bigcap_{i \in J} E_i$. Then $p^{-1}(\bar{W})$, even if it does not remain irreducible, still contains a component of $\bigcap_{i \in J} p^{-1}(E_i)$. Hence for some J'

$\phi(p^{-1}(\bar{W})) \supseteq$ a component of $\bigcap_{i \in J'} q^{-1}(\bar{\mathcal{O}}_i) = q^{-1}(\bigcap_{i \in J'} (\bar{\mathcal{O}}_i))$. But $\bigcap_{i \in J'} \bar{\mathcal{O}}_i$ is in fact, the closure of an orbit $\bar{\mathcal{O}}$ (i.e., let τ'= least face of τ containing the vertices in J': then let $\mathcal{O}$ correspond to τ'). And we have seen that $q^{-1}(\bar{\mathcal{O}})$ is irreducible. Therefore

$$\phi(p^{-1}(\bar{W})) \supset q^{-1}(\bar{\mathcal{O}})$$

Since $\mathcal{O}^* \subset \bar{\mathcal{O}}$, $t^* \in q^{\bar{\mathcal{O}}}$. therefore $x^* \in \bar{W}$.

This proves that $Z_\tau = \text{Star } \tilde{Y}$. Finally, to compare τ and $\sigma^{\tilde{Y}}$, pass to the completion of $Z_\tau \times_{X_\sigma} \hat{\mathcal{O}}_{t,X_\sigma}$ at (t^*, t) to get local models for X and Z_τ simultaneously:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{Z_\tau,x^*} & \cong & \hat{\mathcal{O}}_{X_\tau,x^*} \\ \uparrow & & \uparrow \\ \hat{\mathcal{O}}_{X,x} & \cong & \hat{\mathcal{O}}_{X_\sigma,t} \end{array}$$

As above let E_i and $\bar{\mathcal{O}}_i$ be corresponding components of $Z_\tau - U$ and $X_\tau - T$.Then as in Section 1 (verificar referencia):

$$\begin{aligned} \sum n_i E_i \text{ is a Cartier dividor in } Z_\tau &\Longrightarrow \sum n_i \bar{\mathcal{O}}_i \text{ is a Cartier divisor in } X_\tau, \\ &\Longrightarrow \sum n_i \bar{\mathcal{O}}_i = \chi^r \text{some } r \in M(T) \\ &\Longrightarrow \sum n_i \bar{\mathcal{O}}_i = g^* D, D \text{ a Cartier divisor on } X_\sigma \\ &\Longrightarrow \sum n_i E_i \text{ agrees formally at } x^* \text{with } f^* D, D \text{a Cartier divisor on } X \\ &\Longrightarrow \sum n_i E_i = f^* D, D \text{ a Cartier divisor on } X \end{aligned},$$

In other words, $D \mapsto f^* D$ sets up

$$M^Y \simeq M^{\tilde{Y}} \tag{2.3}$$

hence

$$N_{\mathbb{R}}^Y \simeq N_{\mathbb{R}}^{\tilde{Y}} \tag{2.4}$$

But also:

$$\begin{aligned} \sum n_i E_i \text{ effective} &\Longleftrightarrow \sum n_i \sigma_i \text{ effective} \\ &\Longleftrightarrow \text{the corresp. linear function } \mathcal{Q}_{\sum n_i \bar{\mathcal{O}}_i} \\ &\quad is \geq 0 \text{ on } \tau \end{aligned}$$

hence in the identification (2.4)

$$M_+^{\tilde{Y}} \simeq \tau^\vee \cap M^{\tilde{Y}}$$

and in (2.4)

$$\sigma^{\tilde{Y}} \simeq \tau$$

It is immediate that isomorphism is compatible with ord since for all λ : Spec $k[[t]] \to Z_\tau$ and $D \in M^Y$,

$$\langle f \circ \lambda, D\rangle \simeq \langle \lambda, f^* D\rangle$$

And if λ : Spec $k[[t]] \to X$ is given, then clearly λ lifts to Z_τ is and only if

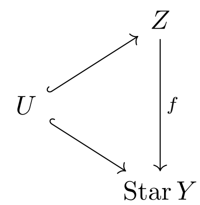
$$\lambda^*(\delta_i) \in k[[t]], (\delta_i \text{ local } eq^n \text{ of } D_i, D_i's \text{ generators of } \tau^\vee \cap M^Y)$$

ie.,

$$\langle \mathrm{Ord}\,\lambda, D_i\rangle \geq 0, 1 \leq i \leq N,$$

and this is equivalent to Ord $\lambda \in \check{\check{\tau}} = \tau$.

It remains to prove that all affine canonical modifications of Star Y are obtained in this way. Let



be a canonical affine modification. For all $D \in M_+^Y$, define a sheaf of ideals:

$$\mathcal{Q}_D = \{a \in \mathcal{O}_X : \frac{a}{\delta} \in f_*\mathcal{O}_Z, \delta = \text{local equation of } D\}$$

Then because $f_*\ \mathcal{O}_Z|_U = \mathcal{O}_{\text{Star } Y}|_U$ and $f_*\mathcal{O}_Z$ is quasi-cohenrent, it follows easily that

$$f_*\mathcal{O}_Z = \bigcup_{D \in M_+^Y} \mathcal{Q}_D \mathcal{O}_{\text{Star } Y}(D)$$

Moreover, because f is canonical, so is $\mathcal{Q}_D$ (i.e., $\mathcal{Q}_D$ in invariant under all formal isomorphism α as in definition 2.1.1. In the way, we can easily reduce the proof that $Z \cong Z_\tau$, some τ, to:

Lemma 2.2.5. *Let $\mathfrak{F}$ be a canonical coherent sheaf of fractional ideals on Star Y. Then $\exists$ $d_1, ..., D_n \in M^Y$ such that*

$$\mathfrak{F} = \sum_{i=1}^n \mathcal{O}_{\mathit{Star}\ Y}(D_i)$$

Proof. Let $\{z_1, ..., z_d\}$ =Ass ($\mathfrak{F}$) and $Z_i = \{\bar{z_i}\}$. Let

$$x \in Y - \bigcup_{\text{all } i \text{ s.t. } \mathbb{Z}_i \not\supset Y} Z_i$$

be a closed point and $\phi : \hat{\mathcal{O}}_{x,X} \to \hat{\mathcal{O}}_{t,X_\sigma}$ a local model. We prove the lemma in 4 steps.

Step I: $\exists x_1, ..., x_n \in M(T)$ s.t $\phi(\mathfrak{F}\hat{\mathcal{O}}_{x,X}) = (\chi^{r_1}, ..., \chi^{r_n})$

Step II: $\exists D_1, ..., D_n \in M^Y$ s.t $\mathfrak{F}_x = \sum \mathcal{O}_{\text{Star } Y}(D_i)_x$

Step III: $\exists D_1, ..., D_n \in M^Y$ and a neighborhood V of x such that:

$$\mathfrak{F}|_V = \sum \mathcal{O}_{\text{Star } Y}(D_i)$$

Step IV: $\exists D_1, ..., D_n \in M^Y$ such that $\mathfrak{F} = \sum \mathcal{O}_{\text{Star Y}(D_i)}$

Step I: The fractional ideal $\mathscr{Q} = \phi(\mathfrak{F} \cdot \hat{\mathcal{O}}_{x,X})$ in $\hat{\mathcal{O}}_{t,X_\sigma}$ is by hypothesis invariant under all automorphism of $\hat{\mathcal{O}}_{t,X_\sigma}$ that leave fixed the componentsof $X_\sigma - T$. Let $\mathcal{O} \subset X_\sigma$ be the orbit through which we can assume closed. Then also forall associated primes $\wp$ of $\mathscr{Q}$, $\wp \subseteq \wp' = I(\mathcal{O}\hat{\mathcal{O}}_t)$.If T_1=stabilixer of t, then we can write

$$\begin{aligned} T &= T_1 \times T_2 \\ X_\sigma &= X_1 \times X_2 \\ \mathcal{O} &= \{t_1\} \times T_2 \\ t &= (t_1, 1) \end{aligned}$$

Now embed

$$T_2 \simeq (\mathbb{A}^1 - \{0\})^k \subset \mathbb{A}^k$$

Define

$$\begin{aligned} \psi : X_\sigma &\rightarrow X_\sigma{}' \\ \psi(x_1, t_2) &= (x_1, t_2 - 1) \end{aligned}$$

taking the point t to the point $t' = (t, 0) \in X'_\sigma$. via ψ on the $\mathcal{O}_{t,X_\sigma}$ and $\hat{\mathcal{O}}_{t,X_\sigma}$ as well. Therefore by assumption $\mathscr{Q}$ is invariant under this action. By the usual argument this means that $\mathscr{Q}$ is generated by characters for this action, i.e., by $\chi^r \circ \psi, r \in M(T)$. But now $\mathscr{Q} = \mathscr{Q}(\hat{\mathcal{O}}_t)_\wp \cap \hat{\mathcal{O}}_t$ and in $(\hat{\mathcal{O}}_t)_\wp$, all the characters $\chi^r \circ \psi, r \in M(T_2)$, became units. Bit $\chi \circ \psi = \chi^r$ is $r \in M(T_1)$, so $\mathscr{Q}$ is generated by characters χ^r.

Step I $\implies$ Step II since if $\delta 1$ is a local equation of E_i, then $\phi(\delta_1) = u_1 \circ \chi^r, u_i$ unit and r_i a basis of $\sigma^\vee \cap M(T)$.

Step II $\implies$ Step III by coherency.

Step III $\implies$ Step IV: In fact, for all closed points $x' \in$ Star Y, choose $x'' \in V$ in the same stratum as x'. Then there exists ans isomorphism

$$\psi : \hat{\mathcal{O}}_{x',X} \simeq \hat{\mathcal{O}}_{x'',X}$$

preserving strata. Sinece $\mathfrak{F}$ is canonical, we get

$$\begin{aligned} \psi(\mathfrak{F}\hat{\mathcal{O}}_{x',X}) &= \mathfrak{F}\hat{\mathcal{O}}_{x'',X} \\ &= \sum \hat{\mathcal{O}}_{x'',X}(D_i) \\ &= \psi(\sum \hat{\mathcal{O}}_{x',X}(D_i)) \end{aligned}$$

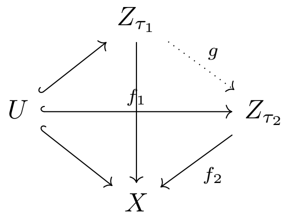
and hence:

$$\mathfrak{F}_{X'} = \mathcal{O}_{x',X} \cap \mathfrak{F}\hat{\mathcal{O}}_{x',X} = \mathcal{O}_{x',X} \cap \sum \hat{\mathcal{O}}_{x',X}(D_i) = \sum \mathcal{O}_{x',X}(D_i)$$

□

Having proven Theorem 2.2.2, we can now make rapid progress: the analog of Theorem 1.3.4 has already been worked out in the first section in the very definition of σ^Y and Ord. We get next:

Theorem 2.2.6. *If $\tau_1, \tau_2 \subset \sigma^Y$ are rational polyhedral cones, then there exists a morphism g:*



if and only if $\tau_1 \subseteq \tau_2$. Moreover g is an open immersion if and only if τ_1 is a face of τ_2.

The proof is completely analogous to that of the orbit cone correspondence replacing orbits by strata, 1-P.S. by elements of $R.S.^U(x)$, and descriptions of the affine rings of X_{τ_1}, X_{τ_2} by descriptions of the sheaves $f_{1,*}\mathcal{O}_{Z_{\tau_1}}, f_{2,*}\mathcal{O}_{Z_{\tau_2}}$

Theorem 2.2.7. *Z_τ is non-singular if and only if $\tau^\vee \cap N^Y$ over $\mathbb{Z}$*

The proof this time uses the known characterization of smooth cones and carries it over to Z_τ by using local models $\widehat{\mathcal{O}_{X,Z_\tau}} \simeq \widehat{\mathcal{O}_{t,X_\tau}}$

Definition 2.2.8. If $\Delta = \{|\delta|, \sigma_\alpha, V_\alpha\}$ is a canonical polyhedral complex, then a finite partial polyhedral decomposition is a second conical polyhedral complex $\Delta' = \{|\Delta'|, \sigma'_\beta, V'_\beta\}$ with

1. $|\Delta'| \subseteq |\Delta|$
2. $\forall \beta, \exists \alpha$ such that $\mathrm{Int}\sigma'_\beta \subseteq \mathrm{Int}\ \sigma_\alpha$
3. If σ'_β, then $V'_\beta = \mathrm{res}_{\sigma'_\beta} V_\alpha$

//

If $\{L_\alpha\}$ is an integral structure on Δ, then Δ' is called rational if whenever $\sigma'_\beta \subseteq \sigma_\alpha$, then σ'_β is defined by inequalities $l \geq 0, l \in L_\alpha$. In this case, $L'_\beta = \mathrm{res}_{\sigma'_\beta} L_\alpha$ is an integral structure on Δ'. This is a *polyhedral complex* decomposition.

Definition 2.2.9. Consider diagrams:

$$\begin{array}{ccc} & & Z \\ & \nearrow & \downarrow f \text{ birational} \\ U & & \\ & \searrow & \\ & & X \end{array}$$

where:

1. Z has an open covering $\{V_i\}$ such that $U \subset V_i$, $f(V_i) \subset \mathrm{Star}\ Y_i$ for some stratum Y_i and V_i is affine and canonical over $\mathrm{Star}\, Y_i$
2. Z normal

We call these allowable modification of X //

We can construct them from f.r.p.p. decomposition Δ' of Δ by reversing the procedure followed at the end of section 2.1: set

$$Z_{\Delta'} == \coprod_\beta Z_{\sigma'_\beta} \Big/ \sim$$

Explicitly, the equivalence relation is

$$x_1 \in Z_{\sigma'_\beta} \sim x_2 \in Z_{\sigma'_\gamma} \iff \left\{ \begin{array}{c} \text{The strata containing } x_1 \text{ and } x_2 \\ \text{correspond to a common face } \tau \text{ of} \\ \sigma'_\beta \text{ and } \sigma'_\gamma \text{ and } x_1 \text{ and } x_2 \text{ come from} \\ \text{the same point of } Z_\tau \end{array} \right\}$$

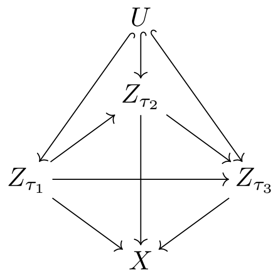
so that $Z_{\Delta'} = \{$disjoint union of the closed strata $Y_\beta$in each $Z_{\sigma'_\beta}\}$. The identification can be carried out like this: $\forall \beta, \gamma \sigma'_\beta \cap \sigma'_\gamma$is a closed subpolyhedron of σ'_β; hence it corresponds to a set of strata forming an open subscheme $Z_{(\beta,\gamma)} \subset Z_{\sigma'_\beta}$. Define

$$\begin{array}{ccccc} & Z_{(\beta,\gamma)} & \xrightarrow[h_{(\beta,\gamma)}]{\approx} & Z_{(\gamma,\beta)} & \\ \rotatebox{45}{$\subseteq$} & & & & \rotatebox{-45}{$\supseteq$} \\ Z_{\sigma'_\beta} & & & & Z_{\sigma'_\gamma} \end{array}$$

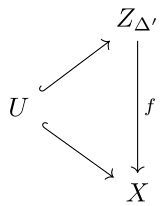
by requiring that for all faces $\tau \subset \sigma'_\beta \cap \sigma'_\gamma, h_{(\beta,\gamma)}$ should be given on the image of Z_τ by:

$$\begin{array}{ccc} & Z_\tau & \\ \approx\swarrow & & \searrow\approx \\ \mathrm{Im}[Z_\tau \longrightarrow Z_{\sigma'_\beta}] & & \mathrm{Im}[Z_\tau \longrightarrow Z_{\sigma'_\gamma}] \\ \cap\!| & & \cap\!| \\ Z_{(\beta,\gamma)} & & Z_{(\gamma,\beta)} \end{array}$$

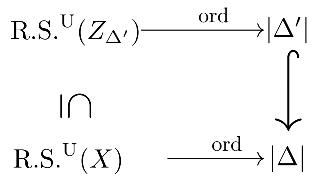
The only compatibility condition here is that when $\tau_1 \subseteq \tau_2 \subseteq \tau_3$, then the middle triangle here:



should commute. This is clear. It is immediate from the affine correspondence of toric varieties and this construction that (1) $Z_{\Delta'}$ fits into a diagram



that (2) $U \hookrightarrow Z_{\Delta'}$ is a toroidal variety without self-intersection, that (3) Δ' is the polyhedral complex associated to $U \hookrightarrow Z_{\Delta'}$,that (4)the diagram:



commutes and that (5) if $\lambda \in R.S.^U(X)$, then $\lambda \in R.S.^U(Z_{\Delta'})$, if and only if Ord $\lambda \in |\Delta'|$. By the valuative criterion for separation, $Z_{\Delta'}$, is separated (i.e., each $Z_{\sigma'_\beta}$ is affine over the separated scheme X, hence is separated; and if λSpec $k((t)) \to U$ has extensions to 2 of these open pieces:

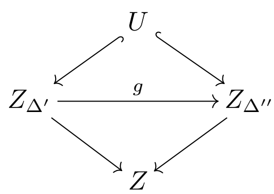
$$\begin{aligned} \mu &: \text{ Spec } k[[t]] \to Z_{\sigma'_\beta} \\ \nu &: \text{ Spec } k[[t]] \to Z_{\sigma'_\gamma} \end{aligned}$$

then in $|\Delta|$, ord $\mu =$ ord ν, so $\mu(o), \nu(o)$ both lie in some $Z_{\tau'}$, $\tau =$ common face of $\sigma'_\beta, \sigma'_\gamma$ and μ, ν both factor through Z_τ.) This proves:

Theorem 2.2.10. *The correspondence $\Delta' \mapsto Z_{\Delta'}$, defines a bijection between the f.r.p.p. decomposition of Δ and the isomorphism classes of allowable modifications of X.*

We also find:

Theorem 2.2.11. *Let Δ', Δ'' be 2 f.r.p.p decomposition of Δ. Then there exists a morphism g:*



if and only if for all polyhedra σ'_β of Δ', $\sigma'_\beta \subseteq \sigma''_\gamma$ for some polyhedron σ''_γ of Δ''

Theorem 2.2.12. *The morphism g is proper if and only if $|\Delta'| = |\Delta''|$*

(proof immediate by valuative criterion for properness.)

Next we can generalize Theorem 1.5.1 along the lines already indicated in lemma 2.1.10. We consider canonical coherent shaves $\mathfrak{F}$ of fractional ideals on X, i.e. $\forall \alpha : \hat{\mathcal{O}}_{x_1,X} \to \hat{\mathcal{O}}_{x_2,X}$, preserving strata, $\alpha(\hat{\mathfrak{F}}_{x_1}) = \hat{\mathfrak{F}}_{x_2}$ By lemma 2.1.10, it follows that for any stratum Y,

$$\mathfrak{F}|_{\text{Star } Y} \simeq \sum_{i=1}^{n} \mathcal{O}_{\text{Star } Y}(-D_i), \text{ some } D_1, ..., D_n \in M^Y.$$

This allows us to define a map

$$\text{ord } \mathfrak{F} : |\Delta| \to \mathbb{R}$$

in the following way:

$$\forall x \in \sigma^Y, \text{ord } \mathfrak{F}(x) = \min_{1 \le i \le n} < D_i, x >$$

Note that:

$$\begin{aligned} \text{Ord}\,\mathfrak{F}(\text{ord } \lambda) &= \min_{1\le i\le n} \langle D_i, \text{Ord}\,\lambda\rangle \\ &= \min_{1\le i\le n} \text{ord }_0\lambda^{-1}(D_i) \\ &= \text{ord}_0\lambda^{-1}(\mathfrak{F}) \end{aligned}$$

and hence the definition is independent of the choice of D_i. Clearly Ord $\mathfrak{F}$ is a function $f : |\delta| \to \mathbb{R}$ such that:

(i) $f(\lambda \cdot x) = \lambda \cdot f(x), \lambda \in \mathbb{R}^+$

(ii) f is continuous, piecewise-linear,

(iii) $f(\sigma^Y \cap N^Y) \subset \mathbb{Z}$ all Y

(iv) f is convex on each σ^Y

A function satisfying this conditions will be called an order function.

Conversely let $f : |\Delta| \to \mathbb{R}$ be an order function. for all Y, put

$$(\mathfrak{F}_f)_Y = \sum_{D\in M^Y, D\ge f \text{on } \sigma^Y} \mathcal{O}_{\text{star } Y}(-D)$$

Theorem 2.2.13. *I. Let $f : |\Delta| \to \mathbb{R}$ be an order function. Then the $(\mathfrak{F}_f)_Y$ can be patched together into a canonical coherent complete sheaf $\mathfrak{F}_f$ of fractional ideals on X.*

II.
- *a) ord $\mathcal{F}_f = f$*
- *b) $\mathfrak{F}_{ord\ f}$ is the completion of f*
- *c) The maps $\mathfrak{F} \mapsto$ ord $\mathfrak{F}$ and $f \mapsto \mathfrak{F}_f$ define bijection between the set of canonical coherent complete sheaves of fractional ideals and the set of order functions f.*
- *d) $\mathfrak{F} \subset \mathfrak{F}_f$ if and only if ord $\mathfrak{F} \ge f$*
- *e) ord $\mathfrak{F}_1 \cdot \mathfrak{F}_2 =$ ord $\mathfrak{F}_1 +$ ord $\mathfrak{F}_2$*
- *f) $\mathfrak{F}|_{Star\ Y} \simeq \mathcal{O}_{Star\ Y}$ if and only if ord $\mathcal{F} \equiv 0$ on σ^Y*

III.
- *a) $\mathfrak{F}^{-1} = \mathfrak{F}_g$ where g is the convexinterpolation of ord $\mathfrak{F}$ on $\bigcap_Y Sk^1(\sigma^Y)$*
- *b) $(\mathfrak{F}^{-1})^{-1} = \mathfrak{F}$if and only if $\mathfrak{F}$ is complete and ord $\mathfrak{F}$ is the convex interpolation offunction of a function $\bigcap_Y Sk^1(\sigma^Y) \to \mathbb{Z}$. Moreover there exists a bijective correspondence between the set of canonical Weil-divisors (i.e., those supported on $X - U$) and the set of intgral functions on $\bigcap_Y Sk^1(\sigma^Y)$.*
- *c) The following are equivalent:*
  - *i) $\mathfrak{F}$ invertible*
  - *ii) $\mathfrak{F} \cdot \mathfrak{F}^{-1} = \mathcal{O}_X$*
  - *iii) ord $\mathfrak{F}$ is linear on each σ^Y*

The proof is similar to that of Theorem 1.5.1.

Theorem 2.2.14. *Let $\mathfrak{F}$ be a canonical coherent sheaf of fractional ideals. Let $B_{\mathfrak{F}(X)}$ be the normalization of the variety obtained by blowing up $\mathfrak{F}$. Then $B_{\mathfrak{F}}(X)$ is an allowable modification of X and is described by the f.r.p.p. decomposition of Δ obtained by subdividing the δ^Y's into the biggest possible polyhedra on which ord $\mathfrak{F}$ is linear.*

Proof. First,$f : B_{\mathfrak{F}}(X) \to X$ is an allowable modification: In fact, if

$$\mathfrak{F}|_{\text{Star } Y} \simeq \sum_{i=1}^{n} \mathcal{O}_{\text{Star } Y}(-D_i)$$

the $f^{-1}(\text{Star } Y)$ is covered by the n relatively affine open places which are the normalizations of:

$$V_i = \underline{\text{Spec }} \mathcal{Q}_i, \mathcal{Q}_i = \{\mathcal{O}_{\text{Star } Y} - \text{algebra generated by } \mathcal{O}(D_i - D_j), 1 \leq j \leq n\}$$

[Sinceif, locally in Spec $R \subset$ Star Y, δ_i is an equation of D_i, then $\mathfrak{F}$ is given by the fractional ideal $\sum \delta_i R$, hece the blow-up is covered by affines with rings

$$S_i = R[\delta_1/\delta_i, ..., \delta_n/\delta_i]$$

and S_i is the $\mathcal{O}_{\text{Star } Y}$-algebra generated by $\mathcal{O}(D_i - D_1), ..., \mathcal{O}_{D_i - D_n}$.] As $B_{\mathfrak{F}}(X)$is also charactezed as the minimal normal variety dominating X such that the pull-back of $\mathfrak{F}$ is invertible, 1.6.1 follows from Theorem 1.5.1. ☐

Finally the proof of Theorem 1.5.1 goes over immediately to prove:

Theorem 2.2.15. *For any toroidal variety $U \subset X$ without self-intersection,there exists a canonical sheaf ofideal $\mathcal{Q} \subset \mathcal{O}_X$ such that $B_{\mathcal{Q}}(X)$ is non-singular.*

There is another situation that we must analyze for the sake of its application to semi-stable reduction. This does not involve any new ideas but rather a slight reformulation of what has been studied so far in a new situation. Suppose that in addition to $U \subset X$, a toroidal variety without self-intersection, we are given a positive Cartier-divisor D with support exactly $X - U$. We can associate to the triple (X, U, D) a compact plyhedral complex Δ_0 with integral structure, where

$$\begin{aligned} |\Delta|_0 &= \{x \in |\Delta| :< D, x >= 1\} \\ \sigma_0^Y &= |\delta_0| \cap \sigma^Y \end{aligned}$$

$$res_{\sigma_0^Y}(M^Y)\text{giving integral structure.}$$

Note that a polyhedral subdivision of Δ_0 gives a conical polyhedral subdivision of Δ and vice versa. When one has a compact polyhedral complex with integral structure $(\Delta_0, \sigma_\alpha, L_\alpha)$, note that one can define several more structure on Δ_0:

a) An increasing series of "lattices" on Δ_0: $\nu \geq 1$

$$(\Delta_0)_{\frac{1}{\nu}\mathbb{Z}} = \{x \in |\Delta_0| : if x \in \sigma_\alpha, then \forall f \in L_\alpha, f(x) \in \frac{1}{\nu}\mathbb{Z}\}$$

Every rational point of Δ_0 lies on one of these lattices but each $(\Delta_0)_{\frac{1}{\nu}\mathbb{Z}}$

b) a volume element on each polyhedron σ_α or even on each rational polyhedron $\tau \subset \sigma_\alpha$(possibly of lower dimension than σ_α): let $L = res_\tau L_\alpha$ and if $k = \dim\tau$, let $1/a, f_1, ..., f_k$ be abasis of L. Use $f_1, ..., f_k$ to define $F : \tau \to \mathbb{R}^k$ and pull-backthe volume elemnt. Since this embedding is unique up to translation and unimodular transformation, the volume element is well-defined.

