Polyhedral, Tropical and Analytic Geometry of Higher Rank

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Abstract

With the aim of starting a systematic development of higher rank tropical geometry, we develop a theory of *higher rank polyhedral geometry* over the ordered ring of *generalized dual numbers* $\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^k)$. We generalize several classical results to this context, including, but not limited to, Fourier-Motzkin Elimination, Farkas' Lemma, the Minkowski-Weyl decomposition and the basic results on the duality theory of cones and the theory of normal fans of polyhedra.

We use this theory to endow tropical hypersurfaces of higher rank with the structure of a polyhedral complex over $\mathbb{D}$. As a first application, we show how the polyhedral structure on tropical hypersurfaces of higher rank is dual to a *layered regular subdivision* of their Newton polytope.

Later on, we introduce a certain number of tools and results suitable for the study of valuations of higher rank on function fields of algebraic varieties. This is be based on a study of higher rank quasi-monomial valuations taking values in the lexicographically ordered group $\mathbb{R}^k$.

We prove a *duality theorem* that gives a geometric realization of higher rank quasi-monomial valuations as *tangent cones of dual cone complexes* acting as *multi-directional derivative operators* on tropical functions.

Tangent cones of dual cone complexes provide an analogue of *skeleton* in higher rank non-archimedean geometry. Generalizing the picture in rank one, we construct retraction maps to tangent cones of dual cone complexes, and use them to obtain limit formulae in which we reconstruct higher rank non-archimedian spaces with their tropical topology as the projective limit of their higher rank skeleta.

Keywords: tropical geometry, polyhedral geometry, analytic geometry, valuations, higher rank, okounkov bodies

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Résumé

Dans le but de commencer un développement systématique de la géométrie tropicale de rang supérieur, nous développons une théorie de la *géométrie polyédrale de rang supérieur* sur l'anneau ordonné des *nombres duaux généralisés* $\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^k)$. Nous généralisons plusieurs résultats classiques dans ce contexte, y compris, mais sans s'y limiter, l'élimination de Fourier-Motzkin, le lemme de Farkas, la décomposition de Minkowski-Weyl et les résultats de base sur la théorie de la dualité des cônes et la théorie des ventilateurs normaux des polyèdres.

Nous utilisons cette théorie pour doter les hypersurfaces tropicales de rang supérieur de la structure d'un complexe polyédral sur $\mathbb{D}$. En guise de première application, nous montrons comment la structure polyédrale sur les hypersurfaces tropicales de rang supérieur est duale à une *subdivision régulière en couches* de leur polytope de Newton.

Plus tard, nous introduisons un certain nombre d'outils et de résultats adaptés à l'étude des valuations de rang supérieur sur les corps de fonctions des variétés algébriques. Ceci est basé sur une étude des valuations quasi-monomiales de rang supérieur prenant des valeurs dans le groupe ordonné lexicographiquement $\mathbb{R}^k$.

Nous démontrons un *théorème de dualité* qui donne une réalisation géométrique des valuations quasi-monomiales de rang supérieur sous forme de *cônes tangents des complexes de cônes duaux* agissant comme des *opérateurs de dérivées multi-directionnelles* sur les fonctions tropicales.

Les cônes tangents des complexes de cônes duaux fournissent une analogue de *squelette* dans la géométrie non archimédienne de rang supérieur. En généralisant l'image en rang un, nous construisons des cartes de rétraction vers les cônes tangents des complexes de cônes duaux, et les utilisons pour obtenir des formules limites dans lesquelles nous reconstruisons des espaces non archimédiens de rang supérieur avec leur topologie tropicale comme la limite projective de leurs squelettes de rang supérieur.

Mots clés : géométrie tropicale, géométrie polyédrale, géométrie analytique, valuations, rang supérieur, corps d'okounkov

Contents

Introduction

The research covered in this manuscript lays between two branches of mathematics: *tropical geometry* and *non-Archimedean geometry*.

The main objective of tropical geometry is to provide tools to translate questions from algebraic geometry into questions about polyhedral geometry and its combinatorics. There have been applications of this line of work in many areas of algebraic geometry such as enumerative geometry [Mik05; Mik06], the study of linear series [Bou14; LM09a], the theory of singularities [Thu07; Ste06], the theory of degenerations [AB15], the theory of moduli spaces [CGP18; ACP15], and many others. Recently, tropical geometry has also allowed going in the opposite direction dealing with combinatorial problems using algebro-geometric ideas. A notable example of this is the combinatorial Hodge theory initiated in [AHK18] that has had striking results in the theory of matroids.

Tropical geometry can be introduced in different ways. One of them is the *synthetic approach* in which one works with the *tropical semifield* or *min-plus algebra*, denoted by $\mathbb{T}$, whose underlying set is $\mathbb{R} \cup \{\infty\}$ and whose addition and multiplication is given by the usual minimum and addition respectively. One can use this to define *tropical varieties*. These are the set of common *zeros* of certain families of *tropical polynomials*, where tropical polynomials are those polynomials whose coefficients lie in $\mathbb{T}$ and the notion of zero is defined in a clever way to make it compatible with its algebraic counterpart.

A different way of looking at tropical geometry is given by the *valuative approach*. From this point of view, one has to consider a field κ endowed with a *valuation* $\mathrm{val}\colon \kappa \to \mathbb{R}$ with a dense image. Then, given a subvariety X of the algebraic torus $\mathbb{G}_\kappa^n$ one can consider its *tropicalization* as the euclidean closure of the set of all images

$$\mathrm{trop}(x) \coloneqq (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n))$$

as $x = (x_1, \dots, x_n)$ goes over the elements of X. A *tropical variety* is then defined as the tropicalization of an algebraic subvariety of a torus.

The fact that these two points of view agree is an important result of the area that earned the name of the *fundamental theorem of tropical geometry* [EKL06; SS04]. We refer to [Mik07; MS15; MR09; IMS09; Gub13; Bru+15] and the references therein for an introduction to tropical geometry.

Having these different points of view has made the theory evolve in many directions. From one part, the synthetic approach gives us a setting to study tropical geometry entirely in combinatorial terms, without reference to algebraic geometry. On the other hand, the valuative approach remains more in touch with its algebraic side and allows to generalize the theory to other settings such as the theory of toric varieties or *non-Archimedean analytic geometry*.

Non-Archimedean analytic geometry is the branch of algebraic geometry dealing with algebraic varieties defined over *valued fields*, that is, fields endowed with valuations $\mathrm{val}\colon \kappa \to \mathbb{R}$. It allows us to consider "analytic version" of our varieties, called *analytifications*, mimicking the construction of the complex manifold associated to an algebraic variety defined over $\mathbb{C}$. It was introduced by Tate in his seminal work [Tat71] and developed in different forms by other authors. We refer to [Con08] for an overview of these ideas.

The approach to non-Archimedean analytic geometry which relates arguably more naturally to tropical geometry is the one introduced by Berkovich [Ber12]. On it, given a variety X defined on a valued field $\mathrm{val}\colon \kappa \to \mathbb{R}$ we can construct its *Berkovich analytification*, denoted by X^{An}. The underlying set[1] of X^{An} corresponds to the family of all the valuations $\nu\colon k(\eta) \to \mathbb{R}$ whose domain is the residue field of a schematic point $\eta \in X$ and such that it extends the base field valuation, in the sense that the following diagram commutes:

$$\begin{array}{ccc} k(\eta) & \xrightarrow{\ \nu\ } & \mathbb{R} \\ \uparrow & \nearrow_{\mathrm{val}} & \\ \kappa. & & \end{array}$$

Any closed point $x \in X$ has κ as residue field and hence induces a unique point ν_x in X^{An} given by $\nu_x(f) = \mathrm{val}(f(x))$. Moreover, for each element $\nu \in X^{\mathrm{An}}$ with domain $\kappa(\eta)$, each regular function f defined on a neighborhood of η can be evaluated

$$f \longmapsto \nu(f).$$

[1] Although the usual construction of the Berkovich analytification uses multiplicative seminorms, here we use the point of view of valuations as it fits better the content of the manuscript.

These evaluation maps allow us to turn X^{An} into a topological space by considering the coarsest topology making the projection $X^{\mathrm{An}} \to X$ sending a valuation $k(\eta) \xrightarrow{\eta} \mathbb{R}$ to η continuous, together with all the evaluation maps at regular functions over affine open subsets.

Now, if we suppose that X has some global non-vanishing coordinates, that is, X is a subvariety of $\mathbb{G}_{\kappa}^{n}$, the considerations above allow us to extend the map $\mathrm{trop}(x) = (\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n))$ to a map defined on its analytification X^{An}. We do this by evaluation on the coordinates

$$\mathrm{trop}(\nu) := (\nu(x_1), \ldots, \nu(x_n)).$$

In this way, we have extended the valuative approach of tropical geometry to those analytic spaces obtained as the Berkovich analytification of an algebraic subvariety of a torus.

There is, on the other hand, an *intrinsic approach* to tropical geometry on Berkovich spaces provided by the notion of *skeleton*. These are polyhedral complexes living inside the Berkovich space which are constructed in terms of local coordinates of the original variety. The idea is that, by putting weights on these local coordinates, one can construct *quasi-monomial valuations* which will naturally be parametrized in terms of polyhedra.

These skeleta provide an approximation of the analytic space in terms of finitely many data and, surprisingly, it gives us a considerable amount of information of it. This is exemplified in the fact that there is a way to connect these different polyhedral complexes in such a way that the entire space is the inverse limit of all these. Moreover, under suitable assumptions, the entire space is a deformation retraction of these skeleta.

This thesis is in the context of generalizing these ideas to the case in which the valuations have a higher rank, that is, they have values in $\mathbb{R}^k$ rather than $\mathbb{R}$. This theory is therefore called *higher rank tropical geometry*.

Organization of the thesis

This thesis is composed of two chapters and an introduction which presents in a succinct way the results obtained during the thesis period. The content of Chapter 1 is based on our paper [Iri21], and Chapter 2 comes from the joint work [AI21] with Omid Amini.

In Chapter 1 we construct a theory of higher rank polyhedral geometry that we expect will play in higher rank tropical geometry the same role as the one that plays the usual theory of polyhedral geometry in tropical geometry. We show how this theory helps us to understand higher rank tropical hypersurfaces from the synthetic point of view. In Chapter 2 we extend the intrinsic approach to tropical geometry on Berkovich analytifications to the case in which we are working with higher rank analytifications with constant coefficients.

In the rest of the introduction, we will go more deeply into the contents of Chapter 1 and Chapter 2.

Chapter 1: Polyhedral Geometry Over the Generalized Dual Numbers

This chapter is concerned with the polyhedral geometry behind a higher rank tropical geometry.

Higher rank tropical geometry is a generalization of tropical geometry adapted to work with a valued field K in which the valuation val is not necessarily of rank 1. It was initiated by Aroca in [Aro10a] with a generalization to this setting of Kapranov's theorem [Kap00], that is, the hypersurface case of the *fundamental theorem of tropical geometry.*

Since then, higher rank tropical geometry has made appearance in the literature at several occasions, mostly in the case in which Γ is contained in $\mathbb{R}^k$ with its lexicographic order. This is however not a big restriction as, by the Hahn's embedding theorem [Hah95], every abelian group of finite rank k can be embedded into $\mathbb{R}^k$ with its lexicographic order. In particular,

1. Banerjee proves in [Ban15] that the Euclidean closure of the tropicalization of a dimension n subvariety of an algebraic torus over a local field of rank k can be endowed with a polyhedral complex structure, in the usual sense, of non-necessarily pure dimension kn.

2. In [FR16b], Foster and Ranganathan prove, by using non-Archimedean geometry in the higher rank setting, that the higher rank tropicalization of a connected variety is a path connected space when endowed with its Euclidean topology. Later, in [FR16a], they use these ideas to construct multi-stage degeneration of toric varieties.

3. Kaveh and Manon outline, in an appendix to their work on Khovanskii bases [KM19a], a theory of higher rank Gröbner fans for ideals for use in a possible higher rank version of tropical geometry. They moreover introduce in [KM19b] a framework

for tropicalization where the valuation takes value in semimodule of piecewise affine functions, and use this in order to describe toric vector bundles.

4. Joswigh and Smith [JS18] use higher rank tropical geometry in the study of stable intersection of tropical hypersurfaces, and discuss connection with potential applications of such a theory in generalizing the works of Allamigeon-Benchimol-Gaubert-Joswig [All+14; All+15; All+18] and that of Develin and Yu [DY07].

5. A higher rank version of geometric tropicalization with connection to higher rank non-Archimedean geometry was established in our work with Amini [AI21] where *tangent cone complexes* were introduced as an analog notion of *skeleton* in higher rank non-Archimedean geometry over trivially valued fields. This will be covered in Chapter 2 of this manuscript.

6. In their recent works on hybrid geometry of curves and their moduli spaces [AN20; AN21], Amini and Nicolussi introduce hybrid curves and higher rank versions of tropical curves, as well as a moduli space of these higher rank geometric objects, and develop a function theory in this higher rank setting.

7. In addition to the above results, we should mention recent model theoretic works of Hrushovski and Loeser [HL16] and of Haskell, Hrushovski, and Macpherson [HHM06] with possible connection to higher rank tropical geometry. (Note however that in these works, the topology in the value group is the ordered topology, in contrast with the Euclidean topology considered here over $\mathbb{R}^k$.) We refer to the Bourbaki seminar by Ducros [Duc12] for the discussion of this point of view and further references.

All these works can be viewed as indicators for the utility of establishing a higher rank version of tropical geometry in which the structure and geometry of these higher rank tropical objects can be described and used in applications.

The content of the first chapter of this manuscript is about the polyhedral geometry underlying such a theory. We develop a polyhedral geometry over the ring of *generalized dual numbers* $\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which we expect will play in higher rank tropical geometry the role the usual polyhedral geometry over $\mathbb{R}$ has in tropical geometry. As an application, we provide a description of the polyhedral structure on higher rank tropical hypersurfaces, answering questions asked by Joswig and Smith [JS18].

In addition to building the basis of higher rank tropical geometry, the higher rank polyhedral geometry developed here is expected to have further connections and applications in other places. For example, it should appear in connection with:

1. A general theory of skeleta for higher rank non-Archimedean geometry. This already happens in the trivially valued case developed in Chapter 2 of this manuscript, in which skeleta is given by higher rank polyhedral cone complexes, corresponding to (iterated-)tangent cones over usual polyhedral cone complexes. An extension of the results of [AI21] to the non-trivially valued situation is of particular interest in the study of families of algebraic varieties.

2. Understanding the geometry in the theory of affine Λ-buildings developed by Bennett and Hébert-Izquierdo-Loisel in [Ben90; HIL20].

3. Understanding the current theory of perturbed polyhedra as a deformation theory of polyhedra, in the sense of algebraic geometry.

4. Developing linear programming methods for problems in which the variable to optimize has a lexicographic nature. One could study, for example, how to generalize the duality theory of linear programming to this setting.

In the following, we provide an overview of the main elements of this chapter.

The Ring of Generalized Dual Numbers

Along this work we work with the ring $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which we call the *ring of generalized dual numbers* of rank k. An important fact about this ring is that it naturally captures the lexicographic order in $\mathbb{R}^k$ by considering ε to be an *infinitesimally small but positive* quantity. In this way, $\mathbb{D}$ becomes an ordered ring whose underlying abelian group corresponds to $\mathbb{R}^k$ with its lexicographic order.

Given a lattice $N \cong \mathbb{Z}^n$ with dual lattice M, we proceed to study the module $N_{\mathbb{D}}$ from a geometric standpoint. Using the ordered ring structure of $\mathbb{D}$, we can say that a subset $X \subseteq N_{\mathbb{D}}$ is *convex* if, for $x, y \in X$ and $\lambda \in \mathbb{D}$ such that $0 \leq \lambda \leq 1$ we have

$$\lambda x + (1 - \lambda)y \in X$$

or a *cone* if, for $x, y \in X$ and $\lambda, \mu \in \mathbb{D}_{\geq 0}$ we have

$$\lambda x + \mu y \in X.$$

Then, given elements $x_1, \ldots, x_n \in N_{\mathbb{D}}$ one can define their convex hull $\mathrm{conv}_{\mathbb{D}}(x_1, \ldots, x_n)$ and cone hull $\mathrm{cone}_{\mathbb{D}}(x_1, \ldots, x_n)$ as the minimal convex set, respectively cone, containing these elements. Sets of these forms are called *polytopes* and *finitely generated cones* respectively.

In the same way, one can define *half-spaces* in $\mathbb{D}$ as sets of the form

$$\{x \in N_{\mathbb{D}} \mid \langle y\,, x\rangle \geq a\}$$

for some $y \in M_{\mathbb{D}}$ and $a \in \mathbb{D}$. Intersection of sets of these forms are called *polyhedra*. Moreover, a face of a polyhedra is a set of the form $P \cap H$ where $H = \{x \in M_{\mathbb{D}} \mid \langle y\,, x\rangle = a\}$ is a hyperplane such that the half-space $\{x \in M_{\mathbb{D}} \mid \langle y\,, x\rangle \geq a\}$ contains P.

The proof of basic properties of these objects, such as the fact that any polytope or any finitely generated cone is a polyhedra, are more involved than in the classical setting. Notice that $\mathbb{D}$ is not an integral domain, so we cannot reduce the problem to its fraction field. Reduction to the integral domain $\mathbb{R}[[\varepsilon]]$ works neither as the inclusion $\mathbb{D} \hookrightarrow \mathbb{R}[[\varepsilon]]$ is not a ring map. For these reasons, we need to develop the technical apparatus directly on $\mathbb{D}$ to work around this.

Also, we remark that the structure of faces of a polyhedron is more complicated in this context than in the usual case. For example, if the polyhedron is given as

$$P = \{x \in N_{\mathbb{D}} \mid \langle y_1\,, x\rangle \geq a_1, \ldots, \langle y_r\,, x\rangle \geq a_r\},$$

then not every face is of the form

$$P \cap \{x \in N_{\mathbb{D}} \mid \langle y_1\,, x\rangle = a_i,\ i \in I\}$$

for some $I \subseteq \{1, \ldots, n\}$. Similarly, if the polyhedron is given as

$$P = \mathrm{conv}_{\mathbb{D}}(x_1, \ldots, x_n),$$

then not every face is of the form

$$\operatorname{conv}_{\mathbb{D}}\left(\{x_i \mid i \in I\}\right)$$

for some $I \subseteq \{1, \ldots, n\}$. See Remark 1.1.11 for more comments on this and Section 1.6 for some results in the structure of faces on this context.

The Fourier-Motzkin Elimination

One core technical result is the Fourier-Motzkin Elimination. Over the real numbers, given a system of linear inequalities in $\mathbb{R}^n$ of the form

$$\mathcal{L} = \{a_1x_1 + \cdots + a_nx_n \geq a\}$$

with solution set $S \subseteq \mathbb{R}^n$, this algorithms produces explicitly a system of linear inequalities in $\mathbb{R}^{n-1}$ of the form

$$\mathcal{L}' = \{b_1x_1 + \cdots + b_{n-1}x_{n-1} \geq b\}$$

whose solution set is $\pi(S)$ for $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$. This algorithm is based on the two observations:

1. Any system of one variable inequalities can be written in the form

$$x \geq a_i, \quad x \leq b_j, \quad c_k \geq 0 \quad \text{for some } i, j, k.$$

2. The system of inequalities above has a solution if and only if $c_k \geq 0$ for each k and $b_j \geq a_i$ for any i, j.

Then, the algorithm consists in looking at $\mathcal{L}$ as a system of inequalities only in the variable x_n.

Over the ring $\mathbb{D}$, the first observation above does not remain true as, for example, a linear inequality of the form $\varepsilon^i x \geq a$ cannot be reduced to one of the form $x \geq a'$. However, any one variable linear inequality can be reduced to one of the form $\pm\varepsilon^i x \geq a$ or $c_i \geq 0$. With this modification in mind, we can obtain an analog of the second observation. This leads to a generalization of the Fourier-Motzkin Elimination to the ring $\mathbb{D}$ in Theorem 1.2.3.

Theorem A1 (Fourier-Motskin Elimination over $\mathbb{D}$)**.** *Given an integer $n \geq 1$ and a system of linear inequalities $\mathcal{L}$ in $\mathbb{D}^{n+1}$, there is another system of linear inequalities $\mathcal{L}'$ in $\mathbb{D}^n$ such* *that* $(x_1, \ldots, x_{n+1})$ *is a solution of* $\mathcal{L}$ *for some* $x_{n+1} \in \mathbb{D}$ *if and only if* $(x_1, \ldots, x_n)$ *is a solution of* $\mathcal{L}'$.

Although the final algorithm to produce the new system of linear inequalities turns out to be more involved (see Lemma 1.2.2), it will be of fundamental theoretical importance for use in what follows.

Farkas' Lemma

Arguably, one of the most important results in giving shape to a theory of polyhedral geometry is Farkas' Lemma. Over the real numbers, one way to state it is as follows.

Theorem (Farkas' Lemma over $\mathbb{R}$). *If* $f, f_1, \ldots, f_r$ *are affine functions over* $N_{\mathbb{R}}$, *then we have the equivalence*

$$\begin{aligned} &\{f(x) \geq 0\} \supseteq \{f_1(x) \geq 0, \ldots, f_r(x) \geq 0\} \\ \iff &\exists\, \lambda_1, \ldots, \lambda_r, c \in \mathbb{R}_{\geq 0} \text{ such that } f = \lambda_1 f_1 + \cdots + \lambda_r f_r + c. \end{aligned}$$

Remark. The exact statement over $\mathbb{D}$ turns out to be false. To obtain a counterexample, notice that it is possible to have $\{f(x) \geq 0\} = N_{\mathbb{D}}$ in a nontrivial way as in

$$\mathbb{D} = \{x \in \mathbb{D} \mid \varepsilon x + 1 \geq 0\} \supseteq \{x \in \mathbb{D} \mid \varepsilon^2 x \geq 0\}.$$

However, there are no $\lambda, c \in \mathbb{D}_{\geq 0}$ such that

$$\varepsilon x + 1 = \lambda(\varepsilon^2 x) + c.$$

Similarly, it is possible to find a counterexample by having $\{f_1(x) \geq 0, \ldots, f_r(x) \geq 0\} = \varnothing$ in a nontrivial way as in

$$\{x \in \mathbb{D} \mid x \geq 0\} \supseteq \{x \in \mathbb{D} \mid -1 + \varepsilon x \geq 0,\ 1 + \varepsilon x \geq 0\} = \varnothing.$$

But there are no $\lambda_1, \lambda_2, c \in \mathbb{D}_{\geq 0}$ such that

$$x = \lambda_1(-1 + \varepsilon x) + \lambda_2(1 + \varepsilon x) + c$$

With these considerations in mind, the correct statement of Farkas' lemma over $\mathbb{D}$ goes as follows (See Theorem 1.3.1).

Theorem B1 (Farkas' Lemma over $\mathbb{D}$)**.** *Let $f_1, \ldots, f_r \colon N_{\mathbb{D}} \to \mathbb{D}$ be a family of affine functions such that the set*

$$P = \{f_1 \geq 0, \ldots, f_r \geq 0\}$$

is non-empty. Then, any affine function $f \colon N_{\mathbb{D}} \to \mathbb{D}$ achieving its minimum over P can be written in the form

$$f - \min_P f = \lambda_1 f_1 + \cdots + \lambda_r f_r$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$.

The novel part of the hypothesis is that f should achieve its minimum over P. The proof is based on an induction argument relying on the Fourier-Motzkin Elimination over $\mathbb{D}$.

Although there are several generalizations of Farkas' Lemma in the literature, the generalization given here, in a context with nilpotent elements, appears to be novel. We refer to [DJ14] for an overview on generalizations of Farkas' Lemma and to [Bar21] for other results in this direction.

General Strucstural Results

With Fourier-Motzkin Elimination and Farkas' Lemma under our belt, we have enough core results to come back to our basic questions about the structure of polyhedra over $\mathbb{D}$. Here is a statement collecting some of the results we obtain.

Theorem C1 (Some Structure Results on Polyhedra over $\mathbb{D}$)**.**

1. *Finitely generated cones are exactly the same as polyhedral cones (See Proposition 1.2.4 and Proposition 1.5.2).*

2. *Polytopes are polyhedra and moreover, a polyhedron is a polytope if and only if any linear function attains its minimum over it (see Proposition 1.2.4 and Corollary 1.9.9)*

3. *The poset of faces of a polyhedron forms an order lattice (see Corollary 1.6.3).*

4. *There are explicit ways to compute the faces of a polyhedron in terms of its generators or, dually, in terms of an intersection of half-spaces (see the content of Section 1.6)*

5. *There is a notion of relative interior defined by algebraic (rather than topological) means (see Definition 1.4.1). This notion preserves the usual properties of the relative*

interior over $\mathbb{R}$. For example, the relative interior of a non-empty polyhedron is non-empty and any polyhedron can be written as the disjoint union of the relatives interiors of its faces (see Proposition 1.4.2).

6. *It is not always true that a polyhedron accepts a* Minkowski-Weyl decompostion*, that is, that it can be written as the Minkowski sum of a polytope and a polyhedral cone (see Remark 1.9.10). However, we have the following characterization: A polyhedron P can be written as the Minkowski sum of a polytope and a polyhedral cone, if and only if, each time two linear functions achieve their minimum over P, their sum also achieves its minimum over P (see Corollary 1.9.9).*

Duality Theory of Polyhedra

We generalize to $\mathbb{D}$ some duality results for polyhedra. In particular, we generalize the duality theorem for cones, which gives an order reversing bijection between the faces of a cone and the faces of its dual (see Theorem 1.5.3).

Theorem D1 (Higher Rank Cone Duality)**.** *Given a polyhedral cone $\sigma \subseteq N_{\mathbb{D}}$ and its dual*

$$\sigma^{\vee} := \{y \in M_{\mathbb{D}} \mid \langle y\,, x\rangle \geq 0, \forall x \in \sigma\},$$

there is an order reversing bijection between the poset of faces of σ and the poset of faces of its dual $\sigma^{\vee}$ given by

$$\tau \longmapsto \tau^{*} := \tau^{\perp} \cap \sigma^{\vee},$$

where

$$\tau^{\perp} := \{y \in M_{\mathbb{D}} \mid \langle y\,, x\rangle = 0,\ \forall\, x \in \tau\}.$$

Given a polyhedron P, each face of P is of the form

$$\mathrm{face}_y(P) := \mathrm{arg.min}_P \langle y\,, \cdot\rangle$$

for some $y \in M_{\mathbb{D}}$ achiving its minimum over $\mathbb{D}$. The *Normal Fan Duality* says that this function is locally constant over the faces of a fan lying on $M_{\mathbb{D}}$. More concretely, in Theorem 1.8.5 we prove the following.

Theorem E1 (Higher Rank Normal Fan Duality). *Given a polyhedron $P \subseteq N_{\mathbb{D}}$, let us consider the set*

$$|\operatorname{NF}(P)| \coloneqq \{y \in M_{\mathbb{D}} \mid \langle y\,,\cdot\rangle \text{ achieves its minimum over } P\}.$$

Moreover, for each face F of P, consider its normal cone *as the set*

$$\mathrm{C}(F) \coloneqq \{y \in |\operatorname{NF}(P)| \mid \operatorname{face}_y P \supseteq F\}.$$

Then, each $\mathrm{C}(F)$ is a polyhedral cone and the family

$$\operatorname{NF}(P) = \{\mathrm{C}(F) \mid F \preceq P\}$$

is a fan whose support is $|\operatorname{NF}(P)|$ and such that the map $y \mapsto \operatorname{face}_y(P)$ is locally constant over the faces of $|\operatorname{NF}(P)|$. In particular, there is an order reversing bijection between the faces of P and the faces of $\operatorname{NF}(P)$.

Remark. One new aspect of this setting is that, unlike over $\mathbb{R}$, the support of the normal fan of a polyhedron is not always convex. Nevertheless, its convex hull is a polyhedral cone whose dual is the *recession cone* of the polyhedron. Moreover, a polyhedron has a convex normal fan if and only if it is the Minkowski sum of a polytope and a polyhedron.

Finally, in Theorem 1.9.6 we generalize a result of Minkowski to higher rank: There is a correspondence between polyhedral cones endowed with piecewise concave functions and polyhedra. In particular, there is a bijection between polytopes and piecewise linear concave functions.