Theorem 2.2.16. *Given $U \subset X$ and D as above, let Δ' be an f.r.p.p. decomposition of Δ and let Δ_0' be the associated decomposition of Δ_0. Let $f : Z_{\Delta'} \to X$ be the corresponding modification. Then*

a) the vertices of Δ_0' are in $(\Delta_0)_{\mathbb{Z}}$ if and only if $f_{-1}(D)$ vanishes to order one on each component of $Z_{\Delta'} - U$

b) If a9 holds, then moreover the volumen of every polyhedron τ_0 in Δ_0' is $1/(dim dim\ \tau_0)$; if and only if $Z_{\Delta'}$ is non-singular.

Proof. To probe a), note that the components E_i of $Z_{\Delta'} - U$ correspond to the one-dimensioal faces $\mathbb{R}^+ \cdot v_i$ of Δ' and hence to the vertices of Δ'_0. If we normalize v_i so that V_i isa primitive vector in N^Y, then $\langle D, v_i \rangle \mathbb{Z}$. Let $\langle D, v_i \rangle = \nu$. Then $\frac{1}{\nu} v_i$ is the corresponding vertex of Δ'_0 and

$$\frac{1}{\nu} v_i \in (\Delta_0)_{\mathbb{Z}} \text{ if and only if } \nu = 1$$

On the other hand

$$\begin{aligned} \nu &= \text{least integer } n \text{such that } \langle D, v \rangle = n, \text{some } v \in N^Y \cap (\mathbb{R}^+ \cdot v_i) \\ &= \text{least } n \text{such that } \langle D, \text{ord } \lambda \rangle = n, \text{some } \lambda \in R.S.^u(X) \text{ with } \lambda(0) \in E_i - \bigcap_{j \neq 1} E_j \\ &= \text{multiplicity to which } f^{-1}(D) \text{vanishes along } E_i \end{aligned}$$

This prove a). As for b) note that if τ is a k-dimensional compact polyhedron with integral vertices, then

$$vol(\tau) = \frac{a}{k!}; a \in \mathbb{Z}, a \geq 1$$

and the certainly $a > 1$ unless τ is a simplex. Thus for either of the 2 conditions in b) to hold, all τ_0 must be simplices. We are now reduced to:

Lemma 2.2.17. *Let $N_{\mathbb{R}}$ be a real vector space, $N \subset N_{\mathbb{R}}$ a lattice, $N^* = Hom(N, \mathbb{Z})$.Let $x_0, ..., x_k \in N$ be independent vectors such that $\exists l \in N^*$ with $l(x_i) = 1$, $D \leq i \leq k$ and let $\tau =$ [convex hull of $x_0, ..., x_k$ in the hyperplane $l = 1$]. Using N^*, induce a volume element on τ. Then (in the notation of Chapter I)*

$$vol(\tau) = \frac{\text{mult } \langle x_0, ..., x_k \rangle}{k!}$$

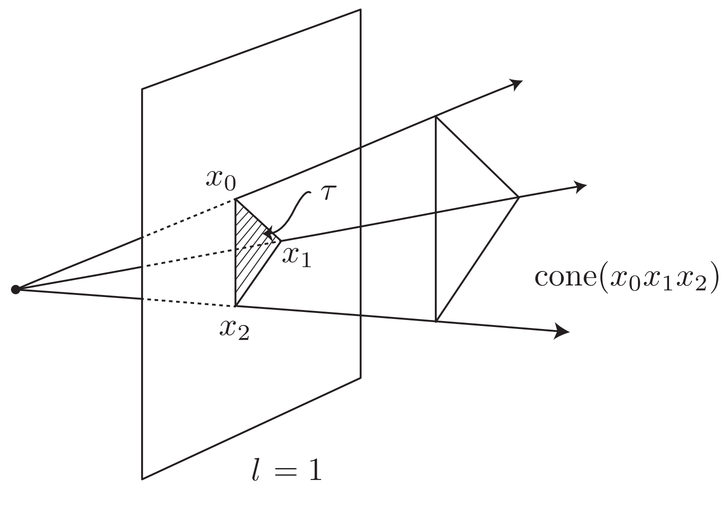
Proof. Choose an isomorphism $N \simeq \mathbb{Z}^{k+1}$ so that $x_0 = (1, 0, ..., 0)$ and $l(a_0, ..., a_k) = a_0$. If $x_i = (1, a_1^{(i)}, ..., a_k^{(i)})$, then

$$\text{mult } \langle x_0, ..., x_k \rangle = \det \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & a_1^{(1)} & \cdots & a_k^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & a_1^{(k)} & \cdots & a_k^{(k)} \end{pmatrix}$$

and

$$\text{Vol}(\tau) = \frac{1}{k!} \begin{pmatrix} a_1^{(1)} & \cdots & a_k^{(1)} \\ \vdots & \ddots & \vdots \\ a_1^{(k)} & \cdots & a_k^{(k)} \end{pmatrix}$$

□



2.3 Reduction of the theorem to a construction

Now return to the situation in the introduction to this chapter:

$$f : X^n \longrightarrow C^1 \ni 0$$

with res $f : X \setminus f^{-1}(0) \longrightarrow C \setminus \{0\}$ smooth. Fix a generator t of $\mathfrak{m}_{0,C}$, and for all $d \geq 1$, let

$$\begin{array}{rcl} C_d & = & \text{normalization of } C \text{ in the field extension generated by } t^{1/d} \\ \pi_d & : & C_d \longrightarrow C, \text{ the canonical morphism,} \\ 0_d & \in & C_d, \text{ the point over } O. \end{array}$$

We will tkae C' to be one of these curves C_d. Now by Hironaka's resolution theorem, we may blou up X by a sequence of monoidal transformations with non-singular centers all lying over $0 \in C$ until we find:

$$g : X' \longrightarrow X \quad \text{birrational and projective}$$

with X' non singular, $(f \circ g)^{-1}(0)_{\text{red}}$ a union of non singular components and $E_1, \ldots, E_n$ crossing transversely.

The only problem is that if $n(i) = \text{Ord}_{E_i}(t)$ so that

$$(f \cdot g)^{-1}(0) = \sum n(i) E_i \quad \text{as divisor on } X'$$

then the $n(i)$ may be bigger than one, hence $(f \cdot g)^{-1}(0)$ may not be reduced. To reduce the $n(i)$, we must replace c by a suitable C_d. It is clear that we may as well renae X' to be the original X and assume that our starting point is a non-singular X with $f^{-1}(0)_{\text{red}}$ good. Now for all $d \geq 1$, let

$$\begin{array}{rcl} X_d & = & \text{normalization of } X \times_C C_d \\ f_d & : & X_d \longrightarrow C_d \text{ the projection} \\ U_d & = & f^{-1}(C_d - (0_d)). \end{array}$$

What does X_d look like? To describe it, choose a point $x \in X$ over $0 \in C$. Suppose $x \in \bigcap_{k=1}^{r} E_{i_k}$ and $x \notin E_l$ for all $j \neq i_1, \ldots, i_r$. Then formally at x, the pair of morphism $X \mapsto c, x \mapsto 0$ is equivalent to the morphism $\mathbb{A}^n \to \mathbb{A}^1$ given by $0 \mapsto 0$.

This is where we use characteristic 0: In fact, if $y_j \in \mathcal{O}_{X,x}$ is a local equation for the divisor E_{i_j}, then

$$t = u \prod y_j^{n(i_j)}, \qquad u \in \mathcal{O}^*_{X,x}.$$

In the completion $\hat{\mathcal{O}}_{X,x}, y_1, \ldots, y_r$ are part of a system of parameters since the E_{i_j} meet transversely. Moreover, since char $k \nmid n(i_1)$, u has an $n(i_1)^{\text{th}}$ root v in $\hat{\mathcal{O}}^*_{X,x}$. So we may replace y_1 by $v \cdot y_1$, and find local equations $y_j \in \hat{\mathcal{O}}_{X,x}$ of E_{i_j} such that $t = \prod y_j^{n(i_j)}$, i.e.,

$$\begin{array}{rcl} \hat{\mathcal{O}}_{X,x} & \overset{\approx}{\longleftrightarrow} & k[[y'_1, \ldots, y'_n]] \\ t & \longleftrightarrow & \prod_{j=1}^{r} y_j^{n(i_j)} \end{array}$$

Therefore, formaly at all points over x:

$$\begin{array}{ccc} X_d & \longrightarrow & X \\ \downarrow & & \downarrow \\ C_d & \longrightarrow & C \end{array}$$

satisfies the universal property of the normalization of $\mathbb{A}^n \times_{\mathbb{A}^1} \mathbb{A}^1$ with projections $\mathbb{A}^1 \overset{d\text{th power}}{\longrightarrow} \mathbb{A}^1$ and $\mathbb{A}^n \overset{p}{\longrightarrow} \mathbb{A}^1$ (Note that normalization commutes with taking completions.).

Let $s = t^{1/d}$. Then

$$\mathbb{A}^n \times_{\mathbb{A}^1} \mathbb{A}^1 = \left[\text{the hypersurface } H : s^d = \prod x_j^{n(i_j)} \text{ in } \mathbb{A}^{n+1}\right]$$

Let

$$e = \gcd(d, n(i_1), \ldots, n(i_r)).$$

Then

$$\begin{aligned} H &= \bigcup_{e^{th}\text{roots of } 1} H_\zeta \\ H_\zeta &= \left[\text{hypersurface } s^{d/e} = \zeta \cdot \prod_{j=1}^{r} x_j^{n(i_j)/e}\right]. \end{aligned}$$

H_ζ is the image of the morphism

$$\begin{array}{ccc} \mathbb{A}^n_{\tilde{x}} & \longrightarrow & \mathbb{A}^{n+1}_{(x,s)} \\ x_i & = (\tilde{x_i})^d & \\ s & = \zeta^{e/d} \cdot \prod_{j=1}^{r} \tilde{x_j}^{n(i_j)} & \end{array}$$

hence is irreducible, and its coordinate ring is the subring:

$$k\left[x_1, \ldots, x_n, \prod_{j=1}^{r} x_j^{n(i_j)/d}\right] \subseteq k\left[x_1^{1/d}, \ldots, x_n^{1/d}\right]$$

Thus if we let

$$M = \mathbb{Z}^n + \left(\frac{n(i_1)}{d}, \ldots, \frac{n(i_r)}{d}, 0, \ldots, 0\right)\mathbb{Z} \subseteq \mathbb{Q}^n,$$

then H_ζ is isomorphic to the affine embedding of the torus T with character group M, given by the semi-group in M generated by

$$\begin{cases} (1, 0, \ldots, 0) \\ \vdots \\ (0, 0, \ldots, 1) \\ \left(\frac{n(i_1)}{d}, \ldots, \frac{n(i_r)}{d}, \ldots, 0\right). \end{cases}$$

The cone generated by this semi-group in $M_\mathbb{R} \cong \mathbb{R}^n$ is just the positive octant $(\mathbb{R}_+)^n$, so we can normalize as in the beginning of chapter 1:

$$\begin{bmatrix} \text{normalization} \\ \text{of } H_\zeta \end{bmatrix} \cong \text{Spec } k[\ldots, x^\alpha, \ldots]_{\alpha \in M \cap (\mathbb{R}_+)^n}$$

Going back, this means that $x \in X$ splits into e distinct points $x' \in X_d$ and that each of them:

$$\hat{\mathcal{O}}_{X_d, x'} \cong k[[\ldots, y^\alpha, \ldots]]_{\alpha \in M \cap (\mathbb{R}_+)^n}$$

Under this formal isomorphism:

$$\begin{array}{ccc} \sum E_i & \text{given by} & \prod y_i = 0 \\ & \text{corresponds to} & \prod x_i = 0 \\ & \text{defining} & H_\zeta \setminus T. \end{array}$$

Therefore $U_d \subseteq X_d$ is a toric variety. Moreover, the components of $X_d \setminus U_d$ are the components of the inverse image in X_d of the various $E_i \subseteq X$. The inverse image of E_{i_j} near x', with reduced structure, is given by ${}^{\mathcal{O}X_d,x'}/_{\sqrt{(y_j)}}$, and

$$\widehat{\mathcal{O}X_{d},x'}/_{\sqrt{(y_j)}} \cong {}^{k[[\ldots,y^\alpha,\ldots]]_{\alpha\in M\cap(\mathbb{R}_+)^n}}/_{\sqrt{(y_j)}}$$

$$\cong {}^{k[[\ldots,y^\alpha,\ldots]]_{\alpha\in M\cap(\mathbb{R}_+)^n}}/_{(\ldots,y^\alpha,\ldots)_{\alpha_j>0}}$$

$$\cong k[[\ldots,y^\alpha,\ldots]]_{\alpha_j=0,\alpha\in M\cap(\mathbb{R}_+)^n}.$$

This again is an integrally closed domain of toroidal type. Thus the inverse image of each E_i is a disjoint union of normal varieties. This proves:

Lemma 2.3.1. *$U_d \subseteq X_d$ is a toroidal variety without self-intersection.*

The next question is: how does X_d vary with d ? Let $\nu = \mathrm{lcm}(n(1), \ldots, n(N))$. We are only interested in d such that $\nu \mid d$. Suppose $d = e \cdot \nu$ and consider $\widehat{\mathcal{O}_{X_d,x'}}$ again at a random point x. Then

$$n(i_j) \cdot m(i_j) = \nu, \qquad 1 \leq j \leq r$$

and

$$M = \mathbb{Z}^n + \left(\frac{1}{e \cdot m(i_1)}, \ldots, \frac{1}{e \cdot m(i_r)}, 0, \ldots, 0\right)\mathbb{Z}.$$

Let

$$M_O = \mathbb{Z}^n + \left(\frac{1}{m(i_1)}, \ldots, \frac{1}{m(i_r)}, 0, \ldots, 0\right)\mathbb{Z}.$$

Suppose $\alpha \in M \cap (R_+)^n$. Then either:

1. $\alpha_1, \ldots, \alpha_r > 0$, in which case

$$\alpha = \left(\frac{1}{e \cdot m(i_1)}, \ldots, \frac{1}{e \cdot m(i_r)}, 0, \ldots, 0\right) + \beta, \qquad \beta \in M \cap (\mathbb{R}_+)^n$$

or

2. some $\alpha_l = 0,\ 1 \leq l \leq r$. Now

$$\alpha = v + p \cdot \left(\frac{1}{e \cdot m(i_1)}, \ldots, \frac{1}{e \cdot m(i_r)}, 0, \ldots, 0\right)$$

where v is an integral vector. Hence $e \cdot m(i_l) \mid p$. It follows $e \mid p$, and

$$\alpha \in M_O \cap \mathbb{R}^n_+$$

This means that the semi-group $M \cap \mathbb{R}^n_+$ is generated by $M_O \cap \mathbb{R}^n_+$ and the vector $\left({}^1/_{e\cdot m(i_1)}, \ldots, {}^1/_{e\cdot m(i_r)}, 0, \ldots, 0\right)$. In therms of rings, this means that $\widehat{\mathcal{O}_{X_d,x'}}$ is generated by $\widehat{\mathcal{O}_{X_\nu,x'}}$ and $t^{1/d}$. Therefore the canonical morphism

$$X_d \longrightarrow X_\nu \times_{C_\nu} C_d$$

induces isomorphisms between the complete local rings of corresponding points, hece it is étale. But it is also finite and birational, hence it is an isomorphism. This proves:

Lemma 2.3.2. [2] *If $\nu \mid d$, then*

$$X_d \xrightarrow{\approx} X_\nu \times_{C_\nu} C_d,$$

is an isomorphism. Hence the closed fibres $f_d^{-1}(0_d)$ is independent of d so long as $\nu \mid d$, and the projection $X_\nu \to X_d$ induces a bijection between the strata of $X_\nu \setminus U_\nu$ and the strata $X_d \setminus U_d$.

[2] In fact, one also checks easily that $t^{\frac{1}{d}}$ vanishes to order 1 on all compononents of $X_d \setminus U_d$, hence $f_d^{-1}(0_d)$ is also a reduced scheme. But we don't actually use this particular fact.

Next, let Δ_d be the polyhedral complex associated to $U_d \subseteq X_d$. I claim that there is a canonical polyhedral isomorphism

$$\Delta_d \xrightarrow{\approx} \Delta_\nu$$

and in fact a commutative diagram:

$$\begin{array}{ccc} \mathrm{R.S}^{U_d}(X_d) & \xrightarrow{\mathrm{Ord}} & \Delta_d \\ \downarrow & & \downarrow \approx \\ \mathrm{R.S.}^{U_\nu}(X_\nu) & \xrightarrow{\mathrm{Ord}} & \Delta_\nu \end{array}$$

(the first vertical arrow given by composing $\phi : \mathrm{Spec}\, k[[t]] \to X_d$ with the projection $p : X_d \to X_\nu$). In fact, let $Y_d \subseteq X_d$ and $Y_\nu \subseteq X_\nu$ be corresponding strata as in Lemma 2.3.2. Then $p^{-1}(\mathrm{Star}(Y_\nu)) = \mathrm{Star}(Y_d)$ and we get homomorphisms:

$$M^{Y_d} \underset{p_*}{\overset{p^*}{\rightleftarrows}} M^{Y_\nu}$$

where p^* is the pull-back of Cartier divisors, whereas p_* is the norm (i.e., apply norm to local defining equations). Then $p_* \circ p^*$ is just multiplication by e. Moreover p_* is injective (in fact, since p is a bijection on $f_d^{-1}(0_d)$, p_* is injective on Weil divisors concentrated on $f_d^{-1}(0_d)$). Thus $p^* \circ p_*$ is multiplication by e as well, and

$$M_{\mathbb{R}}^{Y_d} \xleftarrow{\approx} M_{\mathbb{R}}^{Y_\nu}$$

is an isomorphism. Clearly p^*D is effective if and only if D is effective, so this induces a dual isomorphism

$$\sigma^{Y_d} \xrightarrow{\approx} \sigma^{Y_\nu}.$$

These clearly patch up into an isomorphism $\Delta_d \xrightarrow{\approx} \Delta_\nu$ commuting with Ord.

However, the one thing which changes when you replace Δ_ν by Δ_d is the integral structure. The integral structures on corresponding polyhedra $\sigma^{Y_d, \sigma^{Y_\nu}}$ are given by the functions defined by M^{Y_d} and M^{Y_ν} respectively. I claim:

$$M^{Y_d} = p^* M^{Y_\nu} + \mathbb{Z} \cdot \left(t^{1/d}\right)$$

or equivalently:

Lemma 2.3.3. *Every Cartier divisor D on Star(Y_d), supported by $f_d^{-1}(0_d)$, is of the form $p^*D_1 + a \cdot \left(t^{1/d}\right)$, $a \in \mathbb{Z}$.*

Proof. In the notation used above, the morphism $p : X_d \longrightarrow X_\nu$ corresponds formally, at every $x' \in X_d$, to the morphism of affine torus embeddings:

$$\mathrm{Spec}\, k[\ldots, x^\alpha, \ldots]_{\alpha \in M \cap \mathbb{R}_+^n} \longrightarrow \mathrm{Spec}\, k[\ldots, x^\alpha, \ldots]_{\alpha \in M_O \cap \mathbb{R}_+^n}$$

But if we choose $x' \in Y_d$, then the formal isomorphisms induce isomorphisms

$$\begin{array}{rcl} M^{Y_d} & \cong & M \\ M^{Y_\nu} & \cong & M_0 \end{array}$$

which lie in a diagram:

$$\begin{array}{rcl} M^{Y_d} & \cong & M \\ {\scriptstyle p^*}\uparrow & & \cup\!\vert \\ M^{Y_\nu} & \cong & M_0 \end{array}$$

Also, (t) corresponds in these isomorphisms to $a_0 = (n(i_1, \dots, n(i_r), 0, \dots, 0))$. Since $M = M_0 + {}^{\alpha_0}/_d \cdot \mathbb{Z}$, this proves the lemma 2.3.3. $\square$

Finally, in all the toric varieties $U_d \subseteq X_d$, we are given a particular positive Cartier divisor with support exactly $X_d \setminus U_d$, namely $\left(t^{1/d}\right)$. Let $\left(t^{1/d}\right)$ define the function $l_d : \Delta_d \longrightarrow \mathbb{R}_+$. Note that via the canonical isomorphism $\Delta_d \xrightarrow{\approx} \Delta_\nu$, $l_d = {}^\nu/_d l_\nu$. As in the end of section 2.2, we can therefore define a compact polyhedral complex

$$\Delta_d^* \doteq \{x \in \Delta_d \mid l_d(x) = 1\}.$$

By restriction, we get an integral structure M_d^* on Δ_d^*. Moreover, by central projection and the canonical isomorphism $\Delta_d \xrightarrow{\approx} \Delta_\nu$, we get a canonical isomorphism

$$\Delta_d^* \xrightarrow{\approx} \Delta_\nu^*,$$

and by Lemma 2.3.3, the integral structures are related by

$$M_d^* = \frac{d}{\nu} M_\nu^* + \mathbb{Z}.$$

In other words, via these isomorphisms

$$\left\{\begin{matrix}\text{the integral lattice} \\ (\Delta_d^*)_{\mathbb{Z}} \text{ in } \Delta_d^*\end{matrix}\right\} = \left\{\begin{matrix}\text{the lattice of } (\Delta_\nu^*)_{\nu/_d\mathbb{Z}} \\ \text{of points in } \Delta_\nu^* \text{ with} \\ \text{coordinates in } {}^\nu/_d\mathbb{Z}\end{matrix}\right\}$$

Now the Main Theorem of the next Chapter, applied to Δ_ν^*, says that there is an integer e and a projective subdivision $[\sigma_\alpha]$ of Δ_ν^* such that:

1. vertices of the subdivision lie in $(\Delta_\nu^*)_{1/_e\mathbb{Z}}$,
2. for all σ_α, with the volume element induced from Δ_ν^*,

$$\mathrm{vol}(\sigma_\alpha) = \frac{1}{e^{d_\alpha} \cdot (d_\alpha)!}, \; d_\alpha = \dim \sigma_\alpha.$$

It follows that if we interpret this as a subdivision of $\Delta_{e\nu}^*$ instead of Δ_ν^*, then:

1. vertices of the subdivision lie in $(\Delta_{e\nu}^*)_{\mathbb{Z}}$
2. for all σ_α of $\dim = d_\alpha$, $\mathrm{vol}(\sigma_\alpha) = {}^1/_{(d_\alpha)!}$

Now apply the results of section 2.2: $\{\sigma_\alpha\}$ defines a proper birational morphism:

$$f : X_{\{\sigma_\alpha\}} \longrightarrow X_{e\nu}, \quad f^{-1}(U_{e\nu}) \cong U_{e\nu}.$$

Since the subdivison is projective, the morphism is defined by blowing up a suitable sheaf of ideals. By Theorem 2.2.2, $t^{1/e\nu}$ vanishes to order 1 on all components of $X_{\{\sigma_\alpha\}} \setminus U_{e\nu}$, and $X_{\{\sigma_\alpha\}}$ is non-singular. It follows from this automatically that the components of $X_{\{\sigma_\alpha\}} \setminus U_{e\nu}$ are non-singular and cross transversely (because $U_{e\nu} \subseteq X_{\{\sigma_\alpha\}}$ is a toric variety without self-intersection). Therefore $X_{\{\sigma_\alpha\}}$ has all the required properties.

Chapter 3

Tropical Geometry

In this chapter we start by introducing tropical hypersurfaces which are an analogue of algebraic varieties over the tropical semiring $\mathbb{T}$. Then we prove structure theorems showing that these are weighted polyhedral complexes that are balanced in a certain sense (see Proposition 3.1.3 and Theorem 3.1.9).