Theorem F1 (Higher Rank Minkowki Theorem). *There is a bijection between polyhedra with convex normal fan and polyhedral cones endowed with concave linear functions. Explicitly, this bijection sends a polyhedron P to the pair $(|\operatorname{NF}(P)|, h_P)$, where h_P is its* support function *defined as*

$$\begin{aligned} h_P : |\operatorname{NF}(P)| &\longrightarrow \mathbb{D} \\ y &\longmapsto \min_{x \in P} \langle y\,, x\rangle. \end{aligned}$$

Iterated Fibrations

For $i = 1, \ldots, k$ let $\mathbb{D}_i \coloneqq \mathbb{R}[\varepsilon]/(\varepsilon^i)$. Then, we have a sequence of projections

$$\mathbb{D} = \mathbb{D}_k \to \mathbb{D}_{k-1} \to \cdots \to \mathbb{D}_1 = \mathbb{R}.$$

Applying the tensor product with N yields a sequence of projections

$$N_{\mathbb{D}} = N_{\mathbb{D}_k} \to N_{\mathbb{D}_{k-1}} \to \cdots \to N_{\mathbb{D}_1} = N_{\mathbb{R}}.$$

In this way, the subsets $X \subseteq N_{\mathbb{D}}$ can be regarded as *iterated fibrations.*

Definition. For a given lattice N, an *iterated fibration of subsets of* $N_{\mathbb{R}}$ or simply, an *iterated fibration*, is a diagram of sets of the form

$$X = X^{[r]} \xrightarrow{\pi_{r-1}} X^{[r-1]} \xrightarrow{\pi_{r-2}} \ldots \xrightarrow{\pi_1} X^{[0]}$$

where each map is surjective, $X^{[0]} \subseteq N_{\mathbb{R}}$, and for each $x \in X^{[i]}$, the fiber $\pi_i^{-1}(x)$ can be identified with a subset of $N_{\mathbb{R}}$, denoted by $X_x^{[i+1]}$.

To obtain an iterated fibration from $X \subseteq N_{\mathbb{D}}$ we just need to define $X^{[i]}$ as the projection of X to $N_{\mathbb{D}^i}$. This gives a new geometrical perspective to polyhedra over $\mathbb{D}$: These are certain iterated fibrations in which the base and each fiber are polyhedra over $\mathbb{R}$.

One particular kind of iterated fibration is the *tangent cone bundle* of a set $X \subseteq N_{\mathbb{R}}$. In order to define it, consider the set $T\mathcal{C}^{k-1} X$ whose elements are vectors of the form $(x, w_1, \ldots, w_{k-1})$ where $x \in X$, w_1 is an element pointing inside of X, that is, such that $x + \delta w_1 \in X$ for every $\delta \in \mathbb{R}_{>0}$ small enough, and more generally, $w_1, \ldots, w_i$ satisfy

$$x + \delta w_1 + \cdots + \delta^i w_i \in X$$

for every $\delta \in \mathbb{R}_{>0}$. Then, the sequence

$$T\mathcal{C}^{k-1} X \longrightarrow T\mathcal{C}^{k-2} X \longrightarrow \cdots \longrightarrow T\mathcal{C}\, X \longrightarrow X.$$

is an iterated fibration.

More generally, for a flag of sets

$$\mathcal{A}\colon A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{k-1}$$

we can consider the tangent cone of $\mathcal{A}$, which we denote by $T\mathcal{C}\,\mathcal{A}$, as the set of all elements $(x, w_1, \dots, w_{k-1}) \in N_{\mathbb{D}}^k$ such that

$$x \in A_0, \quad x + \delta w_1 \in A_1, \quad \dots \quad x + \delta w_1 + \cdots + \delta^{k-1} w_{k-1} \in A_{k-1}$$

Theorem G1 (Tangent Cone of Polyhedra)**.**

1. *If $P \subseteq N_{\mathbb{R}}$ is a usual real polyhedron, then $T\mathcal{C}^{k-1}\,P$ is a polyhedron over $\mathbb{D}$. Conversely, any polyhedron over $\mathbb{D}$ which is defined completely in terms of real numbers is of the form $T\mathcal{C}^{k-1}\,P$ for some real polyhedron P.*

2. *If $\mathcal{P}\colon P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{k-1}$ is a flag of real polyhedra, then $T\mathcal{C}\,\mathcal{P}$ is a polyhedron over $\mathbb{D}$. Conversely, any polyhedron which is defined completely in terms of elements of the form $\varepsilon^i a$ with $a \in \mathbb{R}$ is of the form $T\mathcal{C}\,\mathcal{P}$ for some flag of polyhedra.*

In the manuscript, this is developed in Theorem 1.11.2 and Corollary 1.11.3.

The first part of the theorem above should be regarded as a polyhedral version of the equality

$$TX(\mathbb{C}) = X(\mathbb{C}[\varepsilon]/(\varepsilon^2))$$

from algebraic geometry. That is, if one does a base change of a real polyhedron P to the ring $\mathbb{D}$, one obtains the tangent cone bundle $T\mathcal{C}^{k-1}\,P$ which we understand as the correct analog of its tangent space. The second part of the theorem has to be regarded as a first instance between the correspondence between *layered objects* and *fibered objects* on this theory.

Tangent cone bundles will appear again in our work in [AI21] which will be covered in Chapter 2.

$\mathbb{R}$-Rational Polyhedra and Regular Subdivisions

We use all this information to study $\mathbb{R}$-rational polyhedra, that is, polyhedra obtained as intersection of half-spaces of the form

$$a_1 x_1 + \cdots + a_n x_n \geq a$$

with $a_1, \dots, a_n \in \mathbb{R}$ and $a \in \mathbb{D}$. These are the kind of polyhedra appearing in higher rank tropical geometry. In order to do this study, we introduce *layered normal fans* over the real numbers and show how the normal fan duality already obtained gives us a combinatorial correspondence between $\mathbb{R}$-rational polyhedra and this layered normal fan. We call this result the *Local Duality Theorem.*

Theorem H1 (Layered normal fans)**.** *To each $\mathbb{R}$-rational polyhedron P we can construct a sequence of fans*

$$\underline{\Delta}(P) \coloneqq \Delta_0 \preceq \Delta_1 \cdots \preceq \Delta_{k-1}. \tag{$\star$}$$

in which each term is a subdivision of the previous one by considering for each $i \in \{0, \dots, k-1\}$ and each $\delta \in \mathbb{R}_{>0}$ the normal fan of the polyhedron

$$P_i(\delta) \coloneqq \left\{ x \in N_{\mathbb{R}} \;\middle|\; \langle y \,, x \rangle \geq a_j^{(0)} + \delta a_j^{(1)} + \cdots + \delta^i a_j^{(i)}, \quad \forall\, 1 \leq j \leq r \right\},$$

which is independent of δ it is small enough. Moreover, any sequence of fans as in ($\star$) in which each fan is a subdivision of the previous one is the layered normal fan of some $\mathbb{R}$-rational polyhedron.

To see this and other two different definitions of the layered normal fan of an $\mathbb{R}$-rational polyhedron we refer to Proposition 1.12.2.

The theorem below (Theorem 1.12.4 in the manuscript) allow us relates the layered normal fan of an $\mathbb{R}$-rational polyhedron with its usual normal fan over $\mathbb{D}$. In particular, it allow us to understand combinatorially the iterated fibration of an $\mathbb{R}$-rational polyhedron.

Theorem I1 (Local Duality)**.** *Given an $\mathbb{R}$-rational polyhedron P, we can recover the normal fan of P from the layered normal fan as*

$$\mathrm{NF}(P) = T\mathcal{C}\,\underline{\Delta}(P).$$

In the sense that, $\mathrm{NF}(P)$ *is the fan consisting of all the polyhedral cones of the form $T\mathcal{C}\,\underline{\delta}$ where*

$$\underline{\delta} \colon \sigma_{k-1} \subseteq \sigma_{k-2} \cdots \subseteq \sigma_0$$

is a layered face of $\underline{\Delta}$.

Regular Subdivisions

In a similar way as we work with normal fan of $\mathbb{R}$-rational polyhedra we can work with regular subivisions of real polyhedra by height functions with coefficients in $\mathbb{D}$.

We extend the notion of regular subdivision of a polytope to the ring $\mathbb{D}$ in two different ways. Given a finite set $A \subseteq M_{\mathbb{R}}$ and a *height function* of the form

$$\begin{aligned} h\colon A &\longrightarrow \mathbb{D} \\ a &\longmapsto h(a). \end{aligned}$$

we define the *layered regular subdivision* of $\mathrm{conv}_{\mathbb{R}}(A)$ with respect to h as the sequence of regular subdivision

$$\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))\colon \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1}$$

in which Δ_i is defined as the usual regular subdivision of A by the height function

$$h(a)^{(0)} + \delta h(a)^{(1)} + \cdots + \delta^i h(a)^{(i)}$$

which is independent of $\delta \in \mathbb{R}_{>0}$ if it is small enough. To see other two equivalent definitions for the layered regular subdivision we refer to Proposition 1.13.3.

On the other hand, we can define a *regular subdivision over* $\mathbb{D}$ of the polytope $\mathrm{conv}_{\mathbb{D}}(A) \subseteq M_{\mathbb{D}}$ by considering its lifting

$$\mathrm{conv}_{\mathbb{D}}\left(\left\{(a, h(a)) \mid a \in A\right\}\right) \subseteq M_{\mathbb{D}} \times \mathbb{D}$$

and projecting the lower faces of this lifted polytope back to $\mathrm{conv}_{\mathbb{D}}(A)$. We denote this regular subdivision by $\Delta^h(\mathrm{conv}_{\mathbb{D}}(A))$.

In this way we have the following result (See Theorem 1.13.4 and Proposition 1.13.3)

Theorem J1 (Layered Regular subdivisions)**.**

1. *Any sequence*
$$\underline{\Delta}\colon \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1}$$
of regular subdivisions of a real polyhedron P in which each term subdivides the previous one is the layered regular subdivision of P with respect to a height function $h : A \to \mathbb{D}$ for a finite set $A \subseteq M_{\mathbb{R}}$ such that $\mathrm{conv}_{\mathbb{R}}(A) = P$.

2. *Consider a finite set of real points $A \subseteq M_{\mathbb{R}}$ and a height function $h : A \to \mathbb{D}$. Then, we have an equality of the form*
$$\Delta^h(\mathrm{conv}_{\mathbb{D}}(A)) = T\mathcal{C}\, \underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A)).$$

In the sense that, the elements of $\Delta^h(\mathrm{conv}_{\mathbb{D}}(A))$ *are exactly the polyhedra of the form* $T\mathcal{C}(\underline{F})$ *for*

$$\underline{F}\colon F_{k-1} \subseteq F_{k-2} \subseteq \cdots \subseteq F_0$$

where F_i *is a face of* Δ_i *for each* i*.*

Higher Rank Tropical Hypersurfaces

In this section we give some applications of tropical geometry to higher rank tropical geometry. For this we work with the *tropical semifield of rank* k, given by $\mathbb{T}_k = (\mathbb{D}\cup\{\infty\}, \min, +)$ in which the addition is the minimum between two elements and the multiplication is given by the usual addition in $\mathbb{D}$. Expressions in this semifield are written between quotation marks and with the usual symbols $+$ and $\cdot$. For example,

$$\text{“}x + y + \varepsilon xy\text{”} = \min\{x, y, \varepsilon + x + y\}.$$

Given a lattice M, a *Laurent tropical polynomial* with coefficients in M is an expression of the form

$$f = \text{“}\sum_{m\in M} a_m T^m\text{”}$$

such that $a_m = \infty$ for all but finitely many m in M. Laurent tropical polynomials are manipulated as usual polynomials following the rules of the ring $\mathbb{T}_k$. An element $x \in N_{\mathbb{D}}$ is a *zero* of f if the minimum in

$$f(x) = \min\{\langle m\,, x\rangle + a_m \mid m \in M\}$$

is achieved at least twice. Moreover, the *tropical hypersurface* induced by f, denoted by $V(f)$, is the set of all zeros of f.

Theorem K1 (Hypersurface Duality)**.** *Given a Laurent tropical polymonial* f*. The tropical hypersurface* $V(f)$ *has naturally the structure of an iterated fibration. Explicitly, it is given by*

$$V(f) = V(f^{[k-1]}) \longrightarrow V(f^{[k-2]}) \longrightarrow \cdots \longrightarrow V(f^{[0]})$$

where $f^{[i]}$ *is the tropical polynomial with coefficients in* $\mathbb{T}_{i+1}$ *whose coefficients has been obtained as the image of the coefficients of* f *under the projection* $\mathbb{T}_k \to \mathbb{T}_{i+1}$*. Moreover, the base of this iterated fibration and each fiber are tropical hypersurfaces of rank one. If* *one consider the* Newton polytope *of* f

$$\mathrm{New}(f) := \mathrm{conv}_{\mathbb{R}}\left(m \in M_{\mathbb{R}} \mid a_m \neq \infty\right)$$

Then, one can read the combinatorics of all these rank one tropical hypersurfaces from the layered regular subdivision of $\mathrm{New}(f)$ *induced by the coefficients of* f.

See Proposition 1.14.5 and Theorem 1.14.12 for the complete statements.

In particular, these show us that there is a combinatorial correspondence between layered regular subdivisions of polytopes and higher rank tropical hypersurfaces, as was conjectured by Joswigh and Smith in [JS18].

Finally, we show that higher rank tropical hypersurfaces can be endowed with a polyhedral structure compatible with the fibration point of view above. To state the result, given a Laurent tropical polynomial f, an element $x \in N_{\mathbb{D}}$ and an integer $0 \leq i \leq k-1$, let us consider its *i-initial part* by

$$\mathrm{in}_x^i(f) = \text{“} \sum_{\substack{m \in M \\ \langle m, x\rangle + a_m^{[i]} = f^{[i]}(x)}} a_m^{(i+1)} T^m \text{”} \in \mathbb{T}[M].$$

where $a^{(i)}$ correspond to the coefficient of ε^i in a_m and $a_m^{[i]}$. The i-initial parts are the rank one tropical polynomials encoding the fibers of the iterated fibration appearing in $V(f)$.

Theorem L1 (Polyhedral Structures on Hypersurfaces). *The tropical hypersurface* $V(f)$ *has a natural polyhedral complex structure over* $\mathbb{D}$ *on which the vector*

$$(\mathrm{in}_x^0(f), \mathrm{in}_x^1(f), \ldots, \mathrm{in}_x^{k-1}(f))$$

remains constant as x vary over the interiors of the cells. In this way, it is compatible with the description of the fibration described in the duality theorem. See Theorem 1.14.17 for the complete statement.

Organization of the Chapter

In Section 1.1, we introduce the ring of generalized dual numbers and the basic polyhedral objects over it. After that, we start developing the technical apparatus by generalizing Fourier-Motzkin Elimination in Section 1.2 and Farkas' Lemma in Section 1.3. This allow us to develop results in the structure of polyhedra and their faces in Sections 1.4 and 1.6

along with some other results in Sections 1.5, 1.7, 1.8 and 1.9, both in the structure of polyhedra and the duality theory of cones and normal fans at different levels.

Starting this point we change the perspective and move to study higher rank polyhedra as fibered objects. In Section 1.10 we introduce iterated fibration and the example of the tangent cone bundle. In Section 1.11 we see how tangent cone bundle of real polyhedra and, more generally, tangent cone bundle of flags of real polyhedra, produce polyhedra over $\mathbb{D}$. We use all this information to study $\mathbb{R}$-rational polyhedra in Section 1.12, where we introduce layered normal fans and show how to understand the combinatorics of faces of an $\mathbb{R}$-rational polyhedron from its layered normal fan.

In Section 1.13 we introduce *regular subdivisions* for polytopes over $\mathbb{D}$ and layered regular subdivision for polytopes over $\mathbb{R}$. We show how they correspond to each other in the real case by means of the tangent cone operator. In section Section 1.14 we use all these ideas to understand tropical hypersurfaces. In particular, using layered regular subdivisions, we understand the combinatorics coming from the fibration point of view and how to put a polyhedral structure over $\mathbb{D}$ compatible with this fibration point of view.

Chapter 2: Geometry of Higher Rank Valuations

Within this chapter, we introduce a certain number of tools and results suitable for the study of valuations of higher rank on function fields of algebraic varieties. This will be based on finite type approximations of the valuation spaces under consideration via a theory of higher rank skeleta that we develop in this chapter, by providing a geometric interpretation of higher rank quasi-monomial valuations in terms of tangent cones of cone complexes.

The motivations behind the undertaken study is multifold:

- On one side, the theory of Newton-Okounkov bodies and their variations [Oko96; KK12; LM09b; Bou12; BC11; Ami14; Cam+18; Cil+17; KM19a; RW19; EH19; CMM21; Bos21; HKW20]. One desires to understand continuity and wall-crossing behavior of convex bodies associated to big line bundles when the corresponding defining valuations vary. We propose a possible answer to a folklore open question in the field by providing a suitable base space for the study of families of Newton-Okoukov bodies.

- On the other, the recent works of Amini-Nicolussi on hybrid geometry of curves and their moduli spaces, concerning the constructions of higher rank hybrid and tropical compactifications [AN20; AN21] and the development of a function theory in higher rank non-Archimedean analysis [AN21]. This highlights the importance of higher rank non-Archimedean and tropical geometry in the study of the asymptotic geometry of multiparameter dependent families of complex varieties.

In the rest of this introduction, we provide an overview of our results and comment on the links to the related works.

All through this chapter we fix a field κ that we can assume to be algebraically closed. For a positive integer $k \in \mathbb{N}$, we set $[k] \coloneqq \{1, \dots, k\}$.

Valuations

We start by fixing ideas and notations on the valuations that will be used through this chapter.

Let $(\Gamma, \preceq)$ be a totally ordered abelian group and let K/κ be a field extension. A *valuation* ν on K over κ with values in Γ is a map $\nu\colon K \to \Gamma \cup \{\infty\}$ which verifies the following properties for any pair of elements $a, b \in K$.

1. $\nu(a) = \infty \iff a = 0$.

2. $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$ and $\nu(ab) = \nu(a) + \nu(b)$.

3. $\nu(a) = 0$ provided that a is in κ.

In this chapter we consider the additive group $(\mathbb{R}^k, \preceq_{\mathrm{lex}})$, for a fixed $k \in \mathbb{N}$, endowed with the lexicographic order $\preceq_{\mathrm{lex}}$. This is the order defined by saying $x \preceq_{\mathrm{lex}} y$, $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$, if either $x = y$ or there is $i \in [k]$ such that $x_j = y_j$ for $j < i$ and $x_i < y_i$. Moreover, we will suppose that K has finite transcendence degree over κ, that is we suppose the existence of a smooth connected variety X over κ such that K is the function field $K(X)$ of X. The integer number k will be regarded as a *bounding rank* for the valuations considered in this chapter. The idea to consider valuations of different rank simultaneously comes from practical situations in the study of degenerations of families of algebraic varieties over higher dimensional bases, see e.g. [AN20; AN21].

Basic examples of valuations in this setting are the followings:

– (Monomial valuations). Let $X = \mathbb{A}^2 = \operatorname{Spec}(\kappa[\mathrm{X}, \mathrm{Y}])$ and $K = \kappa(\mathrm{X}, \mathrm{Y})$. For $(a, b) \in \mathbb{R}_+$, there is a unique valuation

$$\nu_{a,b} \colon K \to \mathbb{R} \cup \{\infty\}$$

called *monomial valuation* with respect to (a, b) and given by

$$\nu_{a,b}(f) := \min\big\{ai + bj \mid a_{ij} \neq 0\big\}, \qquad f = \sum_{(i,j)\in\mathbb{Z}^2} a_{ij} x^i y^j \in \kappa[x, y].$$

Here and all through the chapter, $\mathbb{R}_+$ is the set of non-negative real numbers.

– (Divisorial and flag valuations). Suppose X/κ is a normal irreducible variety and let $F \subsetneq X$ be a closed irreducible subvariety of codimension one. The *order of vanishing along* F denoted by ord_F is a rank one valuation on $K = K(X)$, and any positive scalar multiple of ord_F is called a *divisorial valuation*. More generally, we can consider a flag of normal irreducible subvarieties

$$\mathcal{F}\colon \qquad F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_k$$

where $F_0 = X$ and $\operatorname{codim}_X(F_\ell) = \ell$, $\ell \in [k]$. Each F_ℓ thus defines a discrete valuation $\operatorname{ord}_{F_\ell}$ over $K(F_{\ell-1})$. This gives rise to a *flag valuation* $\nu_{\mathcal{F}}$ of rank bounded by k defined as

$$\begin{aligned} \nu_{\mathcal{F}} \colon K(X)^* &\to \mathbb{R}^k \\ f &\mapsto (\operatorname{ord}_{F_1}(f_1), \operatorname{ord}_{F_2}(f_2), \ldots, \operatorname{ord}_{F_k}(f_k)) \end{aligned} \tag{1}$$

where $f_1 = f$ and $f_{\ell+1} \in K(F_\ell)$ is the restriction of $f_\ell \cdot t_{\ell+1}^{-\operatorname{ord}_{F_{\ell+1}}(f_\ell)}$ to $F_{\ell+1}$ for $t_{\ell+1}$ a uniformizer for the valuation $\operatorname{ord}_{F_{\ell+1}}$, see for example [LM09b; KK12] for more details and for the link to the theory of Newton-Okounkov bodies.

– (Quasi-monomial valuations) We can generalize the first example above by replacing $\mathbb{A}^2$ by any normal irreducible variety X and taking a simple normal crossing (SNC) divisor $D = D_1 \cup \cdots \cup D_r$ on X. This leads to the concept of *quasi-monomial valuations*, which generalizes monomial, divisorial, and flag valuations, as we will see later in Theorem 2.3.12.

Consider the *dual cone complex* of the divisor D. This is a simplicial cone complex $\Sigma(D)$ in which there is a ray ρ_i corresponding to each component D_i of D, and for each subset $I \subseteq [r]$, each connected component (if any) of the intersection $D_I := \bigcap_{i\in I} D_i$ gives rise to a face σ with generating rays $\{\rho_i\}_{i\in I}$. More details can be found in Construction

2.1.6. Each face σ of $\Sigma(D)$ thus corresponds to a connected component of D_I, for $I \subset [r]$, that we denote by D_σ. In this case, we set $I_\sigma = I$ identified as the set of elements $i \in [r]$ such that D_i contains D_σ. The divisor D being SNC, D_σ is normal irreducible and has a generic point η_σ. Moreover, we can choose local equations $\{z_i\}_{i\in I}$ for the components $\{D_i\}_{i\in I}$ around η_σ.

Just as we did for the case of monomial valuations, for the totally ordered abelian group $(\Gamma, \preceq)$, we can pick a vector $\underline{\alpha} = (\alpha_i)_{i\in I}$ with $\alpha_i \in \Gamma_{\succeq 0}$, and define a unique valuation $\nu_{\underline{\alpha}}$ on $K = K(X)$ by requiring

$$\nu_{\underline{\alpha}}(\prod_{i\in I} z_i^{\gamma_i}) := \sum_{i\in I} \alpha_i \gamma_i$$

for any $\gamma = (\gamma_i) \in \mathbb{Z}_+^I$. We can then naturally extend this, first, to the local ring $\mathcal{O}_{X,\eta_\sigma}$ by taking the minimum over terms of a power series expansion, and then to the full function field. This is the quasi-monomial valuation associated to D and the *weights* $\underline{\alpha}$. For more details we refer to Section 2.3.

We denote by $\mathscr{M}^k(D) = \mathscr{M}^k(X, D)$ the set of all *quasi-monimial valuations of rank bounded by* k with $(\Gamma, \preceq) = (\mathbb{R}^k, \preceq_{\text{lex}})$. For $k = 1$, we further simplify the notation to $\mathscr{M}(D)$. From the above description, it follows that elements of $\mathscr{M}(D)$ are in bijection with the pair $(\sigma, \underline{\alpha})$ with $\underline{\alpha} = (\alpha_i)_{i\in I_\sigma} \in \mathbb{R}_+^{I_\sigma}$. This means $\mathscr{M}(D)$ can be naturally identified with $\Sigma(D)$. The above sets come with a natural tower of projection maps

$$\mathscr{M}(D) \leftarrow \mathscr{M}^2(D) \leftarrow \cdots \leftarrow \mathscr{M}^{k-1}(D) \leftarrow \mathscr{M}^k(D) \leftarrow \dots$$

induced by the projection map to the first $k-1$ coordinates $\mathbb{R}^k \to \mathbb{R}^{k-1}$.

Tropicalization

Let X be a normal irreducible variety and let D be an SNC divisor on X. The elements of the dual cone complex $\Sigma(D)$ correspond to quasi-monomial valuations of rank bounded by one on the function field $K(X)$ of X. For each rational function $f \in K = K(X)$, we thus get by evaluation a function

$$\text{trop}(f) \colon \Sigma(D) \to \mathbb{R}, \qquad \underline{\alpha} \in \sigma \to \nu_{\underline{\alpha}}(f),$$

called the tropicalization of f. This is a piece-wise integral linear function on each cone σ of $\Sigma(D)$.

In this chapter we provide an extension of this picture to the case of higher rank quasi-monomial valuations. This will be based on a duality theorem we state in the next section which will allow to give a geometric meaning to the space $\mathscr{M}^k(D)$ and the tropicalization map

$$\mathrm{trop}(f)\colon \mathscr{M}^k(D) \to \mathbb{R}^k, \qquad \underline{\alpha} \in \sigma \to \nu_{\underline{\alpha}}(f),$$

for $\underline{\alpha} \in (\mathbb{R}^k_{\succeq_{\mathrm{lex}} 0})^{I_\sigma}$

Tangent cone bundles and duality theorem

The first contribution of this paper is the duality theorem below which provides a geometric realization of the set $\mathscr{M}^k(\Sigma(D))$ of quasi-monomial valuations of rank bounded by k as a *tangent cone bundle* on $\Sigma(D)$.

Notations as above, recall that for $k > 1$ there is a natural projection map $\mathscr{M}^k(D) \to \mathscr{M}(D)$ which allows to view $\mathscr{M}^k(D)$ as a bundle over $\mathscr{M}(D) = \Sigma(D)$. We give a geometric characterization of this bundle in terms of $\Sigma(D)$.

Theorem A2 (Duality Theorem)**.** *Notations as above, there is an isomorphism of bundles over* $\mathscr{M}(D) \simeq \Sigma(D)$

$$\begin{array}{ccc} \mathscr{M}^k(D) & \xrightarrow{\simeq} & T\mathcal{C}^{k-1}\,\Sigma(D) \\ \downarrow & & \downarrow \\ \mathscr{M}(D) & \xrightarrow{\simeq} & \Sigma(D) \end{array} \tag{2}$$

where $T\mathcal{C}^{k-1}\,\Sigma(D)$ *is the* $(k-1)$-tangent cone *of* $\Sigma(D)$ *defined as the set of all elements of the form* $(\underline{x}; \underline{w}_1, \dots, \underline{w}_{k-1})$ *where*

- *the* base point $\underline{x}$ *is a point of* $\Sigma(D)$*, and*

- $\underline{w}_1, \dots, \underline{w}_{k-1}$ *is an ordered set of tangent vectors to* $\Sigma(D)$ *at* $\underline{x}$ *such that we have*

$$x + \varepsilon \underline{w}_1 + \varepsilon^2 \underline{w}_2 + \dots + \varepsilon^r \underline{w}_r \in \Sigma(D),$$

for any $r \in [k-1]$ *and any small enough* $\varepsilon > 0$.

For a more precise meaning to the above taken sum, we refer to Section 2.3. We call $T\mathcal{C}^{k-1}\,\Sigma(D)$ the *tangent cone bundle of* $\Sigma(D)$ *of order* $k-1$.

Using the above correspondence, we give an explicit realization of higher rank quasi-monomial valuations as *directional derivative operators* defined in terms of the corresponding tangent vectors. In order to do this, we equip the cone complex $\Sigma(D)$ with its *structure* *sheaf* $\mathcal{O}_{\Sigma(D)}$ which is the sheaf of *tropical functions*. These are continuous functions whose restrictions on each cone σ of Σ coincides with a piecewise integral linear function defined on that cone.

A rational function $f \in K = K(X)$ induces a global section $\mathrm{trop}(f)$ of the structure sheaf.

Theorem B2 (Duality Theorem, second form)**.** *Let $(x;\underline{w})$ be an element of the tangent cone $T\mathcal{C}^{k-1}\,\Sigma(D)$ with $\underline{w} = (w_1, \dots, w_{k-1})$. The valuation $\nu_{x;\underline{w}}$ given by the duality theorem above is described as*

$$\begin{aligned} \upsilon_{x;\underline{w}} \colon K(X) &\longrightarrow \mathbb{R}^k \\ f &\longmapsto \big(\mathrm{trop}(f)(x), D_{w_1}\,\mathrm{trop}(f)(x), \dots, D_{w_1,\dots,w_k}\,\mathrm{trop}(f)(x)\big) \end{aligned}$$

where

- $D_{w_1}\,\mathrm{trop}(f)(x)$ *is the directional derivative of the function* $\mathrm{trop}(f)$ *at x in the direction w_1, and*
- *recursively,* $D_{w_1,\dots,w_{r+1}}\,\mathrm{trop}(f)(x)$ *is the directional derivative of* $D_{w_1,\dots,w_r}\,\mathrm{trop}(f)(x)$ *seen as a function on the variable w_r in the direction w_{r+1}.*

Topologies on the tangent cone of dual complexes

Notations as before, let $D = D_1 \cup \dots \cup D_r$ be an SNC divisor on a smooth quasi-projective variety X, and let $\Sigma(D)$ be the corresponding dual cone complex. Let k be an integer and consider the tangent cone bundle $T\mathcal{C}^{k-1}\,\Sigma(D)$. There are four natural topology one can define on the tangent cone of a dual complex. They all coincide in the case $k = 1$, but differ fundamentally for larger values of k. We now discuss these topologies.