Later we extend the definition of tropical hypersurface to higher codimension using the process of tropicalization for a subvariaty of an algebraic torus defined over a valued field. Then after introduce the necessary machinery we generalize the structure theorems of tropical hypersurfaces to arbitrary tropical varieties.

For the definitions coming from polyhedral geometry used in this chapter we refer the reader to the appendix.

3.1 Tropical Hypersurfaces

Tropical geometry is the study of tropical varieties which are geometric objects constructed from polynomials with coefficient in the *tropical semiring*:

Definition 3.1.1. The tropical semiring (or semiring of tropical numbers) is the set $\mathbb{T} = \mathbb{R} \cup \{\infty\}$ together with two binary operations $\oplus$ and $\odot$ given by $a \oplus b := \min\{a, b\}$ and $a \odot b := a + b$. //

It is called a semiring because it satisfy all the axioms of a ring except the one that assure the existence of inverse for the addition.

As with usual commutative rings, we can consider polynomials or more generally Laurent polynomials over $\mathbb{T}$. These are going to be called *tropical polynomials* and they will be expressions of the form

$$f(x_1, \ldots, x_n) = \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \odot x_1^{\mathbf{u}_1} \ldots x_n^{\mathbf{u}_n}$$

where the sum over $\mathbb{Z}$ has only finite support, the x_i are variables and

$$x_1^{\mathbf{u}_1} \ldots x_n^{\mathbf{u}_n} = x^{\mathbf{u}} = \underbrace{x_1 \odot \cdots \odot x_1}_{\mathbf{u}_1 \text{ times}} \odot \cdots \odot \underbrace{x_n \odot \cdots \odot x_n}_{\mathbf{u}_n \text{ times}}$$

Remark 3.1.2. The formal polynomial above induce naturally the tropical function

$$x \mapsto \min_{\mathbf{u} \in \mathbb{Z}} \{a_{\mathbf{u}} + \mathbf{u}_1 x_1 + \cdots + \mathbf{u}_n x_n\} = \min_{\mathbf{u} \in \mathbb{Z}} \{a_{\mathbf{u}} + \mathbf{u} \cdot x\}$$

but it is not determined by it. For example $\min\{x + y, 2x, 2y\} = \min\{2x, 2y\}\ \forall x, y \in \mathbb{R} \cup \{\infty\}$ but as polynomials we have

$$xy \oplus x^2 \oplus y^2 \neq x^2 \oplus y^2$$

//

Now given a polynomial f, an element $\mathbf{w} \in \mathbb{R}^n$ such that the minimum in $f(\mathbf{w}) = \min_{\mathbf{u} \in \mathbb{Z}}\{a_{\mathbf{u}} + \mathbf{u} \cdot \mathbf{w}\}$ is attained two times is called a zero of f and the set

$$V(f) = \{x \text{ is a zero of } f\} \tag{3.1}$$

is called the *tropical hypersurface* attached to f. Notice that this set only depend on the tropical function induced by f. We will define tropical varieties in general in the next section after we see the concept of tropicalization of algebraic varieties.

When f is a tropical polynomial in two variable with at least two monomials, $V(f)$ is called a *plane tropical curve.*

Proposition 3.1.3. *A tropical hypersurface induced by a tropical polynomial f in n-variables is the support of a Γ_{val}-rational polyhedral complex of pure dimension $n-1$ in $\mathbb{R}^n$.*

Proof. Denoting by the same letter f the function induced by the tropical polynomial we define Σ_f as the coarsest polyhedral complex such that f is linear on each cell in Σ_f. The support of Σ_f is all $\mathbb{R}^n$ and the cells of dimension n have the form

$$\sigma_{\mathbf{u}} = \{\mathbf{w} \in \mathbb{R}^{n+1} \mid f(\mathbf{w}) = c_{\mathbf{u}} + \mathbf{w} \cdot \mathbf{u}\}$$

where $c_{\mathbf{u}}$ moves through the coefficients of f.

We have that $V(f)$ is the support of the $(n-1)$-skeleton of Σ_f and the proposition follows directly from this. □

Example 3.1.4. A tropical line is a tropical curve given by a polynomial of the form $a \odot x \oplus b \odot y \oplus c$. It consist in 3 rays with slopes $1, 0$ and ∞ coming from the point $(c-a, c-b)$ as Figure 3.1 shows. //

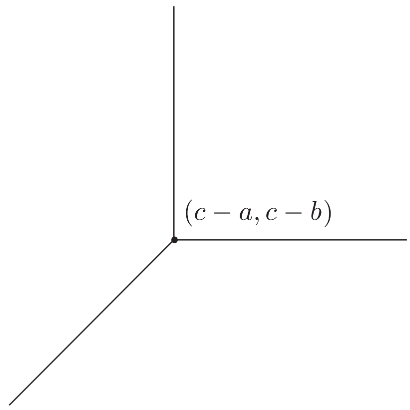


Figure 3.1: A tropical line

Next we will pass to understand this polyhedral complex in terms of a dual construction. To make this precise we need to introduce a new concept.

Given any multivariate Laurent polynomial (not necessary over the tropical semiring)

$$f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$$

we can attached a simple geometric object in $\mathbb{R}^n$ to it called its *Newton polytope* given by

$$\text{Newt}(f) = \text{conv}\{\mathbf{u} \in \mathbb{Z}^n \mid c_{\mathbf{u}} \neq 0\} \subseteq \mathbb{R}^n$$

Of course, as the additive identity of the tropical semiring is ∞ the Newton polytope of a tropical polynomial is given by the convex hull of all $\mathbf{u}$ such that $c_{\mathbf{u}} \neq \infty$.

As a starting point let us notice that when f is a tropical polynomial its Newton polytope depend only on the function induced by f and it can be computed by a simple formula from it.

Theorem 3.1.5. *For a tropical polynomial $f = \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} \odot x^{\mathbf{u}}$ we have*

$$\textit{Newt}(f) = \{\mathbf{v} \in \mathbb{R}^n \mid \exists a \in \mathbb{R} \ \ s.t \ \ f(r\mathbf{w}) \leq a + \mathbf{v} \cdot r\mathbf{w} \ \ \forall \mathbf{w} \in \mathbb{R}^n \ \textit{and} \ r \gg 0\}$$

Proof. Let us called A the set in the right. Is not difficult to see that A is convex. Also as for every $c_{\mathbf{u}} \neq \infty$ we have $f(\mathbf{w}) = \min_{\mathbf{u}}\{c_{\mathbf{u}} + \mathbf{u}\cdot\mathbf{w}\} \leq c_{\mathbf{u}} + \mathbf{u}\cdot\mathbf{w} \;\; \forall\, \mathbf{w}$ we get $\mathbf{u} \in A$. So we conclude $\mathrm{Newt}(f) \subseteq A$.

In the other hand if $\mathbf{v} \in A$ we have $f(r\mathbf{w}) \leq a + \mathbf{v}\cdot r\mathbf{w}$ for large r and so dividing by r and taking limits we get

$$\min_{\mathbf{u} \text{ s.t } c_{\mathbf{u}} \neq \infty}\{\mathbf{u}\cdot\mathbf{w}\} = \lim_{r\to\infty} \frac{1}{r} f(r\mathbf{w}) \leq \lim_{r\to\infty} \frac{1}{r}(a + \mathbf{v}\cdot r\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$$

And from this we get that $\mathbf{v}$ is in the Newton polytope. In fact, if this is not true then by the hyperplane separation theorem we can separate $\mathbf{v}$ from the Newton polytope with a linear functional and hence there is some $\mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{w}\cdot\mathbf{v} < a < \mathbf{w}\cdot\mathbf{u}$ $\forall \mathbf{u}$ with $c_{\mathbf{u}} \neq \infty$ and some constant $a \in \mathbb{R}$. □

Next we have to notice that there is a natural subdivision of the Newton polytope induced by the polynomial f, this is the regular subdivision Δ of $\mathrm{Newt}(f)$ given by the weight vector $(c_{\mathbf{u}} \mid \mathbf{u} \in \mathrm{Newt}(f) \cap \mathbb{Z})$. The construction is given in Definition A.9 in the appendix.

Using this regular subdivision we can state the duality result in which we are interested.

Theorem 3.1.6. *The polyhedral complex defining $V(f)$ with f a tropical polynomial in n variables is dual to the polyhedral complex given by the regular decomposition Δ of the Newton polytope of f.*

Proof. Let $P = \mathrm{conv}\{(\mathbf{u}, x) \in \mathbb{R}^{n+1} \mid x \geq c_{\mathbf{u}},\, c_{\mathbf{u}} \neq 0\}$. The regular subdivision of $\mathrm{Newt}(f)$ induced by the weights $\mathrm{val}(c_{\mathbf{u}})$ consist of the polytopes $\pi(F)$ as F varies over all bounded faces of P and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection into the first n coordinates.

The bounded faces of P are of the form

$$F = \mathrm{face}_{\mathbf{y}}(P) = \{(\mathbf{v}, c) \in P \mid (\mathbf{v}, c)\cdot\mathbf{y} \leq (\mathbf{w}, d)\cdot\mathbf{y} \text{ for all } (\mathbf{w}, d) \in P\}$$

for some $\mathbf{y} \in \mathbb{R}^{n+1}$ with last coordinate positive, without lost of generality we can take this last coordinate to be 1. Then $\mathbf{y} = (\mathbf{z}, 1)$ with $\mathbf{z} \in \mathbb{R}^n$ and replacing we get that the polytope in the regular decomposition is

$$\begin{aligned}\pi(F) &= \{\mathbf{v} \in \mathrm{Newt}(f) \mid \exists c \text{ with } (\mathbf{v}, c) \in P \text{ and } \mathbf{v}\cdot\mathbf{z} + c \leq \mathbf{w}\cdot\mathbf{z} + d \text{ for all } (\mathbf{w}, d) \in P\}\\ &= \mathrm{conv}\{\mathbf{u} \in \mathbb{Z}^n \mid c_{\mathbf{u}} \neq 0 \text{ and } \mathbf{u}\cdot\mathbf{z} + c_{\mathbf{u}} \leq \mathbf{w}\cdot\mathbf{z} + c_{\mathbf{w}} \text{ for all } c_{\mathbf{w}} \neq 0\}\\ &= \mathrm{conv}\{\mathbf{u} \in \mathbb{Z}^n \mid \mathbf{u}\cdot\mathbf{z} + c_{\mathbf{u}} = f(\mathbf{z})\}\end{aligned}$$

If we let $I(F) = \pi(F) \cap \mathbb{Z}^n$ then the polytope $\pi(F)$ of the regular decomposition above will correspond to the set

$$\begin{aligned}F(\sigma) =&\{\mathbf{z} \in \mathbb{R}^n \mid \mathrm{face}_{(\mathbf{z},1)}(P) = F\}\\ =&\{\mathbf{z} \in \mathbb{R}^n \mid f(\mathbf{z}) = \mathbf{u}\cdot\mathbf{z} + c_{\mathbf{u}} \text{ for at least all } \mathbf{u} \in I(F)\}\end{aligned}$$

Is easy to see that $F(\sigma)$ is a cell of the polyhedral complex of $V(f)$ and the union of all this sets is $V(f)$. So the dual of the regular decomposition is exactly $V(f)$. □

An important case of the theorem above is given when the tropical polynomial f has *constant coefficients*, i.e, all its monomials have coefficient 0 (the multiplicative identity of $\mathbb{T}$) attached.

Corollary 3.1.7. *Let $f = \oplus_{\mathbf{u}\in\mathbb{Z}^n} x^{\mathbf{u}}$ be a tropical polynomial with constant coefficients. Then $V(f)$ is a $(n-1)$-dimensional fan in $\mathbb{R}^n$ and it is the $(n-1)$-skeleton of the normal fan to the Newton polytope of f.*

Proof. When f has constant coefficients the decomposition of the Newton polytope associated by f is given by the vector $(c_{\mathbf{u}})_{\mathbf{u}} = 0$, and so the decomposition is the trivial one with only one cell equal to $\mathrm{Newt}(f)$.

Hence, cells of the decomposition are equal to faces F of $\mathrm{Newt}(f)$ and by the proof of 3.1.6 above these correspond exactly to the sets $F(\sigma) = \{\mathbf{z} \in \mathbb{R}^n \mid \mathrm{face}_{\mathbf{z}}(P) = F\}$ so $F(\sigma) = \mathcal{N}_{\mathrm{Newt(f)}}(F)$

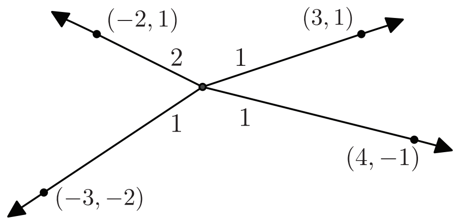


Figure 3.2: A balanced fan of dimension 1.

and then $V(f)$ is the normal fan of Newt(f). □

Next we will prove that the polyhedral complex underlying the tropical hypersurface can be endowed with some natural weights in the cells of maximal dimension such that it become a *balanced polyhedral complex.*

To do this first we introduce the concept of balanced polyhedral complex. The basic idea is given by the case in which the polyhedral complex is a fan Σ of pure dimension 1. Then it consist in a collection of rays σ coming from the origin. Let $\mathbf{v}_\sigma$ denote the first lattice point in the ray σ coming out from zero and $m(\sigma)$ the weight corresponding to σ, then the fan Σ is balanced if

$$\sum_{\sigma\in\Sigma} m(\sigma)\mathbf{v}_\sigma = 0$$

We generalize this concept to fans of higher pure dimension. Once we do it we can use the fans $\text{star}_\Sigma(P)$ (defined in A.8) to extend the definition to general polyhedral complexes. We do all this as follows.

Definition 3.1.8. Let Σ be a fan in $\mathbb{R}^n$ of pure dimension d endowed with weights $m(\sigma) \in \mathbb{N}$ on all maximal cones $\sigma \in \Sigma$. For a cone $\tau \in \Sigma$ of dimension $d-1$ let L be the linear space generated by τ. As τ is rational $L_\mathbb{Z} = L \cap \mathbb{Z}^n$ is a lattice of rank $d-1$, we used this to define the lattice

$$N(\tau) := \mathbb{Z}^n / L_\mathbb{Z} \cong \mathbb{Z}^{n-d+1}$$

Now for each $\sigma \in \Sigma$ containing τ as a proper face the set $(\sigma + L)/L$ is a one dimensional cone in $N(\tau)_\mathbb{R}$. Let $\mathbf{v}_\sigma$ the first lattice point in this ray. The fan Σ will be balanced at τ if

$$\sum_{\sigma\supsetneq\tau} m(\sigma)\mathbf{v}_\sigma = 0 \tag{3.2}$$

and it will be balanced if it is balanced at all $\tau \in \Sigma$ with dimension $d-1$.

Now let Σ be a Γ_{val}-rational polyhedral complex of pure dimension d with weights $m(P) \in \mathbb{N}$ on each d-dimensional cell P. The fan $\text{star}_\Sigma(Q)$ inherits a weighting function m. The complex Σ is balanced if $\text{star}_\Sigma(Q)$ is balanced for all $Q \in \Sigma$ of dimension $d-1$. //

With this definition we are ready to prove the *balancing condition* for tropical hypersurfaces, this is the fact that the polyhedral complex of a tropical hypersurface is balanced with some natural weights.

This weights are defined using Theorem 3.1.6. As the polyhedral complex Σ of a tropical hypersurfaces is dual to the regular subdivisions Δ of its Newton polytope, every facet σ of Σ correspond to an edge $e(\sigma)$ in Δ. We define the multiplicity $m(\sigma)$ of this facet as the lattice length of the edge $e(\sigma)$, i.e, the number of lattice points in the edge minus one.

With this weights we proceed to prove the balancing condition for hypersurfaces.

Theorem 3.1.9. *Any tropical polynomial f in n variables define a tropical hypersuface $V(f)$ that is the support of a $(n-1)$-dimensional polyhedral complex balanced with respect to the weights $m(\sigma)$ defined above.*

Proof. When $n = 1$ the set $V(f)$ is finite and the statement becomes trivial.

For $n = 2$ we have that $V(f)$ is a polyhedral complex of pure dimension 1 and we need to prove that $\text{star}_{V(f)}(\tau)$ is balanced for all its vertices τ. The cell τ is dual to a convex polygon Q in the regular subdivision of the Newton polytope Δ. Each vector $\mathbf{v}_\sigma$ in (3.2) is a primitive lattice vector perpendicular to an edge of Q and the vector $m(\sigma)\mathbf{v}_\sigma$ is precisely an edge of Q rotated by 90 degrees. Then $\sum_{\sigma \supsetneq \tau} m(\sigma)\mathbf{v}_\sigma = 0$ because the sum of the edges vectors of any polygon is 0.

For $n \geq 3$ we can proceed in a similar way to $n = 2$. We need to prove that for any cell τ of codimension 1 the fan $\text{star}_{V(f)}(\tau)$ is balanced. Noticed that the only cone of codimension 1 in $\text{star}_{V(f)}(\tau)$ is τ so we only need to prove that it is τ-balanced. For this we need to pass from the vector space $\mathbb{R}^n$ to the vector space $N(\tau)_\mathbb{R}$ of dim 2 by working modulo L. In terms of the dual space this means to pass to the plane containing the polygon Q dual to τ in Δ. As L is orthogonal to Q we get that the fan given by the cones $(L + \sigma)/L$ in $N(\tau)_\mathbb{R}$ is the normal fan to the polygon Q. So in the same way as for $n = 2$ we can conclude because the sum of the edges vectors of a polygon is 0. □

3.2 Valued Fields and Gröbner Basis

In this section we introduce some commutative algebra that will be useful in the sections to follow. We start with a brief reminder valuations over fields and then we developed a version of Gröbner basis specially designed for polynomial rings over such fields.

3.2.1 Valued Fields

Definition 3.2.1. A rank one valuation or simply a valuation over a field K is a function $\text{val}\colon K \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties:

- $\text{val}(a) = \infty \iff a = 0$
- $\text{val}(ab) = \text{val}(a) + \text{val}(b)$
- $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$

In this context we said that K is a *valued field.* We introduce the *valuation group* given by $\Gamma_{\text{val}} := \text{val}(K^*)$, it is an abelian totally ordered group. The *valuation ring* given by

$$R = \{a \in R \mid \text{val}(a) \geq 0\}$$

that is a local ring with maximal ideal

$$\mathfrak{m}_K = \{c \in K \mid \text{val} \geq 0\}$$

and we also consider the *residue field* given by $\Bbbk = R/\mathfrak{m}_K$. //

Example of valued fields are any field k with the trivial valuation given by $\text{val}(a) = 1 \;\; \forall a \in K^*$, the field $\mathbb{Q}_p$ of p-adic numbers together with its p-adic valuation and the function field $k(C)$ of an algebraic curve C with the valuation given by computing the order of poles at a fixed point p. But along this document the most important valued field will be the next one.

Example 3.2.2 (Field of Puiseux series)**.** For any field k we defined the field $k\{\{t\}\}$ of Puiseux series as the set of all the formal power series

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_2 t^{a_2} + \dots$$

where each $c_i \in k$ and $a_1 < a_2 < a_3 < \dots$ are rational with a common denominator. In other words

$$k\{\{t\}\} = \bigcup_{n \geq 1} k((t^{1/n}))$$

where $k((t^{1/n}))$ is the field of Laurent series over $t^{1/n}$.

This field has a natural valuation defined by

$$\begin{aligned} \text{val} \colon k\{\{t\}\}^* &\to \mathbb{R} \\ \sum_{n\geq 1} c_n t^{a_n} &\mapsto a_1 \end{aligned}$$

//

One reason why this field is important is because of the following result whose proof can be seen in [Rib99] p. 186.

Theorem 3.2.3. *When k is algebraically closed of characteristic 0 the field $k\{\{t\}\}$ is the algebraic closure of the field $k((t))$ of Laurent series and in particular is algebraically closed.*

The valuation defined over the field $K\{\{t\}\}$ has a canonical *split* given by the map

$$\begin{aligned} \psi \colon \Gamma_{\text{val}} = \mathbb{Q} &\to k\{\{t\}\}^* \\ a &\mapsto t^a \end{aligned}$$

this means that ψ satisfy $\text{val}(\psi(a)) = a$. In general such a split ψ exists for any algebraically closed field as we prove now.

Proposition 3.2.4. *For any algebraically closed field K with a valuation val there is group homomorphism $\psi\colon \Gamma_{val} \to \mathbb{R}$ such that $val \circ \psi = id_{\Gamma_{val}}$*

Proof. As K is algebraically closed both the multiplicative group $(K^*, \cdot)$ and the valuation group $(\Gamma_{\text{val}}, +)$ are divisible:

- From $(\Gamma_{\text{val}}, +)$ being divisible, as it is also torsion free, we get that it should be a $\mathbb{Q}$ vector space and hence it has a basis $\{w_i\}_{i\in I}$

- From $(K^*, \cdot)$ being divisible we get that for any $a \in K^*$ there is a group homomorphism $\psi_a \colon \mathbb{Q} \to K^*$ such that $\psi_a(1) = a$.

Now for each $w_i \in \Gamma_{\text{val}}$ we can construct a homomorphism $\psi_i \colon w_i\mathbb{Q} \to K^*$ such that if $\text{val}(a) = w_i$ then $\psi_i(w_i) = a$ and hence $\psi = \oplus_i \psi_i \colon \Gamma_{\text{val}} \to K^*$ is the desired splitting homomorphism. □

By an abuse of notation we are going to denote this map by $a \mapsto t^a$ as in the case of Puiseux series.

3.2.2 Gröbner Basis for Valued Fields

In what follows we will introduce a theory of Gröbner basis specially designed for homogeneous ideals in a polynomial ring with coefficients in a valued field. For technical reasons we will assume that the valuation splits. Because of Proposition 3.2.4 above we know that this is true whenever K is algebraically closed, it is also true when K has the trivial valuation.

We start in the same way as for usual Gröbner basis defining the ideal of initial terms. For this we need to define the initial term of a single polynomial, this depend on a weight vector $\mathbf{w} \in \mathbb{R}^{n+1}$ who takes the roll of the monomial order in the usual Gröbner basis.

For a polynomial $f \in K[x_0, \ldots, x_n]$ we can construct its *tropicalization* as the real function $\mathbb{R}^{n+1} \to \mathbb{R}$ given by

$$\text{Trop}(f)(\mathbf{w}) = \min\{\text{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} \mid \mathbf{u} \in \mathbb{N}^{n+1} \text{ and } c_{\mathbf{u}} \neq 0\}$$

Taking $W = \text{Trop}(f)(\mathbf{w})$ the *initial form* of f with respect to $\mathbf{w}$ is the polynomial in $\Bbbk[x_0, \ldots, x_n]$ defined as

$$\mathrm{in}_{\mathbf{w}}(f) \;=\; \sum_{\mathbf{u}\in\mathbb{N}^{n+1} \text{ s.t } \mathrm{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=W} \overline{c_{\mathbf{u}} t^{-\mathrm{val}(c_{\mathbf{u}})}} x^{\mathbf{u}}$$

Notice that f and $\mathrm{in}_{\mathbf{w}}(f)$ are in different polynomial rings and that the monomials appearing in $\mathrm{in}_{\mathbf{w}}(f)$ are a certain subset of the monomials in f. We can expressed $\mathrm{in}_{\mathbf{w}}(f)$ also as follows.

$$\begin{aligned}\mathrm{in}_{\mathbf{w}}(f) &= \overline{t^{-W} \sum_{\mathbf{u}\in\mathbb{N}^{n+1}} c_{\mathbf{u}} t^{\mathbf{w}\cdot\mathbf{u}} x^{\mathbf{u}}} \\ &= \overline{t^{-\mathrm{Trop}(f)(\mathbf{w})} f(t^{\mathbf{w}_0}x_0, \ldots, t^{\mathbf{w}_n}x_n)}\end{aligned}$$

Now for a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ we set its *initial ideal* as

$$\mathrm{in}_{\mathbf{w}}(I) \;=\; \langle \mathrm{in}_{\mathbf{w}}(f) \mid f \in I\rangle \subseteq \Bbbk[x_0, \ldots, x_n]$$

A finite set $\mathcal{G} = \{g_1, \ldots, g_n\} \subseteq I$ will be called *Gröbner basis* for I with respect to w if

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(g_1), \ldots, \mathrm{in}_{\mathbf{w}}(g_n)\rangle$$

Remark 3.2.5.

1. Even though we don't put in the definition that the Gröbner basis generate the ideal I this is always true as shown in [Cha13]. But if we apply the same definition of Gröbner basis to ideals that are not homogeneous then this is no longer true.
2. The usual Gröbner basis with respect to the monomial weight order determined by $-\mathbf{w}$ correspond to the particular case of a field with trivial valuation of the Gröbner basis treated here with respect to $\mathbf{w}$.

//

The first result is that Gröbner basis do exist and we can take it homogeneous.

Proposition 3.2.6. *Let $I \subseteq K[x_0, \ldots, x_n]$ be a homogeneous ideal. Then there is a Gröbner basis of I consisting of homogeneous polynomials, in particular the initial ideal $in_{\mathbf{w}}(I)$ is homogeneous. Also given $g \in in_{\mathbf{w}}(I)$ there is $f \in I$ such that $g = in_{\mathbf{w}}(f)$.*

Proof. Noticed that if $f = \sum_{i\geq 0} f_i$ with each f_i homogeneous we have $\mathrm{in}_{\mathbf{w}}(f) = \sum \mathrm{in}_{\mathbf{w}}(f_i)$ where the sum goes through the f_i such that $\mathrm{Trop}(f)(\mathbf{w}) = \mathrm{Trop}(f_i)(\mathbf{w})$ and hence we have

$$\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f) \mid f \in I \text{ with } f \text{ homogeneous}\rangle$$

So taking a finite subset of this generating set, as $\mathrm{in}_{\mathbf{w}}(f)$ is homogeneous when f is homogeneous, we get that I has a Gröbner basis made of homogeneous ideals.

Now given $g \in \mathrm{in}_{\mathbf{w}}(I)$ we have $g = \sum a_{\mathbf{u}} x^{\mathbf{u}} \mathrm{in}_{\mathbf{w}}(f_{\mathbf{u}})$ for some $f_{\mathbf{u}} \in I$, $a_{\mathbf{u}} \in \Bbbk^*$. Taking a lift $c_{\mathbf{u}}$ of $a_{\mathbf{u}}$ to K^* we can define

$$f = \sum_{\mathbf{u}} c_{\mathbf{u}} t^{-W_{\mathbf{u}}} x^{\mathbf{u}} f_{\mathbf{u}} \quad \text{for} \quad W_{\mathbf{u}} = \mathrm{Trop}(f_{\mathbf{u}})(\mathbf{w}) + \mathbf{w}\cdot\mathbf{u}$$

Then by construction $\mathrm{Trop}(f)(\mathbf{w}) = 0$ and $\mathrm{in}_{\mathbf{w}}(f) = \sum a_{\mathbf{u}} x^{\mathbf{u}} \mathrm{in}_{\mathbf{w}}(f_{\mathbf{u}}) = g$. □

Next we iterate this construction by taking initial forms of initial forms. For this as $\mathrm{in}_{\mathbf{w}}(f)$ is a polynomial with coefficients in $\Bbbk$ we need to endowed this field with the trivial valuation.

Lemma 3.2.7. *Let $f \in K[x_0, \ldots, x_n]$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. There for every $\varepsilon > 0$ small enough we have*

$$in_{\mathbf{v}}(in_w(f)) = in_{\mathbf{w}+\varepsilon\mathbf{v}}(f) \tag{3.3}$$

Proof. If $f = \sum_{\mathbf{u}} c_{\mathbf{u}} x^{\mathbf{u}}$ and $W = \mathrm{Trop}(f)(\mathbf{w})$ we have

$$\mathrm{in}_{\mathbf{w}}(f) \;=\; \sum_{\substack{\mathbf{u}\in\mathbb{N}^{n+1} \text{ s.t} \\ \mathrm{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=W}} \overline{c_{\mathbf{u}} t^{\mathbf{w}\cdot\mathbf{u}-W}} x^{\mathbf{u}}$$

and if $W' = \min\{\mathbf{v}\cdot\mathbf{u} \mid \mathrm{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u} = W\}$ then

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f)) = \sum_{\substack{\mathbf{u}\in\mathbb{N}^{n+1} \text{ s.t} \\ \mathbf{v}\cdot\mathbf{u}=W'}} \overline{c_{\mathbf{u}} t^{\mathbf{w}\cdot\mathbf{u}-W}} x^{\mathbf{u}} \tag{3.4}$$

Now for small ε small enough we have that

$$\mathrm{Trop}(f)(\mathbf{w}+\varepsilon\mathbf{v}) = \min(\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u} + \varepsilon\mathbf{v}\cdot\mathbf{u}) = W + \varepsilon W'$$

and the minimum has equality if an only if $\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot\mathbf{u} = W$ and $\varepsilon\mathbf{v}\cdot\mathbf{u} = \varepsilon W'$. Hence we have $\mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}} = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))$ as we wanted. □

The idea now is to extend equation (3.3) from a fixed polynomial f to an entire homogeneous ideal I. We will do this by steps, the first step being the following lemma.

Recall that a monomial ideal is an ideal that can be generated by monomials and if I is a monomial ideal then $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in I \implies x^{\mathbf{u}} \in I$.

Lemma 3.2.8. *Let I be a homogeneous ideal in $K[x_0, \ldots, x_n]$ and fix $\mathbf{w} \in \mathbb{R}^{n+1}$. Then there exists $\mathbf{v} \in \mathbb{R}^{n+1}$ such that for every $\varepsilon > 0$ small enough the ideals $in_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ and $in_{\mathbf{v}}(in_w(I))$ are monomial ideals and*

$$in_{\mathbf{v}}(in_{\mathbf{w}}(I)) \subseteq in_{\mathbf{w}+\varepsilon\mathbf{v}}(I) \tag{3.5}$$

Proof. First notice that from Proposition 3.2.6 the ideal $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous and so equation (3.5) makes sense.

Now assuming that there is a $\mathbf{v}$ such that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal we will show it satisfy (3.5). For this take generators $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \langle x^{\mathbf{u}_1}, \ldots, x^{\mathbf{u}_s}\rangle$. By Proposition 3.2.6 again there are $f_i \in I$ such that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f_i)) = x^{\mathbf{u}_i}$. By Lemma 3.2.7 we have for ε small enough $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f_i)) = \mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(f_i)$ for all all i at the same time and this imply $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \subseteq \mathrm{in}_{w+\varepsilon\mathbf{v}}(I)$.

Next we will prove that there is such a $\mathbf{v}$ with $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ a monomial ideal. For $\mathbf{v} \in \mathbb{R}^n$ consider the ideal $M_{\mathbf{v}}$ generated by all the monomials in $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$. As the polynomial ring is Noetherian we can chose $\mathbf{v}$ such that $M_{\mathbf{v}}$ is not contained in any other $M'_{\mathbf{v}}$. If $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is not a monomial ideal then there is $f \in I$ such that no term of $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))$ is in $M_{\mathbf{v}}$. Take $\mathbf{v}' \in \mathbb{R}^{n+1}$ such that $\mathrm{in}_{\mathbf{v}'}(\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f)))$ is a monomial, to find such a $\mathbf{v}'$ use equation (3.4) and consider $\mathbf{v}'$ such that $\mathrm{face}_{\mathbf{v}'}(\mathrm{Newt}(\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(f))))$ is only a vertex so $\mathbf{v}\cdot\mathbf{u}$ is minimized only for one $\mathbf{u}$. By Lemma 3.2.7 for ε small enough $\mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{v}'}(\mathrm{in}_{\mathbf{w}}(f))$ is this monomial. On the other hand each monomial in $M_{\mathbf{v}}$ can be written as $x^{\mathbf{u}} = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(g))$ for some g and then $x^{\mathbf{u}} = \mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{v}'}(\mathrm{in}_{\mathbf{w}}(g))$ for epsilon small enough, this implies that $M_{\mathbf{v}} \subseteq M_{\mathbf{v}+\varepsilon\mathbf{v}'}$ but because of the monomial constructed above we actually have $M_{\mathbf{v}} \subsetneq M_{\mathbf{v}+\varepsilon\mathbf{v}'}$. This contradicts our choise of $M_{\mathbf{v}}$ and so $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal.

We can now modify our $\mathbf{v}$ such that $\mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{w}}(I)$ is at the same time also a monomial ideal. To see this we work in a similar way as before, define $M^{\varepsilon}_{\mathbf{v}}$ as the ideal generated by all the monomials in $\mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{w}}(I)$ and between all $\mathbf{v}$ such that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is a monomial ideal take it such that $M^{\varepsilon}_{\mathbf{v}}$ is not contained in any other $M^{\varepsilon}_{\mathbf{v}'}$. If $\mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{w}}(I)$ is not a monomial ideal we take $f \in I$ such that no term of $\mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{w}}(f)$ is in $M^{\varepsilon}_{\mathbf{v}}$. Then we take $\mathbf{v}'$ so that $\mathrm{face}_{\mathbf{v}'}(\mathrm{Newt}(\mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(f)))$ is a vertex. In the same way as above this implies $M^{\varepsilon}_{\mathbf{v}} \subsetneq M^{\varepsilon}_{\mathbf{v}+\varepsilon\mathbf{v}'}$. Taking ε small we have $M_{\mathbf{v}+\varepsilon\mathbf{v}'} = M_{\mathbf{v}}$ so that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I))$ is still a monomial ideal. This contradiction show us that $\mathrm{in}_{\mathbf{v}+\varepsilon\mathbf{w}}(I)$ is also a monomial ideal. □

From now on we will fixed the notation $S_K = K[x_0, \ldots, x_n]$ and $S_{\Bbbk} = [x_0, \ldots, x_n]$.

Given an ideal $I \subseteq S_K$ we can measure its size using its *Hilbert function.* These is the function $\mathbb{N} \to \mathbb{N}$ given by $d \mapsto \dim_K(S_K/I)_d$. For $d \gg 0$ the Hilbert function agrees with a polynomial, called the *Hilbert polynomial* of I, that encodes important invariant of the ideal. We will show next that the Hilbert function of an ideal coincide with the Hilbert function of the initial ideal. First we do if for monomial initial ideals.

Lemma 3.2.9. *Let $I \subseteq S_K$ be a homogeneous ideal and take $\mathbf{w} \in \mathbb{R}^{n+1}$ such that $in_{\mathbf{w}}(I)$ is a monomial ideal. Then the monomials $x^{\mathbf{u}}$ of degree d that are not in $in_{\mathbf{w}}(I)$ form a K-basis for $(S_K/I)_d$.*

Proof. Consider the set $\mathcal{B}_d$ of all monomials x^u not contained in $\text{in}_{\mathbf{w}}(I)$. The image of this set in $(S_K/I)_d$ is linearly independent over K. To see this noticed that a linear combination gives a polynomial $f = \sum c_{\mathbf{u}} x^{\mathbf{u}} \in I_d$ with no term belonging to $\text{in}_{\mathbf{w}}(I)$. But from $f \in I$ we get $\text{in}_{\mathbf{w}}(f) \in \text{in}_{\mathbf{w}}(I)$ and as this ideal is a monomial ideal we get $x^{\mathbf{u}} \in \text{in}_{\mathbf{w}}(I)$ for some $\mathbf{u}$. This contradiction shows that $\dim_K(S_K/I)_d$ is bigger that the amount of monomials not contained in $\text{in}_{\mathbf{w}}(I)_d$ and hence $\dim_{\Bbbk} \text{in}_{\mathbf{w}}(I)_d \geq \dim_K I_d$.

Now for each monomial $x^{\mathbf{u}} \in \text{in}_{\mathbf{w}}(I)_d$, choose $f_{\mathbf{u}} \in I_d$ with $\text{in}_{\mathbf{w}}(f_{\mathbf{u}}) = x^{\mathbf{u}}$. The set $\{f_{\mathbf{u}} \mid x^{\mathbf{u}} \in \text{in}_{\mathbf{w}}(I)_d\}$ is linearly independent in S_K. Indeed, if not there are $a_{\mathbf{u}} \in K$ not all zero with $\sum_{\mathbf{u}} a_{\mathbf{u}} f_{\mathbf{u}} = 0$. Let $f_{\mathbf{u}} = x^{\mathbf{u}} + \sum c_{\mathbf{uv}} x^{\mathbf{v}}$ and looking at the coefficient $\mathbf{u}'$ in the linear combination above we get $a_{\mathbf{u}'} + \sum_{\mathbf{u} \neq \mathbf{u}'} a_{\mathbf{u}} \mathbf{c}_{\mathbf{uu}'} = 0$. Suppose $\mathbf{u}'$ is taken such that $\text{val}(a_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u}$ is minimal. Then, looking at the valuation in the left-hand side of this equation, as the minimum should be attained two times in particular there is $\bar{\mathbf{u}}$ different from $\mathbf{u}'$ such that $\text{val}(a_{\bar{\mathbf{u}}}) + \text{val}(c_{\bar{\mathbf{u}}\mathbf{u}'}) \leq \text{val}(a_{\mathbf{u}'})$. And from here

$$\text{val}(a_{\bar{\mathbf{u}}}) + \text{val}(c_{\bar{\mathbf{u}}\mathbf{u}'}) + \mathbf{w} \cdot \mathbf{u}' \leq \text{val}(a_{\mathbf{u}'}) + \mathbf{w} \cdot \mathbf{u}' \leq \text{val}(a_{\bar{\mathbf{u}}}) + \mathbf{w} \cdot \bar{\mathbf{u}}$$

Which contradicts the fact that $\text{in}_{\mathbf{w}}(\text{f}_{\bar{\mathbf{u}}}) = x^{\bar{\mathbf{u}}}$ (we no longer have that $\text{Trop}(f_{\bar{\mathbf{u}}})(w)$ attains the minimum at $\bar{\mathbf{u}}$).

This shows $\dim_K I_d \geq \dim_{\Bbbk} \text{in}_{\mathbf{w}}(I)_d$. Hence the dimension of this ideals are equal and then

$$\dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\text{in}_{\mathbf{w}}(I))_d$$

With $\mathcal{B}_d$ a K basis for $(S_K/I)_d$. □

Now we drop the hypothesis of $\text{in}_w(I)$ being monomial and get the last equality of dimensions anyway.

Proposition 3.2.10. *For any $\mathbf{w} \in \mathbb{R}^{n+1}$ and any homogeneous ideal I in S_K, the Hilbert function of I agrees with that of its initial ideal $\text{in}_{\mathbf{w}}(I) \subseteq S_{\Bbbk}$, i.e,*

$$\dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\text{in}_{\mathbf{w}}(I))_d$$

for all $d \geq 0$. In particular the Krull dimension of the rings S_K/I and $S_{\Bbbk}/\text{in}_{\mathbf{w}}(I)$ coincide.

Proof. By Lemma 3.2.8 we can take $\mathbf{v}$ such that for small $\varepsilon > 0$ we have $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) \subseteq \text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ and both are monomial ideals and then by Lemma 3.2.9 the set of monomials that are not in $\text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)_d$ span $(S_K/I)_d$. Hence given $x^{\mathbf{u}} \in \text{in}_{w+\varepsilon\mathbf{v}}(I)_d \setminus \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))_d$ there are polynomials $f_{\mathbf{u}}$ and $f'_{\mathbf{u}}$ with $f_{\mathbf{u}} = x^{\mathbf{u}} - f'_{\mathbf{u}} \in I_d$ and no monomial of $f'_{\mathbf{u}}$ is in $\text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)_d$. But then $\text{in}_{\mathbf{w}}(f_{\mathbf{u}})$ contains only monomials not in $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))$ which contradicts $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(f)) \in \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))$. This implies $\text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)_d = \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))$. Now Lemma 3.2.9 applied to this monomial ideals give us

$$\dim_{\Bbbk}(S_{\Bbbk}/\text{in}_{\mathbf{w}}(I))_d = \dim_{\Bbbk}(S_{\Bbbk}/\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)))_d \text{ and } \dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I))_d$$

From which we get

$$\dim_{\Bbbk}(S_{\Bbbk}/\text{in}_{\mathbf{w}}(I))_d = \dim_K(S_K/I)_d$$

As we wanted. □

With this we can deduce the result about iterated initial ideals we were looking for.

Corollary 3.2.11. *Let I be a homogeneous ideal in $K[x_0, \ldots, x_n]$. For any $\mathbf{w}, \mathbf{v} \in \mathbb{R}^{n+1}$ if we take $\varepsilon > 0$ small enough we have*

$$\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$$

Proof. Take a basis $g_1, \ldots, g_s \in \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))$ and write each g_i as $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(f_i))$. By Lemma 3.2.7 above for ε small enough we have $g_i = \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(f_i)) = \text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(f_i)$ and so $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) \subseteq \text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ but by the proposition above, both $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I))$ and $\text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ have the same Hilbert function so the inclusion is not strict. □

3.2.3 Gröbner Complexes

Now we will construct a polyhedral complex for each homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$. The polyhedra in our complex will be given by the topological closure of the sets

$$C_I[\mathbf{w}] = \{\mathbf{v} \in \mathbb{R}^{n+1} : \text{in}_{\mathbf{v}}(I) = \text{in}_{\mathbf{w}}(I)\}$$

we denote by $\overline{C_I[\mathbf{w}]}$ this closure and by $\mathbf{1}$ the vector $(1, \dots, 1) \in \mathbb{R}^{n+1}$.

Proposition 3.2.12. *The set $\overline{C_I[\mathbf{w}]}$ is a Γ_{val}-rational polyhedron whose lineality space contains the line $\mathbb{R}\mathbf{1}$. If $in_{\mathbf{w}}(I)$ is not a monomial ideal, then there exists $\mathbf{w}' \in \Gamma_{val}^{n+1}$ such that $in_{\mathbf{w}'}(I)$ is a monomial ideal and $\overline{C_I[\mathbf{w}]}$ is a proper face of the polyhedron $\overline{C_I[\mathbf{w}']}$.*

Proof. Given $\mathbf{w}$ we will start finding $\mathbf{w}' \in \Gamma_{\text{val}}^{n+1}$ such that $\text{in}_{\mathbf{w}'}(I)$ is a monomial ideal and $\overline{C_I[\mathbf{w}]} \subseteq \overline{C_I[\mathbf{w}']}$. For this use Lemma 3.2.8 to find a $\mathbf{v}$ such that for small epsilon $\text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ is a monomial ideal and choose $\mathbf{w}' = \mathbf{w} + \varepsilon\mathbf{v}$. As ε can be arbitrarily small we have $\mathbf{w} \in \overline{C_I[\mathbf{w}']}$ and then $C_I[\mathbf{w}] \subseteq \overline{C_I[\mathbf{w}']}$ so we can conclude.

Next we prove that $\overline{C_I[\mathbf{w}']}$ is a polyhedral complex. We will do this by choosing an adequate Gröbner basis on I as follows: Take a monomial basis $\text{int}_{\mathbf{w}}(I) = \langle x^{\mathbf{u}_1}, \dots, x^{\mathbf{u}_s}\rangle$, by Lemma 3.2.9 for each i the monomial of degree $d = \deg(\mathbf{u}_i)$ not contained in $\{\mathbf{u}_1, \dots, \mathbf{u}_s\}$ form a basis of $(S_K/I)_d$. Writing $x^{\mathbf{u}_i}$ in this basis we find a polynomial g_i' such that $g_i := x^{\mathbf{u}_i} - g_i' \in I$. By construction $\text{in}_{w'}(g_i) = x^{\mathbf{u}_i}$ and hence $\{g_1, \dots, g_s\}$ is a Gröbner basis. Now $\overline{C_I[\mathbf{w}']}$ has a Γ_{val}-rational polyhedral structure shown by the equality

$$\overline{C_I[\mathbf{w}']} = \{\mathbf{z} \in \mathbb{R}^{n+1} \mid \mathbf{u}_i \cdot \mathbf{z} \leq \text{val}(c_{i\mathbf{v}}) + \mathbf{v} \cdot \mathbf{z} \ \text{ for } 1 \leq i \leq s,\ \mathbf{v} \in \mathbb{N}^{n+1})\}$$

where $c_{i\mathbf{v}}$ is the coefficient of the monomial $x^{\mathbf{v}}$ in $g_i - x^{\mathbf{u}}$. Let us prove this equality.

Suppose $\tilde{\mathbf{w}} \in C_I[\mathbf{w}']$ but one of the inequalities $\mathbf{u}_i \cdot \mathbf{z} \leq \text{val}(c_{i\mathbf{v}}) + \mathbf{v} \cdot \mathbf{z}$ is not valid when $\mathbf{z} = \tilde{\mathbf{w}}$. For that i we have $\text{in}_{\tilde{\mathbf{w}}}(g_i) \neq x^{\mathbf{u}_i}$ but then $\text{in}_{\tilde{\mathbf{w}}}(g_i)$ contains other monomials of g_i which is not possible because $\text{in}_{\tilde{\mathbf{w}}}(g_i) \in \text{in}_{\tilde{\mathbf{w}}}(I) = \text{in}_{\mathbf{w}'}(I)$ and by construction the other monomials of g_i are not in $\text{in}_{\mathbf{w}'}(I)$. This proves that $C_I[\mathbf{w}']$ is contained in the right-hand side.

For the other inclusion, it is enough to prove that if $\mathbf{u}_i \cdot \tilde{\mathbf{w}} < \text{val}(c_{i\mathbf{v}}) + \mathbf{v} \cdot \tilde{\mathbf{w}}$ for all i, then $\tilde{\mathbf{w}} \in \overline{C_I[\mathbf{w}']}$, as such set of $\tilde{\mathbf{w}}$ is dense in the right-hand side. Notice that $\mathbf{u}_i \cdot \tilde{\mathbf{w}} < \text{val}(c_{i\mathbf{v}}) + \mathbf{v} \cdot \tilde{\mathbf{w}}$ is equivalent to $\text{in}_{\tilde{\mathbf{w}}}(g_i) = x^{\mathbf{u}_i}$ and so this for all i implies $\text{in}_{\tilde{\mathbf{w}}}(I) \subseteq \text{in}_{\mathbf{w}'}(I)$ and as this two ideals have the same Hilbert function we conclude $\text{in}_{\tilde{\mathbf{w}}}(I) = \text{in}_{\mathbf{w}'}(I)$ from which $\tilde{\mathbf{w}} \in C_I[\mathbf{w}']$. This proves the equality.

Now we show that $\overline{C_I[\mathbf{w}]}$ is actually a face of the polyhedron $\overline{C_I[\mathbf{w}]}$, in particular it is a Γ_{val}-rational polyhedron. To see this note that $\text{in}_{\tilde{\mathbf{w}}}(I) = \text{in}_{\mathbf{w}}(I)$ implies $\text{in}_{\tilde{\mathbf{w}}}(g_i) = \text{in}_{\mathbf{w}}(g_i)$ because if this is not the case, as $x^{\mathbf{u}_i}$ is the only monomial of $\text{in}_{\tilde{\mathbf{w}}}(g_i)$ and of $\text{in}_{\mathbf{w}}(g_i)$ that is contained in $\text{in}_{\mathbf{w}'}(I)$, then $\text{in}_{\tilde{\mathbf{w}}}(g_i) - \text{in}_{\mathbf{w}}(g_i) \in \text{in}_{\mathbf{w}}(I)$ would be a polynomial without monomials in $\text{in}_{\mathbf{w}'}(I)$ which contradicts that $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}'}(I)$. Also as the set $\{\text{in}_{\mathbf{w}}(g_1), \dots, \text{in}_{\mathbf{w}}(g_s)\}$ is a Gröbner basis for $\text{in}_{\mathbf{w}}(I)$ with respect to $\mathbf{v}$ we have

$$\begin{aligned} \tilde{\mathbf{w}} \in C_I[\mathbf{w}] &\iff \text{in}_{\tilde{\mathbf{w}}}(I) = \text{in}_{\mathbf{w}}(I) \\ &\iff \text{in}_{\tilde{\mathbf{w}}}(g_i) = \text{in}_{\mathbf{w}}(g_i) \ \ \forall 1 \leq i \leq s \end{aligned}$$

So $\overline{C_I[\mathbf{w}]}$ is exactly the set of points $\mathbf{z}$ in the polyhedron $\overline{C_I[\mathbf{w}']}$ that satisfy $\mathbf{u}_i \cdot \mathbf{z} = \text{val}(c_{i\mathbf{v}}) + \mathbf{v} \cdot \mathbf{z}$ whenever $x^{\mathbf{v}}$ appears in $\text{in}_{\mathbf{w}}(g_i)$.

Finally, note that for a homogeneous polynomial f we have $\text{in}_{\mathbf{w}}(f) = \text{in}_{\mathbf{w}+\lambda\mathbf{1}}(f)$ from which $\text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{w}+\lambda\mathbf{1}}(I)$ and then $\mathbf{v} \in \overline{C_I[\mathbf{w}]} \implies \mathbf{v} + \lambda\mathbf{1} \in \overline{C_I[\mathbf{w}]}$ so the lineality space of $\overline{C_I[\mathbf{w}]}$ contains $\mathbb{R}\mathbf{1}$. □

Since the line $\mathbb{R}\mathbf{1}$ is always contained in $\overline{C_I[\mathbf{w}]}$ we can regard this polyhedron as a polyhedron in the quotient space $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$. The idea is to prove that all this polyhedra form a polyhedral complex in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ as $\mathbf{w}$ varies. In order to prove this we will need some lemmas.