First note that by the definition of the monomial valuations, we have a surjection $(\mathbb{R}^k)^r_{\succeq 0} \to \mathscr{M}^k(D)$. Via the duality theorem, the two first topologies on the tangent cone $T\mathcal{C}^{k-1}\,\Sigma(D)$ are induced by this surjection. Namely,

• (*Ordered topology*) This is the topology on $T\mathcal{C}^{k-1}\,\Sigma(D) \simeq \mathscr{M}^k(D)$ induced by the ordered topology of $(\mathbb{R}^k)_{\succeq 0}$.

• (*Euclidean topology*) This is the topology on $T\mathcal{C}^{k-1}\,\Sigma(D)$ induced by the Euclidean topology of $(\mathbb{R}^k)_{\succeq 0} \subseteq \mathbb{R}^k$. Equivalently, this is the topology induced by the Euclidean topology of $\Sigma(D)$.

• (*Hahn-Berkovich topology*) This is the natural topology which appears usually in the context of non-Archimedean geometry, that is the coarsest topology which makes continuous all the tropicalization maps

$$\mathrm{Trop}(f)\colon\ T\mathcal{C}^{k-1}\,\Sigma(D)\to\mathbb{R}^k_{\mathrm{lex}},\qquad f\in K(X)$$

where $\mathbb{R}^k_{\mathrm{lex}}$ refers to $\mathbb{R}^k$ equipped with its lexicographically ordered topology. Note that it makes sense for any ordered abelian group Γ as the value group for the space of valuations.

• (*Tropical topology*) This is arguably the most interesting topology one can define on the tangent cone, as it happens to mix the properties of the Euclidean topology on $\mathbb{R}^k$ with those coming from the lexicographic order used in order to define the valuations. By definition, this is the coarsest topology which makes continuous all the tropicalization maps

$$\mathrm{Trop}(f)\colon\ T\mathcal{C}^{k-1}\,\Sigma(D)\to\mathbb{R}^k,\qquad f\in K(X)$$

in which $\mathbb{R}^k$ is equipped with its Euclidean topology. This topology might be called as well the Hahn-Euclidean topology.

In this chapter we provide an explicit description of the tropical topology. This is obtained as a consequence of the tropical weak approximation theorem proved below, as we describe next.

Refined tropicalization and tropical weak approximation

Let D be an SNC divisor on X. For each cone $\sigma\in\Sigma(D)$ and for each $i\in I_\sigma$, consider a local equation z_i for D_i around η_σ. Then, the family $\{z_i\}_{i\in I_\sigma}$ provides a system of local parameters for the local ring $\widehat{\mathcal{O}}_{X,\eta_\sigma}$ obtained as the completion of $\mathcal{O}_{X,\eta_\sigma}$ at its maximal idea. Each element of the local ring $\widehat{\mathcal{O}}_{X,\eta_\sigma}$ admits an *admissible expansion* in the terminology of [JM12], that is an expansion of the form

$$f=\sum_{\beta\in\mathbb{Z}^r_+}c_\beta z^\beta,\ \ c_\beta\in\widehat{\mathcal{O}}_{X,\eta_\sigma},\tag{3}$$

in which the right hand side is a convergent series with each coefficient c_β either zero or a unit element in $\widehat{\mathcal{O}}_{X,\eta_\sigma}$. Here and in what follows, the notation z^β stands for the product $z_1^{\beta_1}\dots z_r^{\beta_r}$ where $\beta_1,\dots,\beta_r$ denote the coordinates of $\beta\in\mathbb{Z}^r$.

The *support* of the admissible expansion is the set given by all $\beta\in\mathbb{Z}^r_+$ such that c_β is not zero.

Although an element f has generally infinitely many admissible expansions, we will show later that the set of initial terms of f is invariant under the choice of the expansion and the local parameters. Here an initial term is an element of the support which is minimal for the partial order $\leq_{\text{cw}}$ in which a vector $x = (x_1, \dots, x_r)$ is less than or equal to $y = (y_1, \dots, y_r)$ if coordinate-wise we have $x_j \leq y_j, j \in [r]$. We denote the initial terms of f by A_f^σ. Note that the terms in A_f^σ form an *antichain* for the partial order $\leq_{\text{cw}}$, that is any pair of distinct elements in A_f^σ is incomparable relative to $\leq_{\text{cw}}$. The antichain A_f^σ determines the restriction of $\mathrm{Trop}(f)|_\sigma$.

$$\mathrm{Trop}(f)(x) = \min_{\beta \in A_f^\sigma} \langle x, \beta \rangle$$

For a rational function $f \in K(X)$ which belongs to all the local rings $\widehat{\mathcal{O}}_{X,\eta_\sigma}$, $\sigma \in \Sigma$, we thus get the *refined tropicalization* of f given by the collection $\mathcal{A}_f := \{A_f^\sigma \,|\, \sigma \in \Sigma(D) \text{ with } f \in \mathcal{O}_{X,\eta_\sigma}\}$, the *family of antichains attached to* f. Such a family verifies the following compatibility property:

- (*Coherence property*) For any inclusion of faces $\tau \subseteq \sigma$, we have the relation

$$A_f^\tau = \min_{\leq_{\text{cw}}} \mathrm{pr}_{\sigma \succ \tau}(A_f^\sigma).$$

Here $\mathrm{pr}_{\sigma \succ \tau}$ is the projection map $\mathbb{R}^{I_\sigma} \to \mathbb{R}^{I_\tau}$.

Theorem C2 (Tropical weak approximation theorem)**.** *Let X be a smooth quasi-projective variety over a field k and let D be an SNC divisor on X. Let $\mathcal{A} = \{A^\sigma \,|\, \sigma \in \Sigma(X, D)\}$ be a family consisting of finite sets $A^\sigma \subset \mathbb{Z}_+^{I_\sigma}$ such that*

- *each A^σ is antichain for the partial order $\leq_{\text{cw}}$, for $\sigma \in \Sigma(X, D)$*

- *the family $\mathcal{A}$ verifies the coherence property, that is for inclusion of faces $\tau \subseteq \sigma$, we have $A^\tau = \min_{\leq_{\text{cw}}} \mathrm{pr}_{\sigma \succ \tau}(A^\sigma)$.*

Then there exists a rational function $f \in K(X)$ such that for each cone σ of $\Sigma(X, D)$, we have $f \in \mathcal{O}_{X,\eta_\sigma}$ and $A^\sigma = A_f^\sigma$.

The above theorem might be regarded as a tropical analogue of the weak approximation theorem in number theory.

From the above theorem we deduce the following result [2].

Corollary. *Let X be a smooth quasi-projective variety over a field κ and let D be a simple normal crossing divisor on X. Any tropical function F on the support of the dual cone complex $\Sigma(D)$ is the tropicalization of a rational function $f \in K(X)$.*

As a consequence of the above result and our analytic description of higher rank valuations as multidirectional derivatives of tropical functions, we infer that both the Hahn-Berkovich and tropical topology are intrinsic to the cone complex $\Sigma(D)$, that is they can be defined more generally for any rational cone complex Σ. (The intrinsic nature of the two other topologies, the ordered and the Euclidean, is obvious from the definition.)

The following theorem provides a description of the tropical topology. Let Σ be a rational cone complex and suppose $\widetilde{\Sigma}$ is a rational subdivision of it. Let k be a positive integer. A set $U \subset T\mathcal{C}^{k-1}\,\Sigma$ is called a $\widetilde{\Sigma}$-*open* if $U \cap T\mathcal{C}^{k-1}\,\sigma$ is open in $T\mathcal{C}^{k-1}\,\sigma$ with respect to the Euclidean topology for every cone σ of $\widetilde{\Sigma}$.

Theorem D2 (Characterization of the tropical topology)**.** *Notations as above, we have*

1. *For each rational subdivision $\widetilde{\Sigma}$ of Σ, the $\widetilde{\Sigma}$-open sets of $T\mathcal{C}^{k-1}\,\Sigma$ are open with respect to the tropical topology.*

2. *The union of all $\widetilde{\Sigma}$-open sets, $\widetilde{\Sigma}$ a rational subdivision of Σ, form a basis of opens sets for the tropical topology.*

Spaces of higher rank valuations

Given a variety X over κ, the *birational analytification of X of bounded rank k* is the set

$$X^{\mathrm{bir},k} := \big\{\nu\colon K(X)^* \to \mathbb{R}^k \mid \nu \text{ is a valuation}\big\}$$

that we endow with the coarsest topology which makes continuous all the evaluation maps, for any $f \in K(X)^*$,

$$\begin{aligned} \mathrm{ev}_f\colon X^{\mathrm{bir},k} &\longrightarrow \mathbb{R}^k \\ \nu &\longmapsto \nu(f). \end{aligned}$$

[2] As it was pointed to us independently by Sébastien Boucksom and by Mirko Mauri and Enrica Mazzon, this corollary can be alternatively obtained by more direct methods.

Here, we equip $\mathbb{R}^k$ with its Euclidean topology. Moreover, we define the following subspaces of $X^{\mathrm{bir},k}$

$$X^{\beth,k} := \left\{\nu \in X^{\mathrm{bir},k} \mid \nu \text{ has a center in } X\right\}$$
$$X^{\daleth,k} := \left\{\nu \in X^{\mathrm{bir},k} \mid \nu \text{ does not have any center in} X\right\}$$

that we endow with the topology induced by that of $X^{\mathrm{bir},k}$. Recall that for a variety X and a valuation $\nu\colon K(X) \to \Gamma$, the center of ν, if it exists, is the unique point $x \in X$ such that ν is non-negative over $\mathcal{O}_{X,x}$ and strictly positive over its maximal ideal.

Notice that $X^{\mathrm{bir},k} = X^{\beth,k} \sqcup X^{\daleth,k}$ and $X^{\mathrm{bir},k} = X^{\beth,k}$ if X is proper. In the terminology of Foster and Ranganathan [FR16b], the space $X^{\mathrm{bir},k}$ coincides with the subspace of all valuations defined over the generic point in the Hahn analytification of X endowed with the *extended Euclidean topology*. Moreover, the notation $X^{\beth,k}$ is used in analogy to the analytic space $X^{\beth}$ of Berkovich [Ber96] and Thuillier [Thu07], where the dot is a reminder that we are considering only tehe birational parts.

We can actually go further and introduce a flag of subspaces of $X^{\mathrm{bir},k}$ called the *the centroidal flag*, which interpolates between $X^{\beth,k}$ and $X^{\mathrm{bir},k}$. This is done as follow. For $0 \leq r \leq k$, we consider the set

$$\mathscr{F}^r X^{\mathrm{bir},k} := \left\{\nu \in X^{\mathrm{bir},k} \mid \mathrm{proj}_r(\nu) \text{ has a center in } X\right\}$$

where $\mathrm{proj}_r(v)$ is the composition of v with the projection $\mathbb{R}^k \to \mathbb{R}^r$ to the first k coordinates. In other words,

$$\mathscr{F}^r X^{\mathrm{bir},k} := \mathrm{proj}_r^{-1} X^{\beth,k}.$$

This give us a decreasing filtration

$$X^{\mathrm{bir},k} = \mathscr{F}^0 X^{\mathrm{bir},k} \supseteq \mathscr{F}^1 X^{\mathrm{bir},k} \supseteq \cdots \supseteq \mathscr{F}^k X^{\mathrm{bir},k} = X^{\beth,k}.$$

For an SNC divisor D on the variety X, the space $T\mathcal{C}^{k-1}\,\Sigma(D)$ endowed with its tropical topology naturally fits inside $X^{\beth,k}$. As we will next explain, the tangent cone bundles provide a higher rank notion of skeleton for the above space of valuations.

Tangent cone bundles as higher rank skeleta

We now discuss the relevance of the tangent cone bundles to higher rank non-Archimedean geometry.

We start by recalling some basic definitions in birational geometry. Let X be a smooth variety over κ. A *log-smooth compactification of* X is a proper variety Y containing X as an open subvariety such that $Y \setminus X$ is a simple normal crossing divisor on Y. A morphism between log-smooth compactifications Y' and Y is a morphism $f\colon Y' \to Y$ between the underlying varieties such that $f^{-1}(X) = X$ and $f|_X$ is an isomorphism. The category of log-smooth compactifications of X will be denoted by $\mathbf{LSC}_X$.

A *compactified log-smooth pair* is the data of a pair $\overline{\mathbf{Y}} = (Y, D)$ consisting of a proper variety Y and a simple normal crossing divisor $D \subset Y$ together with a birational map $\varphi\colon Y \dashrightarrow X$ such that the divisor D can be decomposed as $D = D^\circ + D^\infty$ where D° and D^∞ does not have any component in common, and such that

(*i*) the domain of definition of φ is $Y \setminus D^\infty$, that is,

$$\varphi\colon Y \setminus D^\infty \longrightarrow X$$

is well-defined and $Y \setminus D^\infty$ is the maximum open set with this property.

(*ii*) the pair $(Y \setminus D^\infty, D^\circ|_{Y\setminus D^\infty})$ is a log-smooth pair for X, i.e., $\varphi|_{Y\setminus D^\infty}$ is a proper morphism from $Y \setminus D^\infty$ to X and the restriction

$$Y \setminus (D^\circ \cup D^\infty) \longrightarrow X \setminus \varphi(D^\circ)$$

is an isomorphism.

Morphisms between compactified log-smooth pairs can be defined in a natural way. The category of compactified log smooth pairs will be denoted by $\mathbf{CLSP}_X$.

For a compactified log-smooth pair $\overline{\mathbf{Y}} = (Y, D)$, we denote by $\Sigma(\overline{\mathbf{Y}}) = \Sigma(Y, D)$ the dual cone complex associated to the divisor D on Y. We denote by $T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$ the corresponding tangent cone bundle that we endow with the tropical topology.

Given a compactified log-smooth pair $\overline{\mathbf{Y}} = (Y, D)$ over X, as above, the decomposition $D = D^\circ \cup D^\infty$ gives the subcomplex $\Sigma(D^\circ)$ inside $\Sigma(\overline{\mathbf{Y}})$ which we denote by $\Sigma(\overline{\mathbf{Y}}^\circ)$. The *centroidal filtration* of $T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$ is by definition the filtration

$$\mathscr{F}^0\, T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \supseteq \mathscr{F}^1\, T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \supseteq \cdots \supseteq \mathscr{F}^k\, T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$$

given for $0 \le r \le k$ by

$$\mathscr{F}^r\, T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \coloneqq \Big\{\big(x;(w_1,\dots,w_{k-1})\big) \in T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \mid \big(x;(w_1,\dots,w_{r-1})\big) \in T\mathcal{C}^{r-1}\,\Sigma(\overline{\mathbf{Y}}^{\circ})\Big\}.$$

We prove the following theorem.

Theorem E2 (Characterization of the tropical topology)**.** *Notations as above, for each compactified log-smooth pair* $\overline{\mathbf{Y}} = (Y, D)$ *over* X*, there is a continuous retraction*

$$r_{\overline{\mathbf{Y}}} \colon X^{\mathrm{bir},k} \longrightarrow T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}).$$

Moreover, the deduced continuous map

$$r\colon X^{\mathrm{bir},k} \longrightarrow \varprojlim_{\overline{\mathbf{Y}}\in\mathbf{CLSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$$

is a homeomorphism. In addition, the limit are compatible with the centroidal filtration on the analytic spaces and on tangent cone bundles. That is, for each $0 \le r \le k$*, we get to a homeomorphism*

$$\mathscr{F}^r X^{\mathrm{bir},k} \longrightarrow \varprojlim_{\overline{\mathbf{Y}}\in\mathbf{CLSP}_X} \mathscr{F}^r\, T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}).$$

We note that the above theorem shows that tangent cones with their tropical topology should be regarded as the *higher rank analogue of skeletons* in non-Archimedean geometry.

We remark that the statements of the above theorem hold as well in the case where the spaces in consideration are equipped with the Hahn-Berkovich topology. However, due to *mixed nature* of the tropical topology, the arguments in the proof in the case of the tropical topology become more subtle. In particular, on the way for getting the result, we need to establish a somehow surprising Topology-Mixing Lemma 2.6.9.

Variations of Newton-Okounkov bodies

The tropical topology and the above spaces of valuations seem to be the right topological spaces for the problem of understanding the variations of Okounkov bodies, as we explain now.

Let X be a smooth projective variety of dimension d and let $L = \mathcal{O}(E)$ be a big line bundle over X. Consider the graded algebra

$$H_\bullet = \bigoplus_{n \geq 0} H_n$$

where $H_n := H^0(X, \mathcal{O}(nE))$ is a finite dimensional κ-vector subspace of $K(X)$.

Each valuation $\nu \in X^{\mathrm{bir},d}$ gives rise to the corresponding Newton-Okounkov body in $\mathbb{R}^d$ denoted by Δ_ν and defined by

$$\Delta_v := \overline{\bigcup_{n \geq 0} \left\{ \frac{\nu(f)}{n} \mid f \in H_n \right\}}.$$

Let now D be a simple normal crossing divisor on X and consider the tangent cone $T\mathcal{C}^{d-1}\,\Sigma(D)$. Consider the space $\mathrm{BC}(\mathbb{R}^k)$ of compact subsets of $\mathbb{R}^k$ endowed with the Hausdorff distance. We get a map

$$\begin{aligned} \Delta\colon\ T\mathcal{C}^{d-1}\,\Sigma(D) &\longrightarrow \mathrm{BC}(\mathbb{R}^n) \\ (x; \underline{w}) &\longmapsto \Delta_{\nu_{x,\underline{w}}}. \end{aligned} \tag{4}$$

The study undertaken in this chapter has as objective to ultimately prove that the above map Δ is continuous when $T\mathcal{C}^{d-1}\,\Sigma(D)$ is endowed with the tropical topology. This topology is actually the only natural one on $T\mathcal{C}^{d-1}\,\Sigma(D)$ for which one can expect this statement to be both true and non-trivial, as can be verified through basic examples.

Conjecture. Let L be a big line bundle on a projective variety X of dimension d. Let D be a simple normal crossing divisor on X with dual cone complex $\Sigma(D)$ of pure dimension d. The variation of Newton-Okounkov bodies on $T\mathcal{C}^{d-1}(\Sigma(D))$ is continuous.

We later provide a heuristic argument for the validity of this conjecture. In particular, on those cones whose augmented semigroups are finitely generated, the conjecture holds.

Related work

In this final section, we make a comparison of our results with the existing ones in the literature.

The contributions of this chapter should be regarded as part of the recent attempts to generalize the framework of tropical and non-Archimedean geometry to higher rank valuations.

Analytification of varieties based on valuations has been developed in the pioneering works of Berkovich [Ber12] and Huber [Hub94]. Both spaces are intimately linked with tropical geometry, in the former by means of usual tropicalization and in the latter by means of adic tropicalization [Fos16]. More recently, Kedlaya [Ked15] and Foster-Ranganathan [FR16b; FR16a] introduced an alternative analytification directly linked to the one of Berkovich based on higher rank valuations. This last point of view is similar to the one we have adapted as the setting for formulating our results in this chapter.

Higher rank tropicalization has been studied by Aroca [Aro10b; Aro10a], Aroca-Garay-Toghani [AGT16], Banerjee [Ban15], Foster-Ranganathan [FR16b; FR16a], Kaveh-Manon [KM19a; KM19c], Escobar-Harada [EH19], and Joswig-Smith [JS18]. Our work can be regarded as the geometric version of higher rank tropicalization. A framework for higher rank polyhedral and tropical geometry related to the set-up introduced in this paper will appear in the forthcoming paper [Iri21]. Tangent cone bundles we introduce in this paper and their refinements play a central role in that work.

Geometric tropicalization in rank one has been studied by Hacking-Keel-Tevelev [HKT09], Thuillier [Thu07], Abramovich-Caporaso-Payne [ACP15], and more recently by Ulirsch [Uli17] and Gross [Gro18], among others.

A more general framework for tropicalization has been developed in the work of Lorscheid on blueprints [Lor15], and in the works of Giansiracusa-Giansiracusa [GG16; GG14] and Maclaghan-Rincón [MR20]. Because of the level of generality in those works, higher rank tropicalization can be treated using any of the former two frameworks. Tropicalization with values in hyperfields is studied by Viro [Vir10], Jun [Jun17] and Jell-Scheiderer-Yu [JSY18].

The link between skeletons and tropicalizations in rank one has been throughly studied in the works of Gubler-Rabinoff-Werner [GRW17; GRW16], Macpherson [Mac20], and Baker-Payne-Rabinoff [BPR16]. Since skeletons play a central role in connecting complex and non-Archimedean geometry, in the study of one-parameter families of complex manifolds, we expect that higher rank analogues of skeleta introduced in this paper, and their polyhedral counterparts further developed in [Iri21], will play a central role in the study of multiparameter families of complex manifolds. A systematic study of multiparameter families of Riemann surfaces is undertaken in the series of works [AN20; AN21].

The link between dual cone complexes and higher rank valuations we provide in this paper should be compared with the work of Kaveh and Manon [KM19a] on Khovanskii bases. In that work, the authors show how to associate to prime cones appearing in the tropicalization of subvarieties of affine spaces a higher rank valuation on the coordinate ring of the variety with a finitely generated semigroup. The construction we consider directly associates higher rank valuations to cone complexes of normal crossing divisors. These cone complexes live in the Berkovich analytification of the variety X. By the work of Gubler-Rabinoff-Werner [GRW17; GRW16], a prime cone appearing in the tropicalization of a variety can be viewed naturally in the Berkovich analytification of the variety. In this regard, it seems possible to retrieve the valuations in the work of Kaveh and Manon as those in the framework of our paper which come from faithful tropicalizations of the variety. We refer to [RW19; Bos21; Bos+17; Bos+18; IW20; EH19] for further results on the connection between tropical geometry, toric degenerations and Khovanskii bases.

The origin of limit theorems goes back to the work of Zariski [Zar39; Zar44] on resolution of singularities in dimension two and three using Riemann-Zariski spaces. For tropicalizations, this has been shown in rank one by Kontsevich and Soibelman [KS06] (unpublished), Payne [Pay09], Foster-Gross-Payne [FGP14], Jonsson-Mustaţă [JM12], Boucksom-Favre-Jonsson [BFJ16], and Boucksom-Jonsson [BJ18]. We have been particularly inspired by the work of [JM12] in establishing our limit theorems. A higher rank version of [FGP14] has been obtained by Foster-Ranganathan [FR16b]. Relative Riemann-Zariski spaces are studied by Temkin [Tem11; Tem10]. We refer to the book of Fujiwara-Kato [FK13] for a detailed discussion of Riemann-Zariski spaces and their applications in rigid geometry.

A version of the duality theorem for the valuative tree was proved by Favre and Jonsson in [FJ04]. For curves over non-trivially valued fields this theorem should be compared with the description of tangent directions at points of type 2 in the Berkovich analytification as valuations of rank two on the function field of the curve, a result which can be traced back to Bosch-Lütkebohmert [BL85] and Berkovich [Ber12]. This is also the main ingredient in Thuillier's non-Archimedean version of Poincaré-Lelong formula for curves [Thu05] and its reformulation as a slope-formula by Baker-Payne-Rabinoff [BPR13].

Finally, let us mention that a version of the approximation theorem for curves for non-trivially valued base fields is proved by Baker-Rabinoff [BR15]. We expect that our theorem should be true in the non-trivially valued case in any dimension and plan to come back to this setting in a future work.

Organization of the chapter

Here is the plan of this chapter. In Section 2.1 we introduce dual cone complexes endowed with the sheaf of tropical functions and their tangent cones. We attach to a simple normal crossing divisor on a variety X its corresponding dual cone complex and its tangent cone. In Section 2.2 we recall the definition of tropicalization of rational functions, and explain how to attach a system of antichains to a rational function, leading to a refinement of the definition of tropicalization.

Section 2.3 introduces quasi-monomial valuations of a given rank and study their basic properties. This section contains the proof of the duality theorem and an analytic description of the monomial valuations in terms of directional derivatives along elements in the tangent cones. Section 2.4 contains the proof of our approximation theorem. In Section 2.5, we study the tropical topology on the tangent cone, and provide an explicit basis of this topology. Section 2.6 introduces several spaces of higher rank valuations on the function field of a smooth variety over κ. The results are used in Sections 2.6.3, 2.7 and 2.8 to prove the continuity of the retraction map and the limit formulae.

Chapter 1

Polyhedral Geometry Over the Generalized Dual Numbers

The aim of this chapter is to provide the foundations of a polyhedral geometry in the higher rank setting, and to give the first applications to a (to-be-developed) theory of higher rank tropical geometry.

1.1 Basic Concepts in Higher Rank Polyhedral Geometry

In the following we will work with the *ring of generalized dual numbers of rank* k defined by $\mathbb{D} := \mathbb{R}[\varepsilon]/(\varepsilon^k)$. For $k = 2$ it recovers the usual ring of dual numbers. Elements of $\mathbb{D}$ have the form

$$x = x^{(0)} + x^{(1)}\varepsilon + \cdots + x^{(k-1)}\varepsilon^{k-1} \tag{1.1}$$

with $x^{(0)}, \ldots, x^{(k-1)} \in \mathbb{R}$. They are manipulated as usual power series with coefficients in $\mathbb{R}$ imposing that $\varepsilon^k = 0$, in the same way as one works with Taylor expansions of the form $a_0 + a_1 z + \cdots + a_{k-1} z^{k-1} + o(z^{k-1})$. If the rank of $\mathbb{D}$ has to be made explicit, we use a lower index $\mathbb{D}_k$.

For an element $x \in \mathbb{D}$ as in (1.1) and for $0 \leq i \leq k-1$, we denote by $x^{[i]}$ the truncated element

$$x^{[i]} = x^{(0)} + x^{(1)}\varepsilon + \cdots + x^{(i)}\varepsilon^i.$$

Moreover, we introduce the *order* of x by $\operatorname{ord}(x) = \min\{j \in \{0, \ldots, k-1\} \mid x^{(j)} \neq 0\}$ if

$x \neq 0$ and $\mathrm{ord}(0) = k$.

Notice that $\mathbb{D}$ is isomorphic to $\mathbb{R}^k$ as an additive group. We endow $\mathbb{D}$ with the lexicographic order

$$a^{(0)} + \varepsilon a^{(1)} + \cdots + \varepsilon^{k-1} a^{(k-1)} < b^{(0)} + \varepsilon b^{(1)} + \cdots + \varepsilon^{k-1} b^{(k-1)} \tag{1.2}$$

$$\iff \quad a^{(i)} < b^{(i)} \text{ for the first } i \text{ such that } a^{(i)} \neq b^{(i)}. \tag{1.3}$$

In this way, we obtain an order on $\mathbb{D}$ that we simply denote by $\leq$. This order turns out to be compatible with the additive and multiplicative structure of $\mathbb{D}$ turning it into an ordered ring.

Remark 1.1.1. The following observation is useful and will be used sometimes in the arguments. Given $a, b \in \mathbb{D}$, we have $a \leq b$ with the lexicographic order introduced in (1.2) iff we have

$$a^{(0)} + \delta a^{(1)} + \cdots + \delta^{k-1} a^{(k-1)} \leq b^{(0)} + \delta b^{(1)} + \cdots + \delta^{k-1} b^{(k-1)}$$

for every $\delta \in \mathbb{R}_{>0}$ small enough.

Given a lattice $N \cong \mathbb{Z}^n$ with dual lattice $M = \mathrm{Hom}(N, Z)$ we consider the base changes $N_{\mathbb{D}} = N \otimes \mathbb{D}$ and $M_{\mathbb{D}} = M \otimes \mathbb{D}$. The pairing $M \otimes N \to \mathbb{Z}$ naturally extends to a pairing $M_{\mathbb{D}} \otimes N_{\mathbb{D}} \to \mathbb{D}$ which we denote by $\langle \cdot \,, \cdot \rangle$.

Remark 1.1.2. Under the pairing $\langle \cdot \,, \cdot \rangle$, the linear functions from $N_{\mathbb{D}}$ to $\mathbb{D}$ correspond exactly to the elements of $M_{\mathbb{D}}$. For this reason, we decide to write y instead of $\langle y \,, \cdot \rangle$ when there is no risk of confusion. More generally, the affine functions from $N_{\mathbb{D}}$ to $\mathbb{D}$ are all of the form $\langle y \,, \cdot \rangle + a$ for some $y \in M_{\mathbb{D}}$ and $a \in \mathbb{D}$.

Using the ordered ring structure on $\mathbb{D}$ we can introduce several geometric concepts over the module $N_{\mathbb{D}}$.

Definition 1.1.3.

1. A set $P \subseteq N_{\mathbb{D}}$ is *convex* if for any $x, y \in P$ and any $t \in \mathbb{D}$ such that $0 \leq t \leq 1$, we have
$$tx + (1-t)y \in P.$$

2. A set $\sigma \subseteq N_{\mathbb{D}}$ is a *cone* if for any $x, y \in \sigma$ and any $t \in \mathbb{D}_{\geq 0}$, we have
$$tx \in \sigma \text{ and } x + y \in \sigma.$$

(Sometimes in the literature, a cone refers to a weaker notion in which one merely asks the sets to be closed under positive scalar multiplication. Along this document every cone is convex.)