Lemma 3.2.13. *Let I be a homogeneous ideal in $K[x_0, \dots, x_n]$. There are only finitely many distinct monomial initial ideals $in_{\mathbf{w}}(I)$ as $\mathbf{w}$ varies over $\mathbb{R}^{n+1}$.*

Proof. By [Mac01] any antichain of monomial ideals on $K[x_0, \ldots, x_n]$ is finite. Hence if the statement is not true there are $\mathbf{w}, \mathbf{w}' \in \mathbb{R}^{n+1}$ such that $\text{in}_{\mathbf{w}}(I) \subsetneq \text{int}_{\mathbf{w}'}(I)$ which is not possible because by Proposition 3.2.10 $\text{in}_{\mathbf{w}}(I)$ and $\text{int}_{\mathbf{w}'}(I)$ have the same Hilbert function so the inclusion above should be an equality. □

Lemma 3.2.14. *Let $r \leq s$ integers, A an $r \times s$ matrix of rank r with entries in K and fix $\mathbf{w} \in \mathbb{R}^s$. There exists $U \in GL(r, K)$ and an index set $J = \{l_1, \ldots, l_r\} \subset \{1, \ldots, s\}$ such that the $r \times r$ submatrix of UA with columns in J, denoted by $(UA)^J$, is the identity matrix and $val((UA)_{ij}) + \mathbf{w}_j \geq \mathbf{w}_{l_i}$ for $j \notin J$.*

Proof. As A has rank r there are J such that $\text{val}(\det(A^J)) + \sum_{j \in J} \mathbf{w}_j$ is finite and we fix J such that this number is minimal. In particular $\det(A^J) \neq 0$ so we take $U := {A^J}^{-1}$ and then the matrix UA has the identity matrix in the columns of J. Next to see the inequality with the valuations noticed that if $J_{ij} = J \backslash \{l_i\} \cup \{j\}$ the matrix $UA^{J_{ij}}$ is an identity matrix with the column i changed by the column j of UA and so $\det(UA^{J_{ij}}) = (UA)_{ij}$, hence

$$\begin{aligned}
\text{val}(UA)_{ij} =& \text{val}(\det(UA^{J_{ij}})) \\
=& \text{val}(\det(U)) + \text{val}(\det(A^{J_{ij}})) \\
=& -\text{val}(\det(A^J)) + (\text{val}(\det(A^{J_{ij}})) + \sum_{j \in J_{ij}} \mathbf{w}_j) - \sum_{j \in J_{ij}} \mathbf{w}_j \\
\leq& -\text{val}(\det(A^J)) + (\text{val}(\det(A^J)) + \sum_{j \in J} \mathbf{w}_j) - \sum_{j \in J_{ij}} \mathbf{w}_j \\
=& \mathbf{w}_{l_i} - \mathbf{w}_j
\end{aligned}$$

□

Now fix a homogeneous ideal $I \subseteq S = K[x_0, \ldots, x_n]$. Let $d \in \mathbb{N}$ and choose a K-basis $\{f_1, \ldots, f_r\}$ of I_d, its homogeneous part of degree d. Let $\mathcal{M}_d$ be the set of monomials of degree d in S and consider A_d the $(r \times |\mathcal{M}_d|)$-matrix whose entry $(A_d)_{i,u}$ is the coefficients in the monomial u of the polynomial f_i. Each $J \subseteq \mathcal{M}_d$ with $|J| = r$ specifies an $r \times r$ minor $\det(A_d^J)$ and the vector with entries all this minors is the vector of plücker coordintates of the point I_d in the Grassmanian $G(r, S_d)$. In particular is independent of the basis f_i we chose.

By Lemma 3.2.13 above there exist $D \in \mathbb{N}$ such that any initial monomial ideal $\text{in}_{\mathbf{w}}(I)$ of I has generators of degree at most D. We define the polynomials

$$g_d := \sum_{\substack{J \subseteq \mathcal{M}_d \\ |J| = r}} \det(A_d^J) \prod_{u \in J} x^{\mathbf{u}} \text{ and } g := \prod_{d=1}^{d} g_d$$

With this notations we have

Lemma 3.2.15. *Let $I \subseteq K[x_0, \ldots, x_n]$ and g_d, g as above. Also let $\Sigma_{Trop}(g)$ be the coarsest polyhedral complex in which $Trop(g)$ is linear. Them if $\mathbf{w} \in \mathbb{R}^{n+1}$ lies in the interior of a maximal cell σ in $\Sigma_{Trop}(g)$ we have $\overline{C_I[\mathbf{w}]} = \sigma$.*

Proof. We have to prove that $\mathbf{w}' \in \mathbb{R}^{n+1}$ lies in the interior of the maximal cell σ if and only if $\text{in}_{\mathbf{w}'}(I) = \text{in}_{\mathbf{w}}(I)$. For proving this we will use that if we denote by $\Sigma_{\text{Trop}}(g_d)$ also the coarsest polyhedral complex in which $\text{Trop}(g_d)$ is linear then $\text{Trop}(g)$ is the common refinement of the $\text{Trop}(g_d)$. So if σ_d is the maximal cell in $\text{Trop}(g_d)$ containing σ it's enough to prove that for all $\mathbf{w} \in \mathbb{R}^{n+1}$ and for all $d \leq D$.

$$\mathbf{w}' \in \text{int}(\sigma_d) \iff \text{in}_{\mathbf{w}'}(I)_d = \text{in}_{\mathbf{w}}(I)_d \tag{3.6}$$

Let us prove the equivalence (3.6). Take $\mathbf{w}' \in \text{in}(\sigma_d)$. As $\mathbf{w}$ and $\mathbf{w}'$ lye in the maximal same cell of σ_d the minimum of $\text{Trop}(g_d)$ is achieved in only one term and this term is the same for $\mathbf{w}$ and $\mathbf{w}'$. This term of $\text{Trop} g_d$ is indexed for some $J \subset \mathcal{M}_d$. Now apply Lemma 3.2.14 to the matrix A_d and the vector $\tilde{\mathbf{w}} \in \mathbb{R}^{|\mathcal{M}_d|}$ such that $\tilde{\mathbf{w}}_u = \mathbf{w} \cdot \mathbf{u}$. We get an $(r \times |\mathcal{M}_d|)$-matrix B with B^J the identity and $\text{val}(B_{\mathbf{uv}}) + \mathbf{w} \cdot \mathbf{v} > \mathbf{w} \cdot \mathbf{u}$ for $x^{\mathbf{u}} \in J, x^{\mathbf{v}} \notin J$. The inequality is strict because the minimum is achieved only once. The rows of B represent polynomials

$\tilde{f}_{\mathbf{u}} = x^{\mathbf{u}} + \sum_{x^{\mathbf{v}}\notin J} B_{\mathbf{uv}} x^{\mathbf{v}}$ indexed by $x^{\mathbf{u}} \in J$. Then the inequality translate in $\text{int}_{\mathbf{w}}(\tilde{f}_{\mathbf{u}}) = x^{\mathbf{u}}$ so $x^{\mathbf{u}} \in \text{in}_{\mathbf{w}}(\tilde{f}_{\mathbf{u}}) = x^{\mathbf{u}}$. As $\dim_K(I)_d = \dim_{\Bbbk}(\text{in}_{\mathbf{w}}(I))_d$ we get $(\text{in}_{\mathbf{w}}(I))_d = \langle J\rangle$. Analogously we have $(\text{in}_{\mathbf{w}'}(I))_d = \langle x^{\mathbf{u}} \mid x^{\mathbf{u}} \in J\rangle$ so we conclude $(\text{in}_{\mathbf{w}'}(I))_d = (\text{in}_{\mathbf{w}'}(I))_d$.

Now for the reverse implication suppose $\mathbf{w}' \notin \text{int}(\sigma_d)$. This means that there is $J' \in \binom{\mathcal{M}_d}{r}$ diferent from J such that $\text{Trop}(g_d)$ is minimized in the term of g_d indexed by J'. That is

$$\text{val}(A_d^{J'}) + \sum_{x^{\mathbf{u}}\in J} \mathbf{w}' \cdot \mathbf{u} \leq \text{val}(A_d^{J''}) + \sum_{x^{\mathbf{u}}\in J''} \mathbf{w}' \cdot \mathbf{u} \quad \forall J'' \in \binom{\mathcal{M}}{r}$$

We can take take J' such that it index a vertex of the polytope

$$\text{conv}(\sum_{x^{\mathbf{u}}\in J''} \mathbf{u} \mid \text{val}(A_d^{J''}) + \sum_{x^{\mathbf{u}}\in J''} \mathbf{w}' \cdot \mathbf{u}) \text{ is minimal}$$

Hence there is $\mathbf{v} \in \mathbb{R}^{n+1}$ with $\mathbf{v} \cdot' \sum_{x^{\mathbf{u}}\in J'} \mathbf{u} < \mathbf{v} \cdot \sum_{x^{\mathbf{u}}\in J''} \mathbf{u}$ for all $J'' \in \binom{|\mathcal{M}_d|}{r} \setminus \{J'\}$ and then $\forall \varepsilon > 0$ we have

$$\text{val}(A_d^{J''}) + \sum_{x^{\mathbf{u}}\in J'} (\mathbf{w}' + \varepsilon\mathbf{v}) \cdot \mathbf{u} < \text{val}(A_d^{J''}) + \sum_{x^{\mathbf{u}}\in J''} (\mathbf{w}' + \varepsilon\mathbf{v}) \cdot \mathbf{u}$$

So $\text{Trop}(g_d)(\mathbf{w}' + \varepsilon\mathbf{v})$ attains its minimum uniquely. Then by the result we get at the end of the "only if" part we get $\text{in}_{\mathbf{w}'+\varepsilon\mathbf{v}}(I)_d = \text{span}\{x^{\mathbf{u}} \mid x^{\mathbf{u}} \in J'\}$ but as $\text{in}_{\mathbf{w}'+\varepsilon\mathbf{v}}(I) = \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}'}(I))$ we get that $\text{in}_{\mathbf{w}'}(I)_d$ cannot be the span of the monomials in J and so $\text{in}_{\mathbf{w}'}(I)_d \neq \text{in}_{\mathbf{w}}(I)_d$ as we wanted. □

Now we are ready to prove that $\overline{C_I[\mathbf{w}]}$ fit together in a polyhedral complex.

Theorem 3.2.16. *The polyhedra $\overline{C_I[\mathbf{w}]}$ as $\mathbf{w}$ varies over $\mathbb{R}^{n+1}$ form a Γ_{val}-rational polyhedral complex in $\mathbb{R}^{n+1}$.*

Proof. By Lemma 3.2.15 all top dimensional cells of the polyhedral complex $\Sigma_{\text{Trop}}(g)$ are of the form $\overline{C_I[\mathbf{w}]}$ for some $\mathbf{w}$ with $\text{in}_{\mathbf{w}}(I)$ a monomial ideal. In the other hand if $\mathbf{w}$ is such that $\text{in}_{\mathbf{w}}(I)$ is a monomial ideal then $\text{in}_{\mathbf{w}}(I) = \text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ for any $\mathbf{v}$ and small ε, so $C_I[\mathbf{w}]$ is open and then $\overline{C_I[\mathbf{w}]}$ is a maximal cell. For this reason the set $\{\overline{C_I[\mathbf{w}]} \mid \text{in}_{\mathbf{w}}(I) \text{ is a monomial ideal}\}$ is exactly the set of maximal cells of $\Sigma_{\text{Trop}}(g)$. If $\text{in}_{\mathbf{w}}(I)$ is not a monomial ideal then by Lemma 3.2.12 $\overline{C_I[\mathbf{w}]}$ is a face of some $\overline{C_I[\mathbf{w}']}$ with $\text{in}_{\mathbf{w}'}(I)$ a monomial ideal and so the polyhedral complex is exactly $\Sigma_{\text{Trop}}(g)$. □

The polyhedral complex in this theorem is denoted by $\Sigma(I)$ and is called the *Gröbner complex* of the homogeneous ideal I.

3.2.4 Gröbner Basis for Laurent Ideals

Finally let us say that the definitions of initial form and initial ideal can be extended to the ring of Laurent polynomials $K[x_1^{\pm 1}, \ldots, x_n^{\pm n}]$ by using the same definitions as for the usual ring of polynomials $K[x_0, \ldots, x_n]$. That is, given $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm n}]$ and $\mathbf{w} \in \mathbb{R}^n$ we define

$$\text{in}_{\mathbf{w}}(f) \;=\; \sum_{\substack{\mathbf{u} \text{ such that} \\ \text{val}(c_{\mathbf{u}})+\mathbf{w}\cdot\mathbf{u}=W}} \overline{c_{\mathbf{u}} t^{-\text{val}(c_{\mathbf{u}})}} \cdot x^{\mathbf{u}} \in \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

and for an ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ we define $\text{in}_{\mathbf{w}}(I) = \{\text{in}_{\mathbf{w}}(f) \mid f \in I\} \subseteq \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

Now given a *Laurent ideal* $I \subseteq K[x_1^{\pm}, \ldots, x_n^{\pm}]$ we define its *homogenization* as the ideal $I_{\text{proj}} \subseteq K[x_0, \ldots, x_n]$ generated by all the polynomials

$$\tilde{f} \;=\; x_0^m \cdot f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$$

as f varies over I and where m is the smallest integer that clear all denominators.

We can compute the initial ideal of a Laurent ideal through the initial ideal of its homogenization.

Proposition 3.2.17. *Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be an ideal and fix $\mathbf{w} \in \mathbb{R}^n$. Then $in_{\mathbf{w}}(I)$ is generated by the image of $in_{(0,\mathbf{w})}(I_{proj})$ in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ obtained by setting $x_0 = 1$. Also every element $f \in in_{\mathbf{w}}(I)$ has the form $f = x^u g|_{x_0=1}$ for some $g \in in_{(\mathbf{0},\mathbf{w})}(I_{proj})$.*

Proof. For $f = \sum_{c_{\mathbf{u}}} x^{\mathbf{u}} \in I \cap K[x_1, \dots, x_n]$ its homogenization is $\tilde{f} = \sum c_{\mathbf{u}} x^{\mathbf{u}} x_0^{j_{\mathbf{u}}}$ where $j_{\mathbf{u}} = (\max_{c_{\mathbf{v}} \neq 0} |\mathbf{v}|) - |\mathbf{u}|$. Then as

$$W = \mathrm{Trop}(f)(\mathbf{w}) = \min(\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u}) = \min(\mathrm{val}(c_{\mathbf{u}}) + (0, \mathbf{w}) \cdot (j_{\mathbf{u}}, 0)) = \mathrm{Trop}(\tilde{f})(0, \mathbf{w})$$

We get $\mathrm{in}_{(0,\mathbf{w})}(\tilde{f}) = \sum_{\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = W} \overline{c_{\mathbf{u}} t^{-\mathrm{val}(c_{\mathbf{u}})}} x^{\mathbf{u}} x_0^{j_{\mathbf{u}}}$ and therefore $\mathrm{in}_{(0,\mathbf{w})}(\tilde{f})|_{x_0=1} = \mathrm{in}_{\mathbf{w}}(f)$.

After multiplying for monomials if necessary we can choose $f_1, \dots, f_s \in K[x_1, \dots, x_n] \cap I$ such that $\mathrm{in}_{\mathbf{w}}(I) = \langle \mathrm{in}_{\mathbf{w}}(f_1), \dots, \mathrm{in}_{\mathbf{w}}(f_s) \rangle$. Then $\mathrm{in}_{\mathbf{w}}(I) \subseteq \mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})|_{x_0=1}$ as $\mathrm{in}_{\mathbf{w}}(\tilde{f}_i)|_{x_0=1} = \mathrm{in}_{\mathbf{w}}(f_i)$. In the other sense taking a homogeneous basis $\langle g_1, \dots, g_r \rangle = I_{\mathrm{proj}} \subseteq K[x_0, \dots, x_n]$ we can write each g_i as $x_0^{j_i} \tilde{f}_i$ for some j where $\tilde{f}_i$ is the homogenization of $f_i(x) = g(1, x)$. Then $\mathrm{in}_{\mathbf{w}}(f_i) = \mathrm{in}_{(0,\mathbf{w})}(\tilde{f}_i)|_{x_0=1} = \mathrm{in}_{(0,\mathbf{w})}(\tilde{g}_i)|_{x_0=1}$ so $\mathrm{in}_{(0,\mathbf{w})}(I_{\mathrm{proj}})|_{x_0=1} \subseteq \mathrm{in}_{\mathbf{w}}(I)$.

Finally if $f \in \mathrm{in}_{\mathbf{w}}(I)$ by Proposition 3.2.6 we have $f = \mathrm{in}_{\mathbf{h}}$ for $h \in I$, then taking $u \in \mathbb{Z}^n$ with $x^u h \in \mathrm{in}_{\mathbf{w}}(I) \cap K[x_1, \dots, x_n]$ we have $\mathrm{in}_{(0,w)}(\widetilde{x^u h}) = \mathrm{in}_{\mathbf{w}}(x^u h) = x^u f$ so taking $g = \mathrm{in}_{(0,w)}(\tilde{h})$ we conclude. □

3.3 Tropicalization of Algebraic Varieties

The connection of tropical geometry with algebraic geometry is given by the process of *tropicalization* of a variety defined over a field with a valuation. Here we introduce this process and use it to define general tropical variety. Then we prove that these tropical varieties are the support of a pure Γ_{val}-rational polyhedral complex.

We will denote by T_K^n the torus $\mathbb{G}_K^n$ along this sections and the next one.

Definition 3.3.1. Let K be a field with a valuation and not necessarily algebraically closed.

1. Given a Laurent polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}} \in K[x_1^{\pm}, \dots, x_n^{\pm 1}]$ its tropicalization is the tropical polynomial given by

$$\mathrm{Trop}(f) = \bigoplus_{\mathbf{u} \in \mathbb{Z}^n} \mathrm{val}(c_{\mathbf{u}}) \odot x^{\mathbf{u}}$$

   or in terms of functions

$$\mathrm{Trop}(f)(\mathbf{w}) = \min\{\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} \mid \mathbf{u} \in \mathbb{Z}^n \text{ and } c_{\mathbf{u}} \neq 0\}$$

2. A closed subvariaty $X \subseteq T_K^n$ is detemined by an ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We define its tropicalization as the set

$$\mathrm{Trop}(X) = \bigcap_{f \in I} V(\mathrm{Trop}(f))$$

   where we are using equation (3.1) to interpret $V(\mathrm{Trop}(f))$.

3. A *tropical variety* in $\mathbb{R}^n$ will be any subset of the form $\mathrm{Trop}(X)$ for some subvariety X of a torus T.

//

Remark 3.3.2.

- In order to compute $\bigcap_{f \in I} V(\mathrm{Trop}(f))$ is not enough to consider only generators of the ideal I because intersections does not commute with tropicalization. For that reasons an intersection of finitely many tropical hypersurfaces is not necessarily a tropical variety. A set of generators $\mathcal{T} = \{f_1, \dots, f_N\} \subseteq I$ such that

$$\mathrm{Trop}(V(I)) = \bigcap_{f \in \mathcal{T}} V(\mathrm{Trop}(f))$$

  is called a *tropical basis* of I. Every ideal has a tropical basis (see [MS15] Theorem 2.6.6).

- As any tropical polynomial can be written as the tropicalization of a usual polynomial over a valued field, and as $\text{Trop}(V(f)) = V(\text{Trop}(f))$ we see that any tropical hypersurface as defined in section 3.1 is a tropical variety as defined here. Also we have $\text{Newt}(f) = \text{Newt}(\text{Trop}(f))$.

//

There are different ways of computing the tropicalization of an algebraic variety. These are summarized in the next result that can be found in [MS15] Theorem 3.2.3.

Theorem 3.3.3 (Fundamental Theorem of Tropical Algebraic Geometry). *Let K algebraically closed field with a nontrivial valuation and let X be a subvariety of the algebraic torus T^n_K. Then the following three subsets of $\mathbb{R}^n$ coincide:*

- *$Trop(X)$*
- *the set of vectors $\mathbf{w} \in \mathbb{R}^n$ with $in_{\mathbf{w}}(I) \neq (1)$ where $X = V(I)$ for some ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$*
- *the closure of the set of coordinatewise valuations of points in X*

$$val(X) = \{(val(y_1), \dots, val(y_n)) \mid (y_1, \dots, y_n) \in X\}$$

Furthemore, if X is irreducible and $\mathbf{w}$ is any point in Γ^n_{val} then the set $\{y \in X \mid val(y) = \mathbf{w}\}$ is Zariski dense in the classical variety X

Remark 3.3.4. Given any field extension $K' \mid K$ and an ideal $I \subseteq K[x_1, \dots, x_n]$, one can always take a tropical basis of $IK'[x_1, \dots, x_n]$ composed of only polynomials defined over K (see [MS15] Lemma 2.6.5). Hence, passing to a field extension does not change the tropical variety associated. Then if K is not algebraically closed we may pass to its algebraic closure $\bar{K}$ with an extension of its valuation or if the valuation is trivial we can pass to the field of Puiseux series $K\{\{t\}\}$. //

The objective of this chapter is to prove the following structure theorem.

Theorem 3.3.5 (Structure Theorem for Tropical Varieties). *Let X be an irreducible d-dimensional subvariety of the torus T^n_K. Then $Trop(X)$ is the support of a weighted Γ_{val}-rational polyhedral complex of pure dimension d satisfying the balancing condition.*

We will devote the rest of this section to prove the first part of this theorem. Theorem 3.3.6 below deal with the fact that $\text{Trop}(X)$ is the support of a rational polyhedral complex and Theorem 3.3.14 below show that this polyhedral complex is of pure dimension.

Theorem 3.3.6. *For any subvariety of the torus X the set $Trop(X)$ is the support of a Γ_{val}-rational polyhedral complex.*

Proof. Let $X = V(I)$ with $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. By Theorem 3.3.3 the underlying set of $\text{Trop}(X)$ is equal to $\{\mathbf{w} \in \mathbb{R}^n \mid \text{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$. By Proposition 3.2.17 we have $\text{in}_{\mathbf{w}}(I) = \langle 1 \rangle$ if and only if $1 \in \text{in}_{(0,\mathbf{w})}(I_{\text{proj}})|_{x_0=1}$ and this happens if and only if there is a polynomial in x_0 times a monomial in $x_1, \dots, x_n$ inside $\text{int}_{(\mathbf{w},0)}(I_{\text{proj}})$ but as $\text{int}_{(\mathbf{w},0)}(I_{\text{proj}})$ is homogeneous this is the happens if and only if it contains a monomial.

Hence $\text{Trop}(X)$ is the set of $\mathbf{w}$ for which $\text{in}_{(\mathbf{w},0)}(I_{\text{proj}})$ does not contain a monomial. This is a union of cells in the Gröbner complex $\Sigma(I_{\text{proj}})$ constructed in 3.2.16. Also the set of $\mathbf{w}$ for which $\text{in}_{(\mathbf{w},0)}(I_{\text{proj}})$ contains a monomial is open because by Lemma 3.2.11 if $\text{in}_{\mathbf{w}}(I)$ contains a monomial, also $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I)$ contains a monomial for small ε. Hence $\text{Trop}(X)$ is closed and then is a subcomplex of the Gröbner complex $\Sigma(I_{\text{proj}})$. □

Remark 3.3.7. Given a subvariety $X \subseteq T^n_K$ with ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ there are more than one possible polyhedral complexes Σ such that $\text{Trop}(X) = |\Sigma|$. The proof of the proposition above tell us that there is always one such that for every $\sigma \in \Sigma$ we have that $\text{in}_{\mathbf{w}}(I)$ is constant for all $\mathbf{w} \in \text{int}(\sigma)$ and between all these the construction above give us the coarsest one. //

As an application of the things developed here we can understand the tropical variety defined by the initial ideals $\text{in}_{\mathbf{w}}(I)$ in terms of the tropical variety defined by I.

Proposition 3.3.8. *Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and let Σ be a polyhedral complex with support $Trop(V(I))$. Then for any $\sigma \in \Sigma$ we have*

$$star_\Sigma(\sigma) = Trop(V(in_{\mathbf{w}}(I)))$$

where $\mathbf{w}$ is in the relative interior of σ. In particular $star_\Sigma(\sigma)$ is a tropical variety.

Proof.

$$\begin{aligned}
\mathrm{Trop}(V(\mathrm{in}_{\mathbf{w}}(I))) =& \{\mathbf{v} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \neq \langle 1 \rangle\} \\
=& \{\mathbf{v} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I) \neq \langle 1 \rangle \text{ for small } \varepsilon\} \\
=& \{\mathbf{v} \in \mathbb{R}^n \mid \mathbf{w} + \varepsilon\mathbf{v} \in \Sigma \text{ for small } \varepsilon\} \\
=& \ \mathrm{star}_\Sigma(\sigma)
\end{aligned}$$

Where we use Theorem 3.3.3 in the first equality, Corollary 3.2.11 in the second equality, then Theorem 3.3.3 again and finally the definition of the star of a polyhedral complex. □

3.3.1 Monomial maps and change of coordinates

Now we will prove that this polyhedral complexes have pure dimension. For this we will need some change of coordinates on the variety defined by I in order to simplify the problems. The natural change of coordinates for a variety inside a torus are given by monomial maps.

A *monomial map* is an algebraic homomorphism between two tori. From Proposition 1.1.1 we have

$$\mathrm{Hom}_{\text{alg. groups}}(T_K^n, T_K^m) \cong \bigoplus_{i,j=1}^{n,m} Hom(\mathbb{G}_{m,K}, \mathbb{G}_{m,K}) \cong \bigoplus_{i,j=1}^{n,m} \mathbb{Z} = \mathrm{Mat}(m \times n, \mathbb{Z})$$

In concrete terms the map attached to a matrix $(\alpha_{i,j})_{i,j} \in \mathrm{Mat}(m \times n, \mathbb{Z})$ is given by

$$(x_1, \dots, x_n) \mapsto (x_1^{\alpha_{1,1}} x_2^{\alpha_{1,2}} \cdots x_n^{\alpha_{1,n}}, \dots, x_1^{\alpha_{m,1}} x_2^{\alpha_{m,2}} \cdots x_n^{\alpha_{m,n}})$$

One also have a notion of tropicalization for monomial maps: Given the monomial map $\varphi : T_K^n \to T_K^m$ it induced a map between the coordinate rings $\varphi^* : K[x_1^{\pm 1}, \dots, x_m^{\pm 1}] \to K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and this map correspond to a group homomorphism $\varphi' : \mathbb{Z}^m \to \mathbb{Z}^n$.

We define the tropicalization of φ as the map

$$\mathrm{Trop}(\varphi) := \varphi' \otimes_{\mathbb{Z}} \mathbb{R} : \mathbb{R}^m \to \mathbb{R}^n$$

If $\varphi^*(x_i) = z^{a_i}$ then $\mathrm{Trop}(\varphi)$ is represented by A^T where A is the matrix with ith column equals a_i.

With this concept we can state the following results.

Proposition 3.3.9.

1. *Let $\varphi : T_K^n \to T_K^m$ be a monomial map. Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm}]$ be an ideal and let $I' = \varphi^{*-1}(I)$. Then for all $\mathbf{w} \in \mathbb{R}^n$*

$$\varphi^*(in_{Trop(\varphi)(\mathbf{w})}(I')) \subseteq in_{\mathbf{w}}(I)$$

In particular if $in_{\mathbf{w}}(I) \neq \langle 1 \rangle$ we also have $in_{Trop(\varphi)(\mathbf{w})}(I') \neq \langle 1 \rangle$ and if ϕ is a monomial automorphism

$$in_{\mathbf{w}}(I) \neq \langle 1 \rangle \iff in_{Trop(\varphi)(\mathbf{w})}(I') \neq \langle 1 \rangle$$

2. *Let $\varphi : T_K^n \to T_K^m$ be a monomial map. Consider any subvariety X of T_K^n and the zariski closure $\overline{\varphi(X)}$ of its image in T_K^m. Then*

$$Trop(\overline{\varphi(X)}) = Trop(\varphi)(Trop(X))$$

Proof.

1. Let us denote the coordinates in T^m by x_i and in T^n by z_m and suppose $\varphi^*(x_i) = z^{a_i}$ where $a_i \in \mathbb{Z}^n$. Then $\varphi^*(x^{\mathbf{u}}) = z^{A\mathbf{u}}$ where A is the matrix with ith column equals a_i. Now if $f = \sum c_{\mathbf{u}} x^{\mathbf{u}}$ we have $\varphi^*(f) = \sum c_{\mathbf{u}} z^{A\mathbf{u}} \in I$ so

$$W = \text{Trop}(f)(A^T\mathbf{w}) = \min_u(\text{val}(c_{\mathbf{u}}) + \mathbf{w}\cdot A\mathbf{u}) = \text{Trop}(\varphi^*(f))(\mathbf{w})$$

and then

$$\begin{aligned}\varphi^*(\text{in}_{\text{Trop}(\varphi)(\mathbf{w})}(f)) &= \varphi^*\left(\sum_{\text{val}(c_{\mathbf{u}})+\mathbf{w}\cdot A\mathbf{u}=W} \overline{t^{-\text{val}(c_{\mathbf{u}})}c_{\mathbf{u}}}\cdot x^{\mathbf{u}}\right)\\ &= \sum_{\text{val}(c_{\mathbf{u}})+\mathbf{w}\cdot A\mathbf{u}=W} \overline{t^{-\text{val}(c_{\mathbf{u}})}c_{\mathbf{u}}}\cdot x^{A\mathbf{u}}\\ &= \text{in}_{\mathbf{w}}(\varphi^*(f))\end{aligned}$$

and from this $\varphi^*(\text{in}_{\text{Trop}(\varphi)(\mathbf{w})}(I')) \subseteq \text{in}_{\mathbf{w}}(I)$ as we wanted. The following part of the lemma follows easily from this.

2. Let I be the ideal of X and $I' := (\varphi^*)^{-1}(I)$ the ideal of $\overline{\varphi(X)}$. By part 1 above if we have $\text{in}_{\mathbf{w}}(I) \neq \langle 1\rangle$ then we have $\text{in}_{\text{Trop}(\varphi)(\mathbf{w})}(I') \neq \langle 1\rangle$ and this shows $\text{Trop}(\varphi)(\text{Trop}(X)) \subseteq \text{Trop}(\overline{\varphi(X)})$.

   For the converse by Theorem 3.3.3 we have that $\text{Trop}(\overline{\varphi(X)}) = \text{cl}(\{\text{val}(z) \mid z \in \overline{\varphi(X)}\})$ where $\text{cl}(\cdot)$ denotes the topological closure. Since $\text{Trop}(\varphi)(\text{Trop}(X))$ is already closed we just have to prove that
$$\Gamma^m_{\text{val}} \cap \text{Trop}(\overline{\varphi(X)}) \subseteq \text{Trop}(\varphi)(\text{Trop}(X))$$
   so let $\mathbf{w}$ in the left-hand side of this. By Theorem 3.3.3 the set of $z \in \overline{\varphi(X)}$ for which $\text{val}(z) = \mathbf{w}$ is Zariski dense in $\overline{\varphi(X)}$ so there is y in X such that if $\varphi(y) = z$ then $\text{val}(z) = \mathbf{w}$. As $\text{val}(\varphi)(y) = \text{Trop}(\varphi)(\text{val}(y))$ we get $\mathbf{w} \in \text{Trop}(\varphi)(\text{Trop}(X))$.

□

We can construct some interesting monomial maps using the following result from linear algebra.

Lemma 3.3.10. *If L is a rank k subgroup of $\mathbb{Z}^n$ such that $\mathbb{Z}^n/L$ is torsion-free, then there is a matrix $U \in GL(n,\mathbb{Z})$ with $U(L) = \langle e_1,\dots,e_k\rangle$. In particular for any primitive vector $\mathbf{v} \in \mathbb{Z}^n$ there is a matrix $U \in GL(n,\mathbb{Z})$ such that $U\mathbf{v} = e_1$.*

Proof. As $L = \langle\mathbf{v}\rangle$ has torsion-free quotient $\mathbb{Z}^n/L$ exactly when $\mathbf{v}$ is primitive we see the last part follows from the first.

Now given L consider the $k\times n$ matrix A with column vectors given by an integral basis of L. The Smith normal of A is a matrix A' with its first $k\times k$ block diagonal and such that $A' = SAT$ for some $S \in GL(n,\mathbb{Z}), T \in GL(k,\mathbb{Z})$. As $\mathbb{Z}/L$ is torsion free we have that the first block of A' is actually the identity and so the rows of A' span the vector subspace $\langle e_1,\dots,e_k\rangle$. Then the rows of AT also span this vector subspace and so we can take $U = T^T$. □

3.3.2 Dimension of tropical varieties

Now we deal with the dimension part of Theorem 3.3.5.

We start by proving that given a $X \subseteq T^n_K$ and $m \leq \dim(X)$ there is a projection map $T^m_K \to T^n_K$ with nice behaviour and preserving the dimension of X.

Proposition 3.3.11. *Fix a subvariety X in T^n_K and $m \leq \dim(X)$. There is a monomial morphism $\pi : T^n_K \to T^m_K$ such that $\pi(X)$ is closed and $\dim(\pi(X)) = \dim(X)$.*

Proof. As composition of maps with this properties have this properties we just deal with the case $n = m+1$. Let I be the ideal defining X. Consider the monomial change of coordinates $\varphi : T^n_K \to T^m_K$ defined by $\varphi^*_l(x_1) = x_1 x_n^{l^{n-1}}, \varphi^*_l(x_2) = x_2 x_n^{l^{n-2}}, \dots \varphi^*_l(x_n) = x_n$. For any fixed $f \in I$, taking l large enough we have that the Laurent polynomial

$$\varphi^*_l(f) = f(x_1 x_n^{l^{n-1}}, x_2 x_n^{l^{n-2}}, \dots, x_n)$$

have all its monomials with different degree in x_n. As φ_l^* is invertible, after replacing I with $\varphi_l^*(I)$, we can suppose that I is generated by a set of polynomials satisfying this property. After this the map $\pi : T_K^n \to T_K^{n-1}$ given by the projection onto the first $n-1$ coordinates have the desired properties:

- We have $\overline{\pi(X)} = V(I \cap k[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}])$ and the difference $\overline{\pi(X)} \setminus \pi(X)$ is contained in the variety defined by the leading coefficients of polynomials in a set of generator of I when viewed as polynomials in x_n. As these leading coefficients are taking to be monomials in $x_1, \dots, x_{n-1}$ we conclude that $\overline{\pi(X)} \setminus \pi(X) =$ so $\pi(X)$ is closed.
- To see $\dim(X) = \dim(\pi(X))$ is enough to show that $K[\pi(X)]$ is a finite extension of $K[X]$, as then they have the same trascendence degree over K. For this notice that I contains a monic in x_n with coefficients in $x_1, \dots, x_{n-1}$ and hence x_n is integral over $K[\pi(X)]$.

□

Using this we can prove the following result about zero dimensional tropical varieties.

Lemma 3.3.12. *Let X be a subvariety of the algebraic torus T_K^n. If the tropical variety Trop(X) is a finite set of points in $\mathbb{R}^n$, then X is a finite set of points in T.*

Proof. Let's do induction on n. For $n = 1$ every proper subvariety of T_K^1 is finite and Trop$T_K^1) = \mathbb{R}$ so the statement is clear.

For $n \geq 2$ we can suppose X is not a hypersurface because in that case Proposition 3.1.3 says that Trop(X) is not finite. As every codimension 1 subvariety of an affine variety with coordinate ring UFD (in particular of a torus) is a hypersurface we can assume $\dim(X) \leq n-2$. Now using Proposition 3.3.11 choose a map $\pi : T_K^n \to T_K^{n-1}$ with $Y := \overline{\pi(X)} = \pi(X)$. Changing coordinates we assume that π is the projection onto the first $n-1$ coordinates. By part 2 of Proposition 3.3.9 above we have that Trop(Y) is finite and then by induction Y is finite. Let $Y = \{y_1, \dots, y_r\} \subset T_K^{n-1}$. As we can assume $\lambda e_n \notin$ Trop(X) for $\lambda \gg 0$, the ideal I of X need to contain at least one polynomial of the form $1 + \sum_{i=1}^s f_i x_n^i$ with $f_i \in K[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$. Then each $\pi^{-1}(y_i)$ can have at most s elements and so X is also finite. □

Now we will use the lemma above plus the following result about commutative algebra to finished the proove

Lemma 3.3.13. *Suppose K is algebraically closed and its valution is non trivial. Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a prime ideal of dimension d and fix $\mathbf{w}$ with in$_{\mathbf{w}}(I) \neq \langle 1 \rangle$. Then in$_{\mathbf{w}}(I)$ has pure dimension d in the sense that every minimal prime ideal of it has dimension d.*

Proof. We will start proving that for every homogeneous prime ideal $J \subseteq K[x_0, \dots, x_n]$ and $\mathbf{w} \in \mathbb{R}^{n+1}$ the initial ideal in$_{\mathbf{w}}(J)$ has pure dimension d. By the hypothesis over K we have that Γ_{val} is dense in $\mathbb{R}$, thus the cell containing $\mathbf{w}$ in the Gröbner complex $\Sigma(J)$ contains a point $\mathbf{w}' \in \Gamma_{\text{val}}^n$ and hence we can assume $\mathbf{w} \in \Gamma_{\text{val}}^n$. Also by Proposition 3.2.10 the dimension of in$_{\mathbf{w}}(J)$ is d and hence any minimal prime of the initial ideal has dimension at most d. We will prove the converse now.

As $\mathbf{w} \in \Gamma_{\text{val}}^{n+1}$ we may use a change of coordinates given by $\varphi^*(x_i) = t^{\mathbf{w}_i} x_i$ for all i, and as in$_{\mathbf{w}}(J) = $ in$_{\mathbf{0}}(\varphi^*(J))$ we can suppose $\mathbf{w} = \mathbf{0}$. Now let $\{g_1, \dots, g_s\}$ be a Gröbner basis. After multiplying by t^{c_i} for some big c_i we can suppose $g_i \in R[x_0, \dots, x_n]$ where R is the valution ring of K and $\bar{g}_i \neq 0$ for all i.

We consider a Noetherian subring R' of R in which the ideal J is defined. It will be constructed as follows. Let $\tilde{R}$ be the subring of R generated by the coefficients of all the g_i and let $\tilde{\mathfrak{m}} = \mathfrak{m}_K \cap \tilde{R}$. Then R' is the localization of $\tilde{R}$ with respect to $\tilde{\mathfrak{m}}$. As R' is a localization of a finitely generated ring it is Noetherian. We denote by K' the fraction field of R', by $\mathfrak{m}' = R'\tilde{\mathfrak{m}}$ the maximal ideal of R' and by $\Bbbk' = R'/\mathfrak{m}'$ the subfield of $\Bbbk$.

Let $c = \dim(R')$. By the converse of Krull's principal ideal theorem (see [Eis95], Corollary 10.5) there are $a_1, \dots, a_c \in \mathfrak{m}'$ for which $\mathfrak{m}'$ is minimal over $\langle a_1, \dots, a_c \rangle$. Since $\mathfrak{m}'$ is the only maximal ideal in R', any other prime ideal containing $\langle a_1, \dots, a_c \rangle$ is equal to $\mathfrak{m}'$. Also let $J' = J \cap R'[x_0, \dots, x_n]$ and $J'' = J \cap K'[x_0, \dots, x_n]$. As $J = J'' \otimes_{K'} K$ we get $(K'[x_0, \dots, x_n]/J'') \otimes_{K'}$

$K \cong K[x_0, \ldots, x_n]/J \otimes_{K'} K$ and then $\dim(K[x_0, \ldots, x_n]/J) = \dim(K'[x_0, \ldots, x_n]/J'')$. Moreover $\dim(R'[x_0, \ldots, x_n]/J') = d + c$ using Theorem 13.8 in [Eis95] with $Q = \langle x_0, \ldots, x_n \rangle + \mathfrak{m}'$.

Now consider P a minimal prime ideal of $J' + \mathfrak{m}'$ inside $R'[x_0, \ldots, x_n]$. As any prime containing $\langle a_1, \ldots, a_c \rangle$ contains $\mathfrak{m}'$ we have that P is also a minimal prime for $J' + \langle a_1, \ldots, a_c \rangle$. Thus, the codimension of P/J' in $R'[x_0, \ldots, x_n]/J'$ is at most c, and hence the dimension of P is at least d. This implies that all minimal primes of $(J' + \mathfrak{m}')/\mathfrak{m}'$ has dimension at least d (as they are of the form $P/\mathfrak{m}'$), so to conclude it's enough to show

$$(J' + \mathfrak{m}')/\mathfrak{m}' \otimes_{\Bbbk'} \Bbbk = \mathrm{in}_{\mathbf{0}}(J)$$

Each g_i in the Gröbner basis lies in $R'[x_0, \ldots, x_n]$ by construction and its image $\bar{g}_i$ in $\Bbbk'[x_0, \ldots, x_n]$ is equal to $\mathrm{in}_{\mathbf{0}}(g_i)$. Hence, $\mathrm{in}_{\mathbf{0}}(J) \subseteq (J' + \mathfrak{m}')/\mathfrak{m}' \otimes_{\Bbbk'} \Bbbk$. For the other inclusion we just notice that $\mathrm{in}_{\mathbf{0}}(f) = \langle \bar{f} \mid f \in I \rangle$ and each $\bar{f}$ is contained in the other side. This end this part of the proof.

Now return to the hypothesis of the problem. The ideal $I_{\text{proj}} \subseteq K[x_0, \ldots, x_n]$ is prime of dimension $d+1$, so because of what we did above every minimal prime of $\mathrm{in}_{(0,\mathbf{w})}(I_{\text{proj}})$ has dimension $d+1$. By Krull's principal ideal theorem all minimal prime ideals of $\mathrm{in}_{(0,\mathbf{w})}(I_{\text{proj}}) + \langle x_0 - 1 \rangle$ have dimension at least d. But in the other hand all minimal prime ideals of $\mathrm{in}_{(0,\mathbf{w})}(I_{\text{proj}}) + \langle x_0 - 1 \rangle$ are homogeneous (because the initial ideal is homogeneous) and then contained in $\langle x_0, \ldots, x_d \rangle$. Thus, the minimal primes of $\mathrm{in}_{(0,\mathbf{w})}(I_{\text{proj}}) + \langle x_0 - 1 \rangle$ have dimension exactly d. Then by 3.2.17 we have $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{(0,\mathbf{w})}(I_{\text{proj}})|_{x_0=1}$. So, the minimal primes of $\mathrm{in}_{\mathbf{w}}(I)$ are the images of the minimal prime over $\mathrm{in}_{(0,\mathbf{w})}(I_{\text{proj}}) + \langle x_0 - 1 \rangle$ that do not contain any monomial $x_1, \ldots, x_n$. Hence, they have dimension d. ☐

Using this we can study the dimension of tropical varieties.

Theorem 3.3.14. *Let X be an irreducible subvariety given by an ideal $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of dimension d in the algebraic torus T_K^n. Then every Γ_{val}-rational polyhedral complex with support Trop(X) has pure dimension d.*

Proof. As having pure dimension d only depend in the support of the polyhedral complex it's enough to prove that the polyhedral complex Σ constructed in Theorem 3.3.6 has pure dimension d.

Let's prove first that each cell has dimension at most d. Recall from the construction of Σ that it is given by the cells in the Gröbner complex $\Sigma(I_{\text{proj}})$ contained in the set $\{\mathbf{w} \in \mathbb{R}^n \mid \mathrm{in}_{\mathbf{w}}(I) \neq \langle 1 \rangle\}$. Now let $\mathbf{w} \in \Gamma_{\text{val}}^n$ lying in the relative interior of a maximal cell $\sigma \in \Sigma$. The affine span of σ is $\mathbf{w} + L$, where L is a subspace of $\mathbb{R}^n$. By Lemma 3.3.10 and part 1 of Proposition 3.3.9 we may assume that L is the span of $e_1, \ldots, e_k$ for k is the dimension of the cell σ. We need to show that $k \leq d$. Since $\mathbf{w}$ lies in the relative interior of σ we have $\mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I) \neq \langle 1 \rangle$ for all $\mathbf{v} \in \mathbb{Z}^n \cap L$ and ε small enough. Lemma 3.2.7 and Proposition 3.2.17 imply $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}}(I)$ for all $\mathbf{v} \in L \cap \mathbb{Z}^n$. Choose a set $\mathcal{G}$ of generators for $\mathrm{in}_{\mathbf{w}}(I)$ so that no generator is the sum of two other polynomials in $\mathrm{in}_{\mathbf{w}}(I)$ having fewer monomials. In particular we have $\mathrm{in}_{e_i}(f) = f$ for $1 \leq i \leq k$, so $f = m\tilde{f}$ where m is a monomial, and $x_1, \ldots, x_k$ do not appear in $\tilde{f}$. Since monomials are units in $\Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, this means that $\mathrm{in}_{\mathbf{w}}(I)$ is generated by elements not containing $x_1, \ldots, x_k$. Hence $k \leq \dim(\mathrm{in}_{\mathbf{w}}(I)) \leq \dim(X) = d$ as we wanted.

Now let's prove that each maximal cell σ in Σ has dimension at least d. Fix $\mathbf{w} \in \mathrm{int}(\sigma)$ and suppose that $\dim(\sigma) = k$. By Lemma 3.3.8 we have $|\mathrm{star}_{\Sigma}(\sigma)| = \mathrm{Trop}(V(\mathrm{in}_{\mathbf{w}}(I)))$ and as σ is a maximal cell this is the linear space parallel to σ. After a change of coordinates using part 2 of Proposition 3.3.9 we can assuma that L is generated by $e_1, \ldots, e_k$. Since $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I) = \mathrm{in}_{\mathbf{w}}(I)$ for all $\mathbf{v} \in L$ and small ε we get $\mathrm{in}_{e_i}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}}(I)$ and so the initial ideal is homogeneous with respect to the grading given by $\deg(x_i) = e_i$ for $1 \leq i \leq k$ and $\deg(x_i) = 0$ for $i > k$. Hence $\mathrm{in}_{\mathbf{w}}(I)$ is generated by Laurent polynomials using only the variables $x_{k+1}, \ldots, x_n$. In particular

$$\dim(\mathrm{in}_{\mathbf{w}}(I)) \leq k + \dim(\mathrm{in}_{\mathbf{w}}(I) \cap \Bbbk[x_{k+1}^{\pm 1}, \ldots, x_n^{\pm 1}])$$

Now let $J = \mathrm{in}_{\mathbf{w}}(I) \cap \Bbbk[x_{k+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$. If we have $\mathrm{Trop}(V(J)) = \{0\}$ we are done because by Lemma 3.3.12 then $V(J)$ is finite and so $\dim(\mathrm{it}_{\mathbf{w}}(I)) \leq k$ but then by Lemma 3.3.13 we known that $\dim(\mathrm{in}_{\mathbf{w}}(I)) = d$ and hence $k = \dim(\sigma) \geq d$. ☐

3.3.3 Transversal Intersection of Tropical Varieties

We finished this section by studying a result related to transverse intersection of tropical varieties will take an important role in the proof of the balancing condition in the next section.

Definition 3.3.15. Let Σ_1 and Σ_2 be two polyhedral complexes in $\mathbb{R}^n$ and $\mathbf{w} \in |\Sigma_1| \cap |\Sigma_2|$. As the relative interior of the cells partition the polyhedron complex we have that $\mathbf{w}$ is in the interior of a unique cell σ_i in Σ_i for $i = 1, 2$. The complexes Σ_1 and Σ_2 *intersect transversaly at* $\mathbf{w}$ if the affine span of σ_1 and the affine spane of σ_2 generate $\mathbb{R}^n$ as affine spaces. Two tropical varieties $\mathrm{Trop}(x)$ and $\mathrm{Trop}(Y)$ *intersect transversaly at* $\mathbf{w}$ if for some choice Σ_1, Σ_2 of Polyhedral complex structures such that $\mathrm{Trop}(x) = |\Sigma_1|$ and $\mathrm{Trop}(Y) = |\Sigma_2|$ we have that Σ_1 and Σ_2 intersect transversaly at $\mathbf{w}$. //

The result we want to prove will use the following lemma.

Lemma 3.3.16. *Let I, J be homogeneous ideals in $K[x_0, \ldots, x_n, y_0, \ldots, y_m]$ and fix $\mathbf{w} \in \mathbb{R}^{n+m+2}$. If $in_{\mathbf{w}}(I)$ has a generating set only involving $x_0, \ldots, x_n$ and $in_{\mathbf{w}}(J)$ has a generating set only involving $y_0, \ldots, y_m$ then*

$$in_w(I + J) = in_w(I) + in_{\mathbf{w}}(J)$$

Proof. The inclusion $\supseteq$ is obvious so by contradiction if the equality does not happen then we can find some homogeneous polynomial $f + g$ in $I + J$ of degree d with $f \in I_d$, $g \in J_d$ and $\mathrm{in}_{\mathbf{w}}(f + g) \notin \mathrm{in}_{\mathbf{w}}(I) + \mathrm{in}_{\mathbf{w}}(J)$. Now fix a monomial order $\prec$ in $\Bbbk[x_0, \ldots, x_n, y_0, \ldots, y_n]$ such that $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f + g)) \notin \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(I) + \mathrm{in}_{\mathbf{w}}(I))$. In particular

$$\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f + g)) \notin \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(I)) + \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(I)) \tag{3.7}$$

Let $x^{u_1}y^{v_1}$ and $x^{u_2}y^{v_2}$ be the monomials in $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f))$ and $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(g))$ and call $\alpha_1, \alpha_2 \in K$ the coefficients of these monomials in f and g respectively. From 3.7 we get $x^{u_1}y^{v_1} = x^{u_2}yv_2$ (i.e, $u_1 = u_2$ and $v_1 = v_2$) and $\mathrm{val}(\alpha_1 + \alpha_2) > \mathrm{val}(\alpha_1) = \mathrm{val}(\alpha_2)$.

We assume that this counterexample is maximal in the following sense: if $f' \in I_d$, $g' \in J_d$ is any other pair with $f+g = f'+g'$, then either $\mathrm{Trop}(f')(\mathbf{w}) < \mathrm{Trop}(f)(\mathbf{w})$ or $\mathrm{Trop}(f')(\mathbf{w}) = \mathrm{Trop}(f)(\mathbf{w})$ and $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f')) \succ \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f))$. But first we must prove that such a maximal pair need to exist. For this suppose there were no such pair, then we could find a sequence $f_i = f + h_i \in I$, $g_i = g - h_i \in J$ with $f_i + g_i = f + g$ for all i and $\mathrm{Trop}(f_i)(\mathbf{w})$ strictly increasing. The strictly increasing part is because if $\mathrm{Trop}(f_i)(\mathbf{w}) = \mathrm{Trop}(f_{i+1})(\mathbf{w})$ then the sequence must stop because there are only finitely many candidates for the monomial $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f'))$.