As an example of how to work with this ring, we will show that for subsets of the ordered ring $\mathbb{D}$, the notion of convexity agrees with its counterpart from order theory.

Proposition 1.1.4. *A set $C \subseteq \mathbb{D}$ is convex iff it has the following property:*

$$\text{For each } x, y, z \in \mathbb{D} \text{ such that } x \leq y \leq z \text{ and } x, z \in C, \text{ we have } y \in C. \qquad (*)$$

Proof. If $x \leq y$ then $x \leq tx + (1-t)y \leq y$ for every $0 \leq t \leq 1$. Hence, if C satisfies property $(*)$, then, for every $x, y \in C$ we have $tx + (1-t)y \in C$. Therefore C is convex.

On the other hand, suppose that $x \leq y \leq z$ and $x, z \in C$. If $z - x$ is invertible we can consider the expression

$$y = \frac{y-x}{z-x} z + \left(1 - \frac{y-x}{z-x}\right) x.$$

More generally, we have $0 \leq y - x \leq z - x$. Hence $\operatorname{ord}(z-x) \leq \operatorname{ord}(y-x)$, so we can take elements $a, b \in \mathbb{D}$ with b invertible such that

$$z - x = b\varepsilon^{\operatorname{ord}(z-x)} \quad \text{and} \quad y - x = a\varepsilon^{\operatorname{ord}(z-x)}.$$

If we define $t = a/b$ we have $0 \leq t \leq 1$ and $tb = a$, hence

$$tz + (1-t)x = t(z-x) + x = tb\varepsilon^{\operatorname{ord}(z-k)} + x = a\varepsilon^{\operatorname{ord}(z-k)} + x = (y-x) + x = y.$$

Therefore, $y \in C$. □

Some elementary examples of convex sets and cones in any dimension are given by the half-spaces which we introduce as follows.

Definition 1.1.5. A *half-space* is a subset of $N_{\mathbb{D}}$ of the form

$$H \coloneqq \{x \in N_{\mathbb{D}} \mid \langle y\,, x\rangle \geq a\}$$

for some $y \in M_{\mathbb{D}}$ and $a \in \mathbb{D}$. To simplify notations, we frequently write this as $H = \{y \geq a\}$. For a given subring $R \subseteq \mathbb{D}$, if we can take $y \in M_R$ we say that H is *R-rational.* If moreover we can take $a \in R$ we say that H is *strongly R-rational.* If $a = 0$, then H is a half-space *going through the origin.*

Of special interest for us are the convex sets and cones which are defined in terms of finitely many data. One approach to this is to represent them from *outside* as an intersection of half-spaces. This leads to the following definition.

Definition 1.1.6.

1. A *polyhedron* is a non-empty finite intersection of half-spaces. We say that a polyhedron is *R-rational* (resp. *strongly R-rational*) for a subring $R \subseteq \mathbb{D}$ if we can take each half-space in the intersection to be R-rational (resp. strongly R-rational).

2. A *polyhedral cone* is a finite intersection of half-spaces going through the origin. A polyhedral cone is *R-rational* for a subring $R \subseteq \mathbb{D}$ if we can take each half-space in the definition to be R-rational itself.

In order to manage the data defining a polyhedron we consider the following.

Definition 1.1.7. Given a polyhedron $P \subseteq N_{\mathbb{D}}$, a *representation* of P is an equality of the form

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}. \tag{1.4}$$

for some $y_1, \ldots, y_r \in M_{\mathbb{D}}$ and $a_1, \ldots, a_r \in \mathbb{D}$. This representation is *non-redundant* if it is not possible to obtain a different representation by removing an inequality of the form $\{y_i \geq a_i\}$ from the intersection.

If we allow ourselves to use affine functions instead of linear functions, Equation (1.4) can be written as

$$P = \{f_1 \geq 0, \ldots, f_r \geq 0\} \quad \text{for } f_i = \langle y_i\,, \cdot\rangle - a_i.$$

Proposition 1.1.8. *Given a non-redundant representation as in (1.4), then for any $1 \leq i \leq r$, the function y_i attains its minimum on P and this minimum is a_i.*

Proof. Of course a_i is a lower bound for the values of y_i over P. We will show that this lower bound is attained. For this consider the set

$$\bigcap_{\substack{1\leq j\leq r\\ j\neq i}} \{x \in N_{\mathbb{D}} \mid \langle y_j\,, x\rangle \geq a_j\}\,.$$

This is a convex set and hence its image under $\langle y_i\,, \cdot\rangle$ is a convex set as well that we denote by $C \subseteq \mathbb{D}$. As P is non-empty we have $C \cap [a_i, \infty) \neq \varnothing$, and as the representation is non-redundant we have $C \cap (-\infty, a_i) \neq \varnothing$. So, by Proposition 1.1.4 we have $a_i \in C$. This shows that $\min_P \langle y_i \, , \cdot \rangle = a_i$. □

One can alternatively construct cones and convex sets from *inside* by means of generators as follows.

Definition 1.1.9. For a non-empty subset $X \subseteq N_{\mathbb{D}}$,

1. The *convex hull* of X, denoted by $\mathrm{conv}_{\mathbb{D}}(X)$, is the smallest convex set containing X. Alternatively, this set equals

$$\left\{ \sum_{i=1}^{r} t_i x_i \in N_{\mathbb{D}} \,\middle|\, r \geq 1, x_1, \ldots, x_r \in X \text{ and } t_1 \ldots, t_r \in \mathbb{D}_{\geq 0} \text{ s.t } \sum_{i=1}^{r} t_i = 1 \right\}.$$

   A convex set P is said to be a *polytope* if there is a finite set $X \subseteq N_{\mathbb{D}}$ such that $\mathrm{conv}_{\mathbb{D}}(X) = P$.

2. The *cone generated by* X, also known as the *cone hull of* X, denoted by $\mathrm{cone}_{\mathbb{D}}(X)$, is the smallest cone containing X. Alternatively,

$$\mathrm{cone}_{\mathbb{D}}(X) = \left\{ \sum_{i=1}^{r} t_i x_i \in N_{\mathbb{D}} \,\middle|\, r \geq 1, x_1, \ldots, x_r \in X \text{ and } t_1 \ldots, t_r \in \mathbb{D}_{\geq 0} \right\}.$$

A cone σ is said to be *finitely generated* if there is a finite set $X \subseteq N_{\mathbb{D}}$ such that $\mathrm{cone}_{\mathbb{D}}(X) = \sigma$.

Definition 1.1.10 (Face)**.** Let P be a polyhedron in $N_{\mathbb{D}}$ and consider an element $y \in M_{\mathbb{D}}$ such that

$$\min_{x \in P} \langle y \, , x \rangle$$

exists. Then, the *face* of P determined by y is the subset consisting of all elements in P for which y attains its minimum, that is,

$$\mathrm{face}_y P := \left\{ x \in P \,\middle|\, \langle y \, , x \rangle = \min_{x' \in P} \langle y \, , x' \rangle \right\}.$$

A face of P is a set of the form $\mathrm{face}_y P$ for some $y \in M_{\mathbb{D}}$.[1] We write $F \preceq P$ if F is a face of P.

[1]Notice that with our definition, the empty-set is not considered to be a face. This differ with the definitions of some authors.

Remark 1.1.11. sec:structure-faces

1. A linear function can be bounded below and still does not achieve its minimum. For example, consider
$$P = \{x \in \mathbb{D} \mid \varepsilon x \geq 0\}.$$
and the linear function $x \mapsto x$. This function is bounded bellow in P but does not attain its minimum. This phenomenon has to be kept in mind as minimizing functions plays an important role along the theory.

2. If σ is a polyhedral cone, then $y \in M_{\mathbb{D}}$ attains its minimum over σ iff y is non-negative over σ, and in this case, the minimum has to be zero.

3. A face of P is given by adding one equality, hence two inequalities, to the expression defining P. Therefore, it is a polyhedron itself.

4. If P has a representation of the form
$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\},$$
then it is not true in general that all faces are obtained by adding equalities in this expression of the form $y_i = a_i$. To obtain some faces we may have to add equalities of the form $\varepsilon^\alpha y_i = \varepsilon^\alpha a_i$. In other words, we not only need to consider the locus of points $x \in P$ in which $\langle y_i \, , x \rangle$ is a minimum, but also the locus of points $x \in P$ in which the first $\alpha - k$ coordinates of $\langle y_i \, , x \rangle$ coincide with the minimum. This is a core reason why the combinatorics of faces in this theory is more subtle and capture more information about the minimization of functions. See Proposition 1.6.1 for a precise statement and a proof of this.

In order to use results about polyhedral cones over polytopes, one can go from the perfect pairing $M_{\mathbb{D}} \times N_{\mathbb{D}} \to \mathbb{D}$ to the *extended perfect pairing* defined as follows.

Definition 1.1.12. The *extended perfect pairing* is given by

$$\begin{aligned} (M_{\mathbb{D}} \times \mathbb{D}) \times (N_{\mathbb{D}} \times \mathbb{D}) &\longrightarrow \mathbb{D} \\ ((y,a),(x,b)) &\longmapsto \langle (y,a) \, , (x,b) \rangle := \langle y \, , x \rangle + ab. \end{aligned}$$

In this context, a *lower face* (resp. upper face) of a polyhedron $P \subseteq N_{\mathbb{D}} \times \mathbb{D}$, is a face of the form $\mathrm{face}_{(y,1)(P)}$ (resp. $\mathrm{face}_{(y,-1)(P)}$) for some $y \in M_{\mathbb{D}}$.

Definition 1.1.13 (Face Poset)**.** The *face poset* of P is the partially ordered set

$$\mathfrak{F}(P) := \{F \subseteq P \mid F \text{ is a face of } P\} \cup \{\varnothing\}$$

where the order is given by the inclusion of sets. Moreover, we denote by $\mathfrak{F}(P)^*$ the *reduced face poset* of P given by $\mathfrak{F}(P) \setminus \{\varnothing\}$.

Remark 1.1.14.

1. If we consider $F, G \in \mathfrak{F}(P)^*$ such that $G \subseteq F$, then G is a face of F. Indeed, if $G = \mathrm{face}_y P$ for some $y \in M_{\mathbb{D}}$ then, $G = \mathrm{face}_y F$ for the same y. This shows that we can replace $\subseteq$ by $\preceq$ as the order relation in the definition of $\mathfrak{F}(P)^*$.

2. In Corollary 1.6.3 we will see that if F is a face of P and G is a face of F, then G is a face of P. Therefore, $\mathfrak{F}(F) \subseteq \mathfrak{F}(P)$ for any face F of P.

3. Also in Corollary 1.6.3 we will show that a non-empty intersection of faces of P is a face of P. This shows that $\mathfrak{F}(P)$ is an order lattice. That is, every pair $\{F, G\} \subseteq \mathfrak{F}(P)$ has an infimum given by $F \wedge G := F \cap G$ and a supremum given by

$$F \vee G := \bigcap_{\substack{H \in \mathfrak{F}(P) \\ H \supseteq F \cup G}} H.$$

We will work with more general families of polyhedra besides the set of faces of a given polyhedron. The properties of these families are captured in the following definition.

Definition 1.1.15. A *polyhedral complex* in $N_{\mathbb{D}}$ is a collection of polyhedra Σ in $N_{\mathbb{D}}$ with the following two properties:

1. Given $F, G \in \Sigma$, the intersection $F \cap G$ is either empty or a face of both F and G.

2. If F is a face of G, and $G \in \Sigma$, then $F \in \Sigma$.

The elements of Σ are called the *cells* or *faces* of Σ. Given a polyhedral complex Σ in $N_{\mathbb{D}}$, its *support* is the set

$$|\Sigma| := \bigcup_{F \in \Sigma} F \subseteq N_{\mathbb{D}}.$$

If $P = |\Sigma|$ is itself a polyhedron, we say that Σ is a *subdivision* of P. More generally, if Σ_1 and Σ_2 are polyhedral complexes such that, $|\Sigma_1| = |\Sigma_2|$ and for every $F \in \Sigma_2$ there is a $G \in \Sigma_1$ such that $F \subseteq G$. Then, Σ_2 is said to be a *refinement* of Σ_1 and we write $\Sigma_1 \preceq \Sigma_2$. If every face of Σ is a polyhedral cone, we say that Σ is a *fan* in $N_{\mathbb{D}}$.

Remark 1.1.16. Some basic results concerning the definitions will come naturally after developing the theory.

1. In Corollary 1.7.9 we will prove that a polyhedron that is a cone in the sense of Definition 1.1.3 is a polyhedral cone.
2. In Proposition 1.2.4 we prove that polytopes are polyhedra and finitely generated cones are polyhedral cones.
3. Conversely, in Proposition 1.5.2 we show that polyhedral cones are finitely generated cones. Hence, the concept of finitely generated cones and polyhedral cones coincide.
4. In Proposition 1.9.9 we obtain a criterion to determine which polyhedra are polytopes: A polyhedron P is a polytope iff every linear function achieve its minimum in P.

1.2 The Fourier-Motzkin Elimination

In the following, we will provide a generalization of the Fourier-Motzkin elimination (Theorem 1.2.3) which allows us to reduce the number of variables in a system of linear inequalities. As an immediate consequence, we get that the projection of a polyhedron is a polyhedron. This result has many applications, the most interesting for us being the fact that polytopes are polyhedra and finitely generated cones are polyhedral cones. A more subtle application is Farkas' Lemma, which will be discussed in the next section.

Let us start with a result about the intersection of convex sets in linear orders which, although very simple, we could not find a reference for it in the literature. Given a linear order L, a subset $C \subseteq L$ is called *order-convex* if for every $x, z \in C$ and every $y \in L$ such that $x \leq y \leq z$ we have $y \in C$. In the style of Helly's theorem, we have the next lemma.