By passing to a subsequence we may assume that the support of each f_i (the set of monomials appearing in f_i) is the same. Since $\mathrm{supp}(f_i + h_i) = \mathrm{supp}(f + h_{i+1})$, there are $\alpha, \beta \in K^*$ for which $\alpha(f + h_i) + \beta(f + h_{i+1}) = (\alpha + \beta)f + (\alpha h_i + \beta h_{i+1})$ has strictly smaller support. Since $f+h_i \neq f+h_{i+1}$ we may assume that one of the monomials removed from $\mathrm{supp}(f+h_i)$ in this manner has different coefficients in h_i and h_{i+1}, and thus $\alpha+\beta \neq 0$. Note that for any two polynomials p_1, p_2 we have $\mathrm{Trop}(p_1 + p_2)(\mathbf{w}) \geq \min(\mathrm{Trop}(p_1)(\mathbf{w}), \mathrm{Trop}(p_2)(\mathbf{w}))$. Since $f_i - f = g - g_i \in I \cap J$ for all i the resulting polynomial $h_i' = (\alpha h_{i+1} + \beta h_i)/(\alpha+\beta)$ is also in $I \cap J$, so $f_i' = f + h'$ lies in I and has $\mathrm{Trop}(f_i')(\mathbf{w}) \geq \mathrm{Trop}(f_i)(\mathbf{w})$ and $\mathrm{supp}(f_i') \subsetneq \mathrm{supp}(f_i)$. By passing to another subsequence we may assume that the sequence $\mathrm{Trop}(f + h_i')(\mathbf{w})$ is again increasing. contuining to iterate this procedure would eventually yield the support of the new f_i being empty which is impossible since $\in_w (f_i + g_i) \neq \mathrm{in}_{\mathbf{w}}(J)$. This shows that the infinite increasing sequence does not exist, so we may assume that the pair f, g is maximal in the required sence.

Now $f \in I$ implies that $x^{\mathbf{u}_1}y^{\mathbf{v}_1} \in \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(I))$, so there is $f_1 \in I$ with $\mathrm{in}_{\mathbf{w}}(f_1) \in \Bbbk[x_0, \ldots, x_n]$, and $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f_1)) = x^{\mathbf{u}_3}$ dividing $x^{\mathbf{u}_1}$. We may assume that the coefficient of $x^{\mathbf{u}_3}$ in f_1 is one so we can write $f = \alpha_1 x^{\mathbf{u}_1-\mathbf{u}_3}y^{\mathbf{v}_1}f_1 + f_2$ where $\mathrm{Trop}(f_2)(\mathbf{w}) \geq \mathrm{Trop}(f)(\mathbf{w})$, and if equality holds, then $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f_2)) \prec \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(f))$. Similarly, $g = \alpha_2 x^{\mathbf{u}_1}y^{\mathbf{v}_1-\mathbf{v}_3}g_1 + g_2$ where $\mathrm{Trop}(g_2)(\mathbf{w}) \geq \mathrm{Trop}(g)(\mathbf{w})$, and if equality holds, then $\mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(g_2)) \prec \mathrm{in}_\prec(\mathrm{in}_{\mathbf{w}}(g))$. Since $\mathrm{val}(\alpha_1 + \alpha_2) > \mathrm{val}(\alpha_1) = \mathrm{val}(\alpha_2)$, we can write $\alpha_2 = \alpha_1(-1 + \beta)$ with $\mathrm{val}(\beta) > 0$. Then

$$\begin{aligned}
f + g &= \alpha_1 x^{\mathbf{u}_1-\mathbf{u}_3}y^{\mathbf{v}_1}f_1 + f_2 + \alpha_2 x^{\mathbf{u}_1}y^{\mathbf{v}_1-\mathbf{v}_3}g_1 + g_2 \\
&= \alpha_1 x^{\mathbf{u}_1-\mathbf{u}_3}y^{\mathbf{v}_1-\mathbf{v}_3}(y^{\mathbf{v}_3}f_1 - x^{\mathbf{u}_3}g_1 + \beta x^{\mathbf{u}_3}g_1) + f_2 + g_2 \\
&= \alpha_1 x^{\mathbf{u}_1-\mathbf{u}_3}y^{\mathbf{v}_1-\mathbf{v}_3}(-(g_1 - y^{\mathbf{v}_3})f_1 + (f_1 - x^{\mathbf{u}_3})g_1 + \beta x^{\mathbf{u}_3}g_1) + f_2 + g_2
\end{aligned}$$

Setting

$$\begin{aligned} f' &= \alpha_1 x^{\mathbf{u}_1-\mathbf{u}_3} y^{\mathbf{v}_1-\mathbf{v}_3}(-(g_1 - y^{\mathbf{v}_3})f_1) + f_2 \\ g' &= \alpha_1 x^{\mathbf{u}_1-\mathbf{u}_3} y^{\mathbf{v}_1-\mathbf{v}_3}((f_1 - x^{\mathbf{u}_3})g_1 + \beta x^{\mathbf{u}_3} g_1) + g_2 \end{aligned}$$

then by construction $f' \in I$, $g' \in J$, and $f' + g' = f + g$. In addition, either $\mathrm{Trop}(f')(\mathbf{w}) > \mathrm{Trop}(f)(\mathbf{w})$ or $\mathrm{in}_\prec(\mathrm{in}_\mathbf{w}(f')) \prec \mathrm{in}_\prec(\mathrm{in}_\mathbf{w}(f))$. This contradicts the maximality of our counterexample, so we conclude that none exists and hence $\mathrm{in}_\mathbf{w}(I+J) = \mathrm{in}_\mathbf{w}(I) + \mathrm{in}_\mathbf{w}(J)$. □

Theorem 3.3.17. *Let X and Y be subvarieties of T_K^n. If Trop(X) and Trop(Y) meet transversaly at $\mathbf{w} \in \Gamma_{val}^n$, then $\mathbf{w} \in$ Trop$(X \cap Y)$. Therefore*

$$\mathit{Trop}(X \cap Y) = \mathit{Trop}(X) \cap \mathit{Trop}(Y)$$

if the intersection is transversal everywhere.

Proof. Let Σ_1, Σ_2 be polyhedral complexes in $\mathbb{R}^n$ with support Trop(X) and Trop(Y) respectively and let I and J be the ideals defining X and Y. Let $\sigma_i \in \Sigma_i$ be the cell containing $\mathbf{w}$ in its relative interior for $i = 1, 2$. Our hypothesis says that if the affine span of σ_i is $\mathbf{w} + L_i$ then $L_1 + L_2 = \mathbb{R}^n$.

We can reduce to the case in which L_1 contains $e_1, \dots, e_r, e_{r+1}, \dots, e_s$ and L_2 contains $e_1, \dots, e_r$, $e_{s+1}, \dots, e_n$. To see this notice that as $L_1 + L_2 = \mathbb{R}$ there is a basis $a_1, \dots, a_n \in \mathbb{R}^n$ such that $a_1, \dots a_r \in L_1 \cap L_2$, $a_{r+1}, \dots, a_s \in L$ and $a_{s+1}, \dots, a_n \in L_2$ and all $a_i \in \mathbb{Z}^n$. If we put this vectors as rows of an $n \times n$ matrix A and let $\varphi : T^n \to T^n$ be the monomial map given by $\varphi(x_i) = x^{a_i}$ then Trop(φ) is given by A^T and it is an isomorphism as A has full rank (even though $\varphi(A)$ is not an isomorphism unless $|\det(A)| = 1$). Let $I' =' varphi^*(I)$, $J' =' varphi^*(J)$, $X' = V(I')$ and $Y' = V(J')$. Then $\varphi(X') = X$ and $\varphi(Y') = Y$. By part 2 of Proposition 3.3.9 we have

$$\begin{aligned} \mathrm{Trop}(X) &= \mathrm{Trop}(\varphi)(\mathrm{Trop}(X')) \\ \mathrm{Trop}(Y) &= \mathrm{Trop}(\varphi)(\mathrm{Trop}(Y')) \\ \mathrm{Trop}(X \cap Y) &= \mathrm{Trop}(\varphi)(\mathrm{Trop}(X' \cap Y')) \end{aligned}$$

By construction we have

$$\begin{aligned} \mathrm{Trop}(\varphi)(\mathrm{span}(e_{r+1}, \dots, e_s)) &\subseteq L_1 \\ \mathrm{Trop}(\varphi)(\mathrm{span}(e_{s+1}, \dots, e_n)) &\subseteq L_2 \\ \mathrm{Trop}(\varphi)(\mathrm{span}(e_1, \dots, e_r)) &\subseteq L_1 \cap L_2 \end{aligned}$$

and Trop(X') and Trop(Y') intersect transversely at $\mathrm{Trop}(\varphi)^{-1}(\mathbf{w})$. So we have reduced the problem to show $\mathrm{Trop}(\varphi)(\mathbf{w}) \in \mathrm{Trop}(X' \cap Y')$. Replacing X and Y with X' and Y' we can assume what we wanted over L_1, L_2.

As $\mathbf{w} \in \mathrm{int}(\sigma_1)$ we have for every $\mathbf{v} \in L_1$ that $\mathbf{w} + \varepsilon\mathbf{v} \in \sigma$ for small ε. Hence $\mathrm{in}_{\mathbf{w}+\varepsilon\mathbf{v}}(I) = \mathrm{in}_\mathbf{v}(\mathrm{in}_\mathbf{w}(I)) = \mathrm{in}_\mathbf{w}(I)$ and we get that $\mathrm{in}_\mathbf{w}(I)$ is homogeneous with respect to a graduation given by $\deg x_i = e_i$ for $i \in \{1, \dots, r, r+1, \dots, s\}$ and $\deg x_i = 0$ for other i. Then we can find polynomials $f_1, \dots, f_l$ in the variables $x_{s+1}, \dots, x_n$ that generate $\mathrm{in}_\mathbf{w}(I)$. Similarly there are generators $g_1, \dots, g_m$ for $\mathrm{in}_\mathbf{w}(J)$ depending only in the variables $x_{r+1}, \dots, x_s$. Let $I_{\mathrm{proj}} \subseteq K[x_0, \dots, x_{n+1}]$ be the ideal obtained by homogenizing $I \cap K[x_1, \dots, x_n]$ using the variable x_{n+1}, and let J_{proj} be the ideal obtained by homogenizing $J \cap K[x_1, \dots, x_n]$ using the variable x_0.

For $\overline{\mathbf{w}} = (0, \mathbf{w}, 0) \in \mathbb{R}^{n+2}$ the initial ideal $\mathrm{in}_{\overline{\mathbf{w}}}(I_{\mathrm{proj}})$ has a generating set only using $x_s, \dots, x_{n+1}$ and $\mathrm{in}_{\overline{\mathbf{w}}}(J_{\mathrm{proj}})$ has a generating set only using $x_0, x_{r+1}, \dots, x_s$. Thus by Lemma 3.3.16 we have $\mathrm{in}_{\overline{\mathbf{w}}}(I_{\mathrm{proj}} + J_{\mathrm{proj}}) = \mathrm{in}_{\overline{\mathbf{w}}}(I_{\mathrm{proj}}) + \mathrm{in}_{\overline{\mathbf{w}}}(J_{\mathrm{proj}})$ and by Proposition 3.2.17 if we set $x_0 = x_{n+1} = 1$ we get

$$\mathrm{in}_\mathbf{w}(I + J) = \mathrm{in}_\mathbf{w}(I) + \mathrm{in}_\mathbf{w}(J)$$

since $\mathrm{in}_\mathbf{w}(I)$ and $\mathrm{in}_\mathbf{w}(J)$ are proper ideals, by the Nullstellenatz there exist $y = (y_{r+1}, \dots, y_s) \in (\Bbbk^*)^{s-r}$ and $z = (z_{s+1}, \dots, z_n) \in (\Bbbk^*)^{n-s}$ with $f_i(y) = g_j(z) = 0$ for all i, j. Now, for any $(t_1, \dots, t_r) \in (\Bbbk^*)^r$, the vector $(t_1, \dots, t_r, y_{r+1}, \dots, y_s, z_{s+1}, \dots, z_n)$ lies in the variety $V(\mathrm{in}_\mathbf{w}(I)) \cap V(\mathrm{in}_\mathbf{w}(J)) = V(\mathrm{in}_\mathbf{w}(I+J))$. We conclude that $\mathrm{in}_\mathbf{w}(I+J) \neq \langle 1 \rangle$, and hence $\mathbf{w} \in \mathrm{Trop}(V(I+J)) = \mathrm{Trop}(X \cap Y)$. □

3.4 Weights of Cells and Balancing Condition

In this section we introduce weights for the maximal cells of an arbitrary tropical variety. We show that this weights coincide with the lattice length of the the dual edges used for tropical hypersurfaces and then we prove the balancing condition for arbitrary tropical varieties.

Along this section we use the notation $S_K = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and $S_{\Bbbk} = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ for the rings of polynomial en n variables.

Let us start by recalling some concepts from commutative algebra.

Every ideal I in a Noetherian ring S has a primary decomposition, i.e, can be written as

$$I = Q_1 \cap \cdots \cap Q_n$$

where each Q_i is primary and no term in the intersection can be remove. This primary decomposition is not unique in general but each prime ideal $\sqrt{Q_i}$ is independent of the primary decomposition. The $\sqrt{Q_i}$ are called the associated primes of I and the set of all these is denoted by $\mathrm{Ass}(I)$. We also have the set $\mathrm{Ass}^{\min}(I)$ of minimal elements of $\mathrm{Ass}(I)$, this set can also be seen as the set of minimal prime ideals containing I.

In this context, the *multiplicity* of a minimal prime $P_i \in \mathrm{Ass}^{\min}(I)$ is the number

$$\mathrm{mult}(P_i, I) := \mathrm{length}(A/Q_i)_{P_i} = \mathrm{length}((I : P_i^{\infty})/I)_{P_i}$$

where $\mathrm{length}(\cdot)$ is the length of a S_{P_i}-module and

$$(I : P^{\infty}) := \{f \in S \mid \exists\, m \in \mathbb{N} \text{ such that } f \cdot M^n \subseteq I\}$$

Using this we can introduce the weight of cells.

Definition 3.4.1. Let I be an ideal of $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and let Σ a polyhedral complex with support $|\Sigma| = \mathrm{Trop}(V(I))$ such that $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w} \in \mathrm{int}(\sigma)\ \ \forall \sigma \in \Sigma$ (see Remark 3.3.7). For a top dimensional cell $\sigma \in \Sigma$ we defined its multiplicity as

$$\mathrm{mult}(\sigma) = \sum_{P \in \mathrm{Ass}^{\min}(\mathrm{in}_{\mathbf{w}}(I))} \mathrm{mult}(P, \mathrm{in}_{\mathbf{w}}(I))$$

for any $\mathbf{w} \in \mathrm{relint}(\sigma)$ //

This definition coincide with the one we saw for hypersurfaces in the first section.

Proposition 3.4.2. *Let $f = \sum c_\alpha x^\alpha \in K[x_1^{\pm 1}, \ldots, x_1^{\pm 1}]$, Δ the general subdivision of Newt(f) induced by (val(c_α)) and Σ the dual polyhedral complex supported on Trop($V(f)$). Given any maximal cell $\sigma \in \Sigma$, the multiplicity mult(σ) defined above is the lattice length of the edge $e(\sigma)$ of Δ dual to σ.*

Proof. Fix $\mathbf{w}$ in the relative interior of σ. We have $\mathrm{in}_{\mathbf{w}}(\langle f \rangle) = \langle \mathrm{in}_{\mathbf{w}}(f) \rangle$ and $\mathrm{in}_{\mathbf{w}}(f) = \sum \overline{t^{-\mathrm{val}(c_u)} c_u} x^{\mathbf{u}}$ where the sum goes over those $\mathbf{u} \in e(\sigma)$ with $\mathrm{val}(c_{\mathbf{u}}) + \mathbf{w} \cdot \mathbf{u} = \mathrm{Trop}(f)(\mathbf{w})$. As $e(\sigma)$ is a segment we can take $\mathbf{v}$ and $\mathbf{u}$ such that the exponent of every monomial in $\mathrm{in}_{\mathbf{w}}(f)$ is of the form $\mathbf{u} + k\mathbf{v}$ and hence $\mathrm{in}_{\mathbf{w}}(f)$ is a Laurent polynomial in the variable $y = x^{\mathbf{v}}$ times a monomial in $x_1, \ldots, x_n$. After multiplying by a monomial we can assume that $\mathrm{in}_{\mathbf{w}}(f)$ is a (non-Laurent) polynomial g in y with nonzero constant term and degree the lattice length of the edge $e(\sigma)$. Hence if $g = (y - c_1)^{\alpha_1} \cdot \cdots \cdot (y - c_r)^{\alpha_r}$ we have

$$\begin{aligned}
\mathrm{mult}(\sigma) &= \sum_{P \in \mathrm{Ass}^{\min}(\mathrm{in}_{\mathbf{w}}(I))} \mathrm{mult}(P, \langle g \rangle) \\
&= \sum_i \mathrm{mult}(\langle y - c_i \rangle, \langle (y - c_1)^{\alpha_1} \cdot \cdots \cdot (y - c_r)^{\alpha_r} \rangle) \\
&= \sum_i \alpha_i = \deg g
\end{aligned}$$

and we conclude that the multiplicity of σ is the lattice length of $e(\sigma)$. □

As the general definition of multiplicity is a bit involved we sometimes use the following proposition for understanding it rather than the definition itself.

Lemma 3.4.3. *Let $X \subseteq T_K^m$ be irreducible of dimension d with ideal $I \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and let Σ be a polyhedral complex on Trop(X) as in Remark 3.3.7. Let $\sigma \in \Sigma$ with affine span parallel to $e_1, \ldots, e_d$, and let $\mathbf{w} \in int(\sigma) \cap \Gamma_{val}^n$. If $S' = \Bbbk[x_{d+1}^{\pm 1}, \ldots, x_n^{\pm 1}]$ then $mult(\sigma) = dim_{\Bbbk} k(S'/(in_{\mathbf{w}}(I) \cap S'))$.*

Proof. Since $\mathbf{w} \in \text{int}(\sigma)$ by Corollary 3.2.11 we have $\text{in}_{\mathbf{w}+\varepsilon e_i}(I) = \text{in}_{\mathbf{w}}(I)$ for small ε and then the initial ideal $\text{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the grading $\deg(x_i) = e_i$ for $i \leq d$ and $\deg(x_i) = 0$ for $i > d$. Hence $\text{in}_{\mathbf{w}}(I)$ has a generating set $\{f_1, \ldots, f_r\}$ not containing the variables $x_1, \ldots, x_d$. Let $\bigcap_{i=1}^s Q_i$ be a primary decomposition of $\text{in}_{\mathbf{w}}(I)$. Each Q_i can be taken homogeneous with respect to the same grading and so they are also generated by polynomials in the variables $x_{d+1}, \ldots, x_n$. Hence it's easy to see that $\text{in}_{\mathbf{w}}(I) \cap S' = \bigcap_{i=1}^s (Q_i \cap S')$ is a primary decomposition for the zero dimensional ideal $\text{in}_{\mathbf{w}}(I) \cap S'$ and $\text{mult}(P_i, Q_I) = \text{mult}(P_i \cap S', Q_i \cap S')$. Hence each P_i is a minimal prime of $\text{in}_{\mathbf{w}}(I)$ and we have

$$\begin{aligned} \text{mult}(\sigma) &= \sum_{i=1}^s \text{mult}(P_i, Q_i) \\ &= \sum_{i=1}^s \dim_{\Bbbk}(S'/(Q_i \cap S')) \\ &= \dim_{\Bbbk} S'/(\text{in}_{\mathbf{w}}(I) \cap S') \end{aligned}$$

□

The hypothesis over the affine spane of σ is not so restrictive as any cell can be put in such a position after a monomial change of coordinates in the torus.

Now we prepare the ground for the proof of the balancing condition. We will reduce the proof to the case of constant coefficient curves. For this we will use results about zero dimensional ideals.

We use the notation $S_K = K[x_1, \ldots, x_n]$ and $\tilde{S}_K = K[x_0, \ldots, x_n]$ with similar notations for $S_{\Bbbk}$ and $\tilde{S}_{\Bbbk}$.

Proposition 3.4.4. *Let $I = \bigcap_y Q_y \subseteq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ where each Q_y is primary and $\sqrt{Q_y} = P_y = \langle x_1 - y_1, \ldots, x_n - y_n \rangle$. Then*

1. *For $\mathbf{w} \in$ Trop(V(I)) let*
$$\bigcap_{y:val(y)=\mathbf{w}} Q_y$$
Then the multiplicity of the point $\mathbf{w}$ is equal to $dim_K S_K/I_{\mathbf{w}}$.

2. *Assume further that all $y \in V(I) \subseteq T_K^n$ have the same tropicalization $val(y) = \mathbf{w} \in \Gamma_{val}^n$. Then*
$$dim_{\Bbbk} S_{\Bbbk}/in_{\mathbf{w}}(I) = \sum_y mult(P_y, Q_y) = dim_K S_K/I$$

Proof. From commutative algebra we know that any zero dimensional ideal I satisfies

$$\dim_K S_K/I = \sum_{y \in V(I)} \text{mult}(P_y, Q_y)$$

where the $I = \bigcap_y Q_y$ is a primary decomposition and $P_y = \text{rad}(Q_y)$. Also $\dim_K S_K/I = \dim_K(\tilde{S}_K/I_{\text{proj}})_d$ for $d >> 0$ where I_{proj} is the homogenization of I. As these facts also hold for $\Bbbk$ and as using Proposition 3.2.10 we have $\dim_K(\tilde{S}_K/I_{\text{proj}})_d = \dim_{\Bbbk}(\tilde{S}_{\Bbbk}/\text{in}_{(0,\mathbf{w})}(I_{\text{proj}}))_d$ then in order to prove $\dim_K S_K/I = \dim_{\Bbbk} S_{\Bbbk}/\text{in}_{\mathbf{w}}(I)$ it is enough to show

$$\text{in}_{(0,\mathbf{w})}(I_{\text{proj}})_d = (\text{in}_{\mathbf{w}}(I)_{\text{proj}})_d \quad \text{for } d >> 0$$

The inclusion $\subseteq$ follows from Proposition 3.2.17, since $J \subseteq (J|_{x_0=|})_{proj}$ for any homogeneous ideal $J \subseteq \Bbbk[x_0, \ldots, x_n]$. For the other inclusion note that the same Proposition 3.2.17 implies that $\text{in}_{\mathbf{w}}(I)_{\text{proj}} = (\text{in}_{(0,\mathbf{w})}(I_{\text{proj}}) : \prod_{i=0}^n x_i)$. Saturating by the irrelevant ideal $\langle x_0, \ldots, x_n \rangle$ does not change $(\text{in}_{(0,\mathbf{w})}(I_{\text{proj}}))_d$ for $d >> 0$ and this saturation has only one-dimensional associated primes. These associated primes have the form $P_{y'} = \langle y'_j x_i - y'_i x_j \mid 0 \leq i < j \leq n \rangle$ for some

$y' = (y'_0 : \cdots : y'_n) \in \mathbb{P}^n$. Write $(\text{in}_{(0,\mathbf{w})}(I_{\text{proj}}) : \langle x_0, \ldots, x_n \rangle^\infty) = \bigcap_{y'} Q_{y'}$ where $Q_{y'}$ is primary with radical $P_{y'}$. Now

$$\left(\bigcap Q_{y'} : \prod x_i^\infty\right) = \bigcap_{y'} (Q_{y'} : \prod x_i^\infty) = \bigcap_{y' : x_y \notin P_{y'}} Q_{y'}$$

so it suffices to show that $x_i \notin P_{y'}$ for all i and all $Q_{y'}$.

Since each primary component Q_y of I is P_y-primary, it contains $(x_i - y_i)^d$ for some $d >> 0$. The product $\prod_y (x_i - y_i x_0)^d$ is thus in I_{proj} for $d >> 0$, and so since $\text{val}(y) = \mathbf{w}$ for all y, we have $\prod_y (x_i - \tilde{y}_i x_0)^d \in \text{in}_{(0,\mathbf{w})}(I_{\text{proj}})$, where $\tilde{y}_i = \overline{t^{-\mathbf{w}_i} y_i}$. This shows that $x_i \notin P_{y'}$ for all y' and $0 \leq i \leq n$. Indeed, for each i the porduct $\prod_y (x_i - \tilde{y}_i x_0)^d \in \text{in}_{(0,\mathbf{w})}(I_{\text{proj}})$, so for each y' there is $\tilde{y}_i$ with $x_i - \tilde{y}_i x_0 \in P_{y'}$. If $x_i \in P_{y'}$ for some i, then $x_j \in P_{y'}$ for $0 \leq j \leq n$, since each $\tilde{y}_i$ is nonzero as $y_i \neq 0$. This contradicts the fact that $y' \in \mathbb{P}^n$, so we conclude that the first claim holds.

For the second part, we claim that $\text{in}_\mathbf{w}(I) = \text{in}_\mathbf{w}(I)$. The inclusion $\subseteq$ is immediate from $I \subseteq I_\mathbf{w}$. For the other inclusion note that for any y with $\text{val}(y) \neq \mathbf{w}$ we have $\mathbf{w} \neq \text{Trop}(Q_y)$, so there is $f_y \in Q_y$ with $1 = \text{in}_\mathbf{w}(f)$- Given $f \in I_\mathbf{w}$, we then hae $g = f \prod_{\text{val}(y) \neq \mathbf{w}} f_y \in I$ with $\text{in}_\mathbf{w}(g) = \text{in}_\mathbf{w}(f)$- This gives the other inclusion. The result now follows from the first part using the interpretation of the multiplicity of Lemma 3.4.3. ☐

Now we proof the case of constant coefficient curves, i.e, curves defined over a field with a trivial valuation.

Proposition 3.4.5. *If C is a curve in $T^n_\Bbbk$, then the one dimensional fan Trop(C) is balanced.*

Proof. Let $\mathbf{u}_1, \ldots, \mathbf{u}_s$ denote the first lattice points on the rays of $\text{Trop}(C)$, let $m_i = \text{mult}(\text{cone}(u_i))$ and set $\mathbf{v} = \sum_{i=1}^s \mathbf{u}_i$. We will prove that $\mathbf{v} = 0$ by proving that for any $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_n) \in \mathbb{Z}^n$ primitive we have $\mathbf{w} \cdot \mathbf{v} = 0$. By Lemma 3.3.10 there is a change of coordinate sending $\mathbf{w}$ to e_1 and by part 1 of Proposition 3.3.9 this change of coordinates doesn't change the multiplicities. Hence it is enough to consider $\mathbf{w} = e_1$ and then the objective is to prove $\mathbf{v}_1 = 0$

Let $I = \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the ideal of C, K' the algebraic closure of $\Bbbk(t)$ and I' the extension of I to $K'[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Also denote $C_{K'} \subseteq T^n_{K'}$ for the variety of I'. Note that as $\Bbbk$ is algebraically closed, K' has residue field $\Bbbk$ and also that by Remark 3.3.4 we have $\text{Trop}(C) = \text{Trop}(C_{K'})$.

Given $\alpha \in K'^*$ consider the ideal $J'_\alpha = I' + \langle x_1 - \alpha \rangle$. There exists $L \in \mathbb{N}$ and a finite subset $\mathcal{D} \subset K'^*$ such that $\dim_K(S_{K'}/J'_\alpha) = L$ for all $\alpha \in K'^* \setminus \mathcal{D}$. To see this we use the classical Gröbner basis with respect to any monomial order $\prec$ in $K[x_0, \ldots, x_n]$: The initial ideal of $(J'_\alpha)_{\text{proj}}$ is constant outside a finite set $\mathcal{D}$ because $x_1 - \alpha$ cannot be a zerodivisor on $S_{K'}/I'$ for infinitely many α and the number L is equal to the degree of the Hilbert polynomial of this inital ideal.

Choose $\alpha_1, \alpha_2 \in K'^* \setminus \mathcal{D}$ with $\text{val}(\alpha_1) = 1$ and $\text{val}(\alpha_2) = -1$. Let $X^+ = V(I' + \langle x_1 - \alpha_1 \rangle) \subseteq T^n_{K'}$ and $X^- = V(I' + \langle x_1 - \alpha_2 \rangle) \subseteq T^n_{K'}$. The desired identity $\mathbf{v}_1 = 0$ will be obtained by computing L tropically.

Set $\beta_1 = \overline{t^{-1}\alpha_1} \in \Bbbk^*$ and $\beta_2 = \overline{t^1 \alpha_2} \in \Bbbk^*$. From 3.3.16 we can conclude

$$\text{in}_\mathbf{w}(I' + \langle x_1 - \alpha_1 \rangle) \;=\; \text{in}_\mathbf{w}(I') + \langle x_1 - \beta_1 \rangle \neq \langle 1 \rangle \;\text{ for } \mathbf{w} \in \text{Trop}(X^+)$$

$$\text{in}_\mathbf{w}(I' + \langle x_1 - \alpha_2 \rangle) \;=\; \text{in}_\mathbf{w}(I') + \langle x_1 - \beta_2 \rangle \neq \langle 1 \rangle \;\text{ for } \mathbf{w} \in \text{Trop}(X^-)$$

We now focus on α_1. Let $H = \text{Trop}(V(x_1 - \alpha_1)) = \{\mathbf{w} \in \mathbb{R}^n \mid \mathbf{w}_1 = 1\}$. We claim that $\text{Trop}(X^+) = \text{Trop}(C) \cap H$. Indeed, for any $\mathbf{w} \in \text{Trop}(C) \cap H$ the cone of $\text{Trop}(C)$ containing $\mathbf{w}$ in its relative interior is $\text{cone}(\mathbf{w})$, so $\text{Trop}(C)$ intersects H transversaly at $\mathbf{w}$. Since $\mathbf{w}$ was an arbitrary intersection point, the claim follows from Theorem 3.3.17. We now decompose $I' + \langle x_1 - \alpha_1 \rangle$ as $\bigcap_y Q_y$, where Q_y is P_y-primary for $y \in T^n_{K'}$. The y appearing here are precisely the points of X^+. Let $X^+_\mathbf{w} = \{y \in X^+ \mid \text{val}(y) = \mathbf{w}\}$. Note that for $\mathbf{w} \in \text{Trop}(X^+)$, we have $\text{in}_\mathbf{w}(\bigcap_{y \in X^+} Q_y) = \text{in}_\mathbf{w}(\bigcap_{y \in X^+_\mathbf{w}} Q_y)$. The inclusion $\subseteq$ is easy, now for the other inclusion note that for all $y \in X^+ \setminus X^+_\mathbf{w}$, there is $f_y \in Q_y$ with $\text{in}_\mathbf{w}(f_y) = 1$. For any $g \in \bigcap_{y \in X^+_\mathbf{w}} Q_y$, we set $g' = \prod_{y \in X^+ \setminus X^+_\mathbf{w}}$ to get $\text{in}_\mathbf{w}(g) = \text{in}_\mathbf{w}(g')$. Combined with the first of the above equations, this gives $\mathrm{in}_{\mathbf{w}}(\bigcap_{y \in X_{\mathbf{w}}^+} Q_y) = \mathrm{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle$.

Now using proposition 3.4.4 we have

$$\dim_{K'} S_{K'} / \left(\bigcap_{y \in X_{\mathbf{w}}^+} Q_y \right) = \sum_{y \in X_{\mathbf{w}}^+} \mathrm{mult}(Q_y, P_y) = \dim_{\Bbbk}(S_{\Bbbk}/\mathrm{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle)$$

And summing these identities over all $\mathbf{w} \in \mathrm{Trop}(X^+)$, we find

$$L = \sum_{y \in X^+} \mathrm{mult}(Q_y, P_y) = \sum_{w \in \mathrm{Trop}(X^+)} \dim_{\Bbbk}(S_{\Bbbk}/(\mathrm{in}_{\mathbf{w}}(I') + \langle x_1 - \beta_1 \rangle))$$

The same identities hold for X^- and β_2.

Let $\mathbf{u_w}$ be the first lattice point on the ray cone($\mathbf{w}$) of Trop(C). Then $\lambda = (\mathbf{u_w})_1$ satisfies $\mathbf{u_w} = \lambda \mathbf{w}$ because $\mathbf{w}_1 = 1$. We now claim that

$$\lambda \cdot \mathrm{mult}(\mathrm{cone}(\mathbf{w})) \;=\; \dim_{\Bbbk}(S_{\Bbbk}/(\mathrm{in}_{\mathbf{w}}(I) + \langle x_1 - \beta_1 \rangle)) \tag{3.8}$$

This would imply

$$L = \sum_{\mathbf{w} \in \mathrm{Trop}(X^+)} \mathrm{mult}(\mathrm{cone}(\mathbf{w})) \cdot (\mathbf{u_w})_1 = \sum_{i:\,(\mathbf{u}_i)_1 > 0} m_i \cdot (\mathbf{u}_i)_1$$

and similarly $L = \sum_{i:\,(\mathbf{u}_i)_1 > 0} -m_i \cdot (\mathbf{u}_i)$ so we have

$$\mathbf{u}_1 = \sum_{i:\,(\mathbf{u}_i)_1 > 0} m_i (\mathbf{u}_i)_1 - \sum_{i:\,(\mathbf{u}_i)_1 < 0} m_i \cdot |(\mathbf{u}_i)_1| = L - L = 0$$

Thus it remains to prove equation (3.8). For this we consider a change of coordinates taking x_1 to $x^{\mathbf{u_w}}$, and thus $\mathbf{w}$ to $\lambda^{-1} e_1$. Then (3.8) becomes $\lambda \cdot \mathrm{mult}(\mathrm{cone}(\mathbf{w})) = \dim_{\Bbbk}(S_{\Bbbk}/(\mathrm{in}_{\lambda^{-1}e_1}(I') + \langle x^{\mathbf{u_w}} - \beta_1 \rangle))$. The initial ideal $\mathrm{in}_{\lambda^{-1}e_1}(I')$ has a generating set that does not contain x_1. Since $V(I')$ is a curve by Lemma 3.2.10 the initial ideal is one dimensional, so for each $2 \le i \le n$ it contains a polynomial in $\Bbbk[x_i]$ with constant term one. After dividing by x_i, we obtain $x_i^{-1} - p_i' \in \mathrm{in}_{\lambda^{-1}e_1}(I')$ for some $p_i' \in \Bbbk[x_i]$. Now $\langle x^{\mathbf{u_w}} - \beta_1 \rangle = \langle x_1^{\lambda} - \beta_1 x^{\mathbf{u}'} \rangle$, where $\mathbf{u}_1' = 0$ and $\mathbf{u}_i' = -(\mathbf{u_w})_i$ for $2 \le i \le n$ this implies $\mathrm{in}_{\lambda^{-1}e_1}(I') + \langle x^{\mathbf{u_w}} - \beta_1 \rangle = \mathrm{in}_{\lambda^{-1}e_1}(I') + \langle x_1^{\lambda} - f \rangle$ for some $f \in \Bbbk[x_2, \ldots, x_n]$. We next use the fact that $\dim_{\Bbbk} \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]/J = \dim_{\Bbbk} \Bbbk[x_1, \ldots, x_n]/J_{\mathrm{aff}}$ for any zero-dimensional Laurent ideal J. Fix the lexicographic term order $x_1 \succ x_2 \succ \cdots \succ x_n$ on $\Bbbk[x_1, \ldots, x_n]$. By Bucherger's criterion, the initial ideal of $(\mathrm{in}_{\lambda^{-1}e_1}(I') + \langle x_1^{\lambda} - f \rangle)_{\mathrm{aff}}$ is generated by x_1^{λ} and the monomial generators of $\mathrm{in}_{\mathrm{lex}}((\mathrm{in}_{\lambda^{-1}e_1}(I'))_{\mathrm{aff}})$. The right-hand side of equation 3.8 is λ times the $\Bbbk$-dimension of $\Bbbk[x_2^{\pm 1}, \ldots, x_n^{\pm 1}]/\mathrm{in}_{\lambda^{-1}e_1}(I')$. But, that last $\Bbbk$-dimension equals the multiplicity of cone($\mathbf{w}$) by Lemma 3.4.3. □

With this we can conclude our result.

Theorem 3.4.6 (Balancing condition)**.** *Let I be an ideal in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that $V(I)$ is of pure dimension d. If Σ is a polyhedral complex with support Trop($V(I)$) such that $\mathrm{in}_{\mathbf{w}}(I)$ is constant for $\mathbf{w}$ in the relative interior of each cell in Σ. Then Σ is balanced with the weights defined at 3.4.1.*

Proof. Finding a primary decomposition for $\sqrt{I}$ we get $\sqrt{I} = \bigcap P_i$ were each prime P_i has dimension d. Then $\mathrm{Trop}(V(I)) = \bigcup \mathrm{Trop}(V(P_i))$ and by Theorem 3.3.14 it is a Trop($V(I)$) has pure dimension.

Fix a $(d-1)$-dimensional cell $\tau \in \Sigma$ then by Lemma 3.3.10 and by part 1 of proposition 3.3.9, after a change of coordinates, the affine span of τ is a translate of the span of $e_1, \ldots, e_{d-1}$. Fix $\mathbf{w} \in \mathrm{int}(\tau)$. As $\mathrm{in}_{e_i}(\mathrm{in}_{\mathbf{w}}(I)) = \mathrm{in}_{\mathbf{w}}(I)$ for $1 \le i < d$ we have that $\mathrm{in}_{\mathbf{w}}(I)$ is homogeneous with respect to the $\mathbb{Z}^{d-1}$ grading given by $\deg(x_i) = e_i$ for $1 \le i < d$, and $\deg(x_i) = 0$ for $i \ge d$. This implies that $\mathrm{in}_{\mathbf{w}}(I)$ has a generating set in which $x_1, \ldots, x_{d-1}$ do not appear.

Let $J = \mathrm{in}_{\mathbf{w}} \cap \Bbbk[x_d^{\pm 1}, \ldots, x_n^{\pm 1}]$. By Lemma 3.3.8 the tropical variety Trop($V(\mathrm{in}_w(I))$) is the star of τ in Σ which has lineality space spanned by $e_1, \ldots, e_{d-1}$. Since $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\mathbf{w}}(I)) \cap \Bbbk[x_d^{\pm 1}, \ldots, x_n^{\pm 1}] =$

$\text{in}_{\bar{\mathbf{v}}}(J)$ for $\bar{\mathbf{v}}$ the projection of $\mathbf{v}$ onto the last $n-d+1$ coordinates. As $\text{Trop}(V(I))$ has pure dimension d we have that $\text{Trop}(V(J))$ is one dimensional.

Let $P_1, \ldots, P_r$ be the minimal associated primes of J. Then $V(J) = \bigcup V(P_i)$ so

$$\text{Trop}(V(J)) = \text{cl}(\text{val}(y) \mid y \in V(J)) = \bigcup_{i=1}^{r} \text{cl}(\text{val}(y) \mid y \in V(P_i)) = \bigcup_{i=1}^{r} \text{Trop}(V(P_i))$$

By theorem 3.3.14 we have $\dim(P_i) \leq 1$ and at least one index i satisfies $\dim(P_i) = 1$. Thus $\dim(V(J)) = 1$.

Suppose $\mathbf{v} \in \mathbb{Q}^n$ satisfies $\mathbf{w} + \varepsilon\mathbf{v} \in \sigma$ for all sufficiently small $\varepsilon > 0$, where σ is a maximal cell of Σ that has τ as a facet. The equality $\text{in}_{\mathbf{v}}(\text{in}_{\mathbf{w}}(I)) = \text{Trop}(V(J))$ together with Lemma 3.4.3 imply that the multiplicity of the cone $\text{cone}(\bar{\mathbf{v}})$ in $\text{Trop}(V(J))$ is equal to the multiplicity of $\sigma in \text{Trop}(X)$. Thus, showing that Σ is balanced at τ is exactly the same as showing that $\text{Trop}(V(J))$ is balanced at 0. But this follows from Proposition 3.4.5 above. $\square$

Appendix A

Overview on Polyhedral Geometry

Here we collect the notions from polyhedral geometry that are needed along this document. We will refer to [Zie95] for the proofs.

A *polyhedron* $P \subseteq \mathbb{R}^n$ is a intersection of finitely many closed half-spaces

$$P = \{v \in \mathbb{R}^n \mid \langle v, x_i \rangle \geq r_i \ \ \forall\, i = 1, \dots, k\} \tag{A.1}$$

as intersection of convex sets is convex any polyhedron is a convex set.

Let Γ be an additive subgroup of $\mathbb{R}$. Then the polyhedron P in (A.1) is said to be Γ-rational if each x_i can be taken defined over $\mathbb{Q}$ and each r_i can be taken in Γ.

When P is bounded we say that it is a *polytope*. We can described polytopes in an alternative way.

Proposition A.1. *For a subset $P \subseteq \mathbb{R}^n$ the following are equivalent:*

- *P is a polytope.*
- *There are $v_1, \dots, v_k \in \mathbb{R}^n$ such that*

$$\begin{aligned} P =& conv(v_1, \dots, v_k) \\ :=& \{\lambda_1 v_1 + \cdots + \lambda_k v_k \in \mathbb{R}^n \mid \lambda_i \in \mathbb{R}_{\geq 0} \ and \ \lambda_1 + \cdots + \lambda_k = 1\} \end{aligned}$$

Proof. See section 1.1 in [Zie95]. □

Given $y \in (\mathbb{R}^n)^*$ we define the *face* of the polyhedron P determined by y as

$$\text{face}_y(\sigma) = \{v \in \sigma \mid \langle v, y \rangle \leq \langle w, y \rangle \ \ \forall w \in \sigma\}$$

when this set is non empty. In other words, the set of all elements of σ in which y attains its minimum. Notice that this set do can be empty if P is unbounded and y is not bounded there. A face of P that is not contained in any larger proper face is called *facet*.

The *dimension* of P is the dimension of its affine span and the *relative interior* of P denoted by $\text{int}(P)$ is its interior computed inside its affine span. The *linear space parallel to* P is the translation of the affine span of P to the origin.

Definition A.2. A *polyhedral complex* is a collection Σ of polyhedra satisfying two conditions:

- If P is in Σ then so is any face of P.
- If P and Q are in Σ then $P \cap Q$ is either empty or a face of both P and Q.

//

The element of Σ are called cells and the maximal cells are called facets. If every facet has dimension d then Σ is called of *pure dimension* d. The support of Σ is the set $|\Sigma| = \bigcup_{P\in\Sigma} P$. The *k-skeleton* of Σ is the polyhedral complex consisting of all cells $\sigma \in \Sigma$ with $\dim(\sigma) \leq k$.

Now we introduce polyhedral cones, for this we need the following result

Proposition A.3. *For a subset $\sigma \subseteq \mathbb{R}^n$ the following conditions are equivalent:*

- *There are linear functionals $x_1, ..., x_k \in (\mathbb{R}^n)^*$ defined over $\mathbb{Z}$ such that*

$$\sigma = \{v \in \mathbb{R}^n \mid \langle v, x_i \rangle \geq 0 \ \ \forall 1 \leq i \leq k\}$$

- *There are vectors $v_1, ..., v_k \in \mathbb{Z}^n$ such that*

$$\sigma = \{\lambda_1 v_1 + \cdots + \lambda_k v_k \in \mathbb{R}^n \mid \lambda_i \in \mathbb{R}_{\geq 0} \ \ \forall 1 \leq i \leq k\}$$

Proof. See section 1.1 in [Zie95]. □

If any of this conditions is satisfied we say that σ is a *convex rational polyhedral cone* in $\mathbb{R}^n$ or simple a *cone* if there is no risk of confusion. In the case of the second condition we will write

$$\sigma = \mathrm{cone}(v_1, \dots, v_k)$$

Now given a cone σ we define its dual cone by

$$\sigma^\vee := \{x \in (\mathbb{R}^n)^* \mid \langle v, x \rangle \geq 0 \ \ \forall v \in \sigma\}$$

If $\sigma = \mathrm{cone}(v_1, \dots, v_k)$ then we have $\sigma^\vee = \{x \in (\mathbb{R}^n)^* \mid \langle v_i, x \rangle \geq 0 \ \ \forall 1 \leq i \leq k\}$ and by Proposition A.3 we see that $\sigma^\vee$ is a rational polyhedral cone in $(\mathbb{R}^n)^*$ and $(\sigma^\vee)^\vee = \sigma$.

As cones are particular cases of polyhedrons we have the definition of dimension, face, facet and relative interior of a cone.

A cone $\sigma \subseteq \mathbb{R}^n$ is of maximal dimension if it is not contained in any proper subspace of $\mathbb{R}^n$. In the other hand, a cone is said to be *strictly convex* if it does not contain any non trivial subspace of $\mathbb{R}^n$. As

$$V \subseteq \sigma \iff V^\perp \supseteq \sigma^\vee$$

we see that σ is strictly convex if and only if $\sigma^\vee$ is of maximal dimension.

Given $y \in (\mathbb{R}^n)^*$ we have

$$\begin{aligned} \mathrm{face}_y(\sigma) \text{ is not empty } &\iff y \text{ attains its minimum in } \sigma \\ &\iff y \text{ is non-negative over } \sigma \\ &\iff y \in \sigma^\vee \end{aligned}$$

So faces of σ are given exactly by

$$\mathrm{face}_y(\sigma) = \{v \in \sigma \mid \langle v, y \rangle = 0\}$$

for y an element of the dual. If we now generators for the dual cone finding the faces is easier.

Proposition A.4. *If $\sigma = \{v \in (\mathbb{R}^n)^* \mid \langle v, x_i \rangle \geq 0 \ \ \forall i = 1, \dots, k\}$, i.e, $\sigma^\vee = cone(x_1, \dots, x_k)$. Then all the faces of σ are of the form*

$$\tau = \{v \in \sigma \mid \langle v, x_i \rangle \geq 0 \ \ \forall i \in I\}$$

for some $I \subseteq \{1, \dots, k\}$.

Given a cone $\sigma \subset \mathbb{R}^n$ we define its orthogonal vector space as

$$\sigma^\perp = \{x \in (\mathbb{R}^n)^* \mid \langle v, x \rangle = 0 \ \ \forall\, v \in \sigma\}$$

Using this we can state

Proposition A.5. *There is an order reversing correspondence between faces of σ and faces of $\sigma^{\perp}$ in which the face $\tau \subseteq \sigma$ correspond to the face $\sigma^{\vee} \cap \tau$.*

Proof. See [?]. □

A cone σ is called *simplicial* if it is of the form $\sigma = \mathrm{cone}(v_1, \dots, v_k)$ for some linearly independent vectors $v_1, \dots, v_n$.

A cone σ is called *smooth* if it is of the form $\sigma = \mathrm{cone}(v_1, \dots, v_k)$ for some integral vectors that can be extended to an integral basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ of the lattice $\mathbb{Z}^n$, i.e, $\det(v_1, \dots, v_n) = \pm 1$.

Now we introduce the analogue of a polyhedral complex for cones.

Definition A.6. A *fan* is a finite collection Σ of strictly convex rational polyhedral cones in $\mathbb{R}^n$ such that:

- If τ is a face of σ and $\sigma \in \Sigma$ then $\tau \in \Sigma$.
- For all $\sigma, \tau \in \Sigma$ we have that $\sigma \cap \tau$ is a face of both σ and τ

//

As every fan is also a polyhedral complex we have the notions of facet, pure dimension, support and k-skeleton for fans. We say that a fan is *smooth* (resp. *simplicial*) if each one of its cones is smooth (resp. simplicial) and it is *complete* if its support is the whole $\mathbb{R}^n$.

Now we introduce two diferent kind of fans contructed from polyhedra.

Definition A.7. Given a polyhedron $P \subseteq \mathbb{R}^n$ the *normal fan* of P is the fan $\mathcal{N}_P$ in $(\mathbb{R}^n)^*$ given by the cones

$$\mathcal{N}_P(F) = \mathrm{cl}(w \in (\mathbb{R}^n)^* \mid \mathrm{face}_w(P) = F)$$

where $\mathrm{cl}(\cdot)$ denotes the topological closure in $(\mathbb{R}^n)^*$. Here $\mathcal{N}_P(F)$ is exactly the inner normal cone of the face F. //

Definition A.8. Given a polyhedral complex Σ in $\mathbb{R}^n$ and a cell $P \in \Sigma$ we define the *star* of P in Σ, denoted by $\mathrm{star}_\Sigma(P)$, as the fan in $\mathbb{R}^n$ consisting of all the cones

$$\sigma_Q \;=\; \{\, \lambda(x-y) \mid \lambda \geq 0,\; x \in Q,\; y \in P \,\}$$

for each $Q \in \Sigma$ containing P as a face. Visually σ_Q is the union of all rays with origin a point of P and looking in the direction of a point in Q. If P is a maximal cell then $\mathrm{star}_\Sigma(P)$ is the affine span of P. //

We end this chapter by describing the concept of *regular subdivision* of a polytope.

Definition A.9. Let $P = \mathrm{conv}(v_1, ..., v_r)$ be a polytope in $\mathbb{R}^n$ and fix a weight vector $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{R}^r$. The *regular subdivision* of P induced by w is a polyhedral complex with support P defined in the following way.

Define $u_i = (v_i, 1) \in \mathbb{R}^{n+1}$ and identify $\mathrm{conv}(u_1, \dots, u_r)$ with P. Now the polytopes on the polyhedral complex are $\mathrm{conv}\{u_i \mid i \in I\}$ for the subsets $I \subset \{1, \dots, r\}$ such that there exists $c \in \mathbb{R}^{n+1}$ with

$$c \cdot u_i = w_i \text{ for } i \in I \text{ and } c \cdot u_i < w_i \text{ for } i \notin I$$

//

The regular subdivision also has the following geometric description. In the context of the definition above consider the polyhedra

$$P_w = \mathrm{cone}\{(v_i, r) \mid r \geq w_i \text{ and } 1 \leq i \leq r\} \subseteq \mathbb{R}^{n+1}$$

then the polytopes in the regular subdivision are the projection of the bounded faces of P_w to the plane $\mathbb{R}^n \times \{1\}$.

To check this fix a face $F = \mathrm{cone}\{(v_i, w_i) \mid i \in I\}$ of P_w and take a vector $(c, 1)$ such that $\mathrm{face}_{(c,1)}(P_w) = F$. Then we have that there is c_0 such that $(c, 1) \cdot (v_i, w_i) \geq 0$ for all i with equality exactly when $i \in \sigma$ and the projection of the face to the plane $\mathbb{R}^n \times \{1\}$ is exactly $\mathrm{cone}\{(v_i, 1) \mid u \in I\}$.

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