Lemma 1.2.1. *Consider a linear order L and a finite family of non-empty order-convex sets $\{C_i\}_{i \in I}$ in L. If we have $C_i \cap C_j \neq \varnothing$ for every $i, j \in I$, then $\bigcap_{i \in I} C_i \neq \varnothing$.*

Proof. For each unordered pair $\{i,j\} \subseteq I$ take an element $x_{ij} \in C_i \cap C_j$ and consider $a_i = \min_{j\in I} x_{ij}$ and $b_i = \max_{i\in I} x_{ij}$. In this way, we have $[a_i, b_i] \subseteq C_i$ for each $i \in I$ and $a_i \leq x_{ij} \leq b_j$ for each $i, j \in I$. Therefore $\max_{i\in I} a_i \leq \min_{i\in I} b_i$ from where

$$\bigcap_{i\in I} C_i \supseteq \bigcap_{i\in I} [a_i, b_i] = [\max_{i\in I} a_i, \min_{i\in I} b_i] \neq \varnothing$$

□

In Proposition 1.1.4 we proved that the order-convex subsets of $\mathbb{D}$ coincide with its convex subsets. Hence, the convex subsets of $\mathbb{D}$ satisfy the Helly property above. This tells us that, to understand if an arbitrary intersection of convex sets in $\mathbb{D}$ is non-empty, we can restrict us to study that each convex set is independently non-empty and each intersection of a pair of convex sets is non-empty. The convex sets of $\mathbb{D}$ in which we will be interested are the solutions to linear inequalities like

$$c + dx \geq 0 \quad \text{or} \quad c + dx > 0.$$

After multiplying by an invertible element, we can suppose that these inequalities are of one of the forms

$$-a + \varepsilon^\alpha x \geq 0, \quad -a + \varepsilon^\alpha x > 0, \quad b - \varepsilon^\beta x \geq 0, \quad b - \varepsilon^\beta x > 0.$$

The next lemma gives us conditions in terms of the coefficients a and b for which, a single inequality has a solution or a pair of inequalities have a common solution.

Lemma 1.2.2 (Fourier-Motzkin reduction)**.** *Let $a, b \in \mathbb{D}$ and consider the inequalities*

$$-a + \varepsilon^\alpha x \geq 0 \qquad \text{(i)}$$

$$-a + \varepsilon^\alpha x > 0 \qquad \text{(i*)}$$

$$b - \varepsilon^\beta x \geq 0 \qquad \text{(ii)}$$

$$b - \varepsilon^\beta x > 0 \qquad \text{(ii*)}$$

Then,

1. *The inequality* (i) *has a solution iff the inequality* (i*) *has a solution iff*

$$-\varepsilon^{k-\alpha} a \geq 0.$$

Analogously, the inequality (ii) *has a solution iff the inequality* (ii*) *has a solution iff*

$$\varepsilon^{k-\beta} b \geq 0.$$

2. *The inequalities* (i) *and* (ii) *have a common solution iff each of them has a solution in their own and*

$$b - \varepsilon^{\beta-\alpha} a \geq 0 \text{ if } \beta \geq \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a \geq 0 \text{ if } \alpha \geq \beta.$$

Similarly, (i*) *and* (ii) *have a common solution iff each of them has a solution in their own and*

$$b - \varepsilon^{\beta-\alpha} a \geq 0 \text{ if } \beta > \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a > 0 \text{ if } \alpha \geq \beta.$$

The inequalities (i) *and* (ii*) *have a common solution iff each of them has a solution in their own and*

$$b - \varepsilon^{\beta-\alpha} a > 0 \text{ if } \beta \geq \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a \geq 0 \text{ if } \alpha > \beta.$$

Finally, (i*) *and* (ii*) *have a common solution iff each of them has a solution in their own and*

$$b - \varepsilon^{\beta-\alpha} a > 0 \text{ if } \beta \geq \alpha \text{ or,}$$
$$\varepsilon^{\alpha-\beta} b - a > 0 \text{ if } \alpha \geq \beta.$$

Proof.

1. If there is an x satisfying (i) or (i*) then by multiplying the inequality on both sides by $\varepsilon^{k-\alpha}$ we get $-\varepsilon^{k-\alpha} a \geq 0$. Conversely, If $-\varepsilon^{k-\alpha} a \geq 0$ then either $-\varepsilon^{k-\alpha} a > 0$ or $-\varepsilon^{k-\alpha} a = 0$. In the first case $-a + \varepsilon^{\alpha} x > 0$ for any x and we are done. In the second case a is of the form $\varepsilon^{\alpha} a'$ and we have $-\varepsilon^{\alpha} a' + \varepsilon^{\alpha} x \geq 0$ (resp. $-\varepsilon^{\alpha} a' + \varepsilon^{\alpha} x > 0$) iff $-a' + x \geq 0$ (resp. $-a' + x > 0$) which always have a solution. The statement about (ii) and (ii*) follows from the previous one by replacing x with $-x$ and a with $-b$.

2. Suppose that both inequalities (i) and (ii) have a solution and moreover $\beta \geq \alpha$. By multiplying (i) by $\varepsilon^{\beta-\alpha}$ and adding (ii) we get $b - \varepsilon^{\beta-\alpha}a \geq 0$. Conversely, suppose that both (i) and (ii) have a solution independently and $b - \varepsilon^{\beta-\alpha}a \geq 0$. Then, by the first part we have $-\varepsilon^{k-\alpha}a \geq 0$ and $\varepsilon^{k-b} \geq 0$. Moreover, if $-\varepsilon^{k-\alpha}a > 0$ then (i) is satisfied for every x and any solution for (ii) works for both inequalities. Hence, we can assume $-\varepsilon^{k-\alpha}a = 0$, and for a similar reason, we can assume $\varepsilon^{k-\alpha}b = 0$. Then, $a = \varepsilon^{\alpha}a'$ and $b = \varepsilon^{\beta}b'$ for some $a', b' \in \mathbb{D}$, which we can replace in the inequality $b - \varepsilon^{\beta-\alpha}a \geq 0$ to obtain $\varepsilon^{\beta}b' - \varepsilon^{\beta}a' \geq 0$. Then, there is an x such that

$$b = \varepsilon^{\beta}b' \geq \varepsilon^{\beta}x \geq \varepsilon^{\beta}a' = \varepsilon^{\beta-\alpha}a. \tag{1.5}$$

If $\beta = \alpha$ we are done. If $\beta > \alpha$ then, notice that for every $x' \in \mathbb{D}$ we have $\varepsilon^{\beta}x = \varepsilon^{\beta}(x + \varepsilon^{k-\beta}x')$. Hence, we can modify the last β coordinates of x and (1.5) remains true. By making them big enough we have $\varepsilon^{\alpha}x > a$, that is, x satisfy (ii*). In particular, x satisfy both (i) and (ii) simultaneously and we are done. The case in which $\beta \geq \alpha$ is done similarly.

Notice that in the argument above we proved that if $\beta > \alpha$ then (i*) and (ii) are satisfied together iff they are satisfied individually and $b - \varepsilon^{\beta-\alpha}a \geq 0$. This is the next part of the proposition. For the other part, if $\alpha \geq \beta$ then, if both (i*) and (ii) are satisfied we can multiply the first equation by $\varepsilon^{\alpha-\beta}$ and add it to the second one to obtain $\varepsilon^{\alpha-\beta}b - a > 0$. Conversely, working in the same way as to obtain (1.5) we get $a', b', x \in \mathbb{D}$ such that

$$b = \varepsilon^{\beta}b' > \varepsilon^{\beta}x > \varepsilon^{\beta}a' = \varepsilon^{\beta-\alpha}a.$$

Again, if $\beta = \alpha$ we are done and if $\beta > \alpha$ we can modify the last β coordinates of x as to get $\varepsilon^{\alpha}x > a$, and such an x satisfy both (i*) and (ii*) (in particular (i*) and (ii)). The remaining cases can be done in the same way.

□

For the following result, it will be necessary to use coordinates, hence we will work with spaces of the form $\mathbb{D}^n$ for $n \geq 0$, instead of $N_{\mathbb{D}}$ for a general lattice N. A *linear inequality in* $\mathbb{D}^n$ is an inequality of one of the forms

$$a_1x_1 + \cdots + a_nx_n \geq a \qquad a_1x_1 + \cdots + a_nx_n > a \tag{1.6}$$

with $a, a_1, \dots, a_n \in \mathbb{D}$. If it is of the first form we say it is *closed*, if it is of the second form, we say it is *open*, and if $a = 0$, we say it is *homogeneous*. A finite family of linear inequalities is called a *system of linear inequalities*. Such a system is said to be *closed*, *open* or *homogeneous* if each inequality is of this form.

Theorem 1.2.3 (Fourier-Motzkin over $\mathbb{D}$). *Given an integer $n \geq 1$ and a system of linear inequalities $\mathcal{L}$ in $\mathbb{D}^{n+1}$, there is another system of linear inequalities $\mathcal{L}'$ in $\mathbb{D}^n$ such that an element $(x_1, \dots, x_{n+1})$ is a solution of $\mathcal{L}$ for some $x_{n+1} \in \mathbb{D}$, if and only if, $(x_1, \dots, x_n)$ is a solution of $\mathcal{L}'$. Moreover, if $\mathcal{L}$ is closed or homogeneous then $\mathcal{L}'$ can also be taken to be closed or homogeneous, respectively. If $\mathcal{L}$ is open, it may not be possible to take $\mathcal{L}'$ open.*

Proof. Let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n) \in \mathbb{D}^n$. There is an element $x_{n+1} \in \mathbb{D}$ such that $(\tilde{x}, x_{n+1})$ is a solution to $\mathcal{L}$ iff the system of inequalities $\mathcal{L}_{\tilde{x}}$ has a non-empty set of solutions, where $\mathcal{L}_{\tilde{x}}$ consists of the inequalities

$$a_1\tilde{x}_1 + \dots + a_n\tilde{x}_n + a_{n+1}x \geq a \quad \text{or} \quad a_1\tilde{x}_1 + \dots + a_n\tilde{x}_n + a_{n+1}x > a \tag{1.7}$$

where $a_1x_1 + \dots + a_nx_n + a_{n+1}x_{n+1} \geq a$ and $a_1x_1 + \dots + a_nx_n + a_{n+1}x_{n+1} > a$ go over all the inequalities of $\mathcal{L}$. After multiplying by a positive invertible element of $\mathbb{D}$ we can suppose that each inequality in $\mathcal{L}_{\tilde{x}}$ is of one of the forms

$$-a(\tilde{x}) + \varepsilon^{\alpha}x \geq 0 \tag{i}$$
$$-a(\tilde{x}) + \varepsilon^{\alpha}x > 0 \tag{i*}$$
$$b(\tilde{x}) - \varepsilon^{\beta}x \geq 0 \tag{ii}$$
$$b(\tilde{x}) - \varepsilon^{\beta}x > 0 \tag{ii*}$$
$$c(\tilde{x}) \geq 0 \tag{iii}$$
$$c(\tilde{x}) > 0 \tag{iii*}$$

Notice that the set of solutions of each of these inequalities is a convex set in $\mathbb{D}$. Hence, by Lemma 1.2.1, $\mathcal{L}_{\tilde{x}}$ has a non-empty set of solutions iff each individual inequality on it has a solution and each pair of inequalities on it has a simultaneous solution. By Lemma 1.2.2 we know that each of these conditions can be translated into a linear inequality in the variable $\tilde{x}$, giving rise to a system of linear inequalities $\mathcal{L}'$ in the variable $\tilde{x}$. Hence, there is an x_{n+1} such that $(\tilde{x}, x_{n+1})$ is a solution to $\mathcal{L}$ iff $\mathcal{L}_{\tilde{x}}$ has at least one solution iff $\tilde{x}$ is a solution to $\mathcal{L}'$, as we wanted. Moreover, the explicit linear equations obtained in Lemma

1.2.2 give us that if each inequality in $\mathcal{L}$ is closed or homogeneous, then each inequality in $\mathcal{L}'$ is also closed or homogeneous as well. Finally, in the case in which $\mathcal{L}$ consists in the single equation $x_1 + \varepsilon x_2 \geq 0$, then $\mathcal{L}'$ should have as solution set $\varepsilon x_1 \geq 0$, so $\mathcal{L}'$ cannot consist in a finite set of open inequalities in $\mathbb{D}$. □

As an immediate consequence of this, we get that the projection of a polyhedron in $\mathbb{D}^k$ to some of its coordinates is still a polyhedron, and if it is a polyhedral cone then the projection is also a polyhedral cone. Less immediate consequences are summarized in the following proposition.

Proposition 1.2.4.

1. *The image of a polyhedron P under a linear map is a polyhedron. If P is a polyhedral cone, then the image is also a polyhedral cone.*

2. *The sum of two polyhedra is a polyhedron. The sum of two polyhedral cones is a polyhedral cone.*

3. *Every polytope is a polyhedron.*

4. *Every finitely generated cone is a polyhedral cone.*

Proof.

1. Without loss of generality we can suppose that P is a polyhedron inside $\mathbb{D}^n$ and the linear map f goes from $\mathbb{D}^n$ to $\mathbb{D}^k$. Now consider

$$\Gamma(f|_P) = \left\{(x,y) \in \mathbb{D}^n \times \mathbb{D}^k \mid x \in P, y = f(x)\right\}.$$

This is a polyhedron in $\mathbb{D}^n \times \mathbb{D}^k$ and $f(P)$ is the projection to the second component. Hence, by Fourier-Motzkin this is a polyhedron. In the same way, if P is a polyhedral cone, then so is $\Gamma(f|_P)$ and then its projection $f(P)$.

2. Given $P, Q \subseteq \mathbb{D}^n$, consider

$$R = \{(x,y,z) \in \mathbb{D}^n \times \mathbb{D}^n \times \mathbb{D}^n \mid x \in P, y \in Q, z = x + y\}.$$

By (1), this is a polyhedron and $P + Q$ is the projection to the last component. Hence, it is a polyhedron as well. As R is a polyhedral cone if each of P and Q are, then so is $P + Q$ in this case.

3. The polytope P is equal to $\operatorname{conv}_{\mathbb{D}}(a_1, \ldots, a_r)$ for some $a_1, \ldots, a_r \in \mathbb{D}^n$. Then, $P = f(Q)$ where Q is the polyhedron

$$Q = \{(x_1, \ldots, x_r) \in \mathbb{D}^r \mid x_1, \ldots, x_r \geq 0, x_1 + \cdots + x_r = 1\}$$

and f is the linear map defined by

$$\begin{aligned} f : \mathbb{D}^r &\longrightarrow \mathbb{D}^n \\ x_i &\longmapsto a_i \end{aligned}$$

By part (2) we get that P is a polyhedron as well.

4. As above, if $\sigma = \operatorname{cone}_{\mathbb{D}}(a_1, \ldots, a_r)$ for some $a_1, \ldots, a_r \in \mathbb{D}^n$. Then, $\sigma = f(\tau)$ for the same function f and

$$\tau = \{(x_1, \ldots, x_r) \in \mathbb{D}^r \mid x_1, \ldots, x_r \geq 0\}.$$

□

1.3 Farkas' Lemma over the Generalized Dual Numbers

In this section we will prove the analogous statement to Farkas' Lemma which works over the ring $\mathbb{D}$. This result is at the heart of the theory from a technical standpoint. Its proof is based on the Fourier-Motzkin elimination developed the previous section.

Theorem 1.3.1 (Farkas' Lemma over $\mathbb{D}$)**.** *Let $f_1, \ldots, f_r : N_{\mathbb{D}} \to \mathbb{D}$ be a family of affine functions such that*

$$P = \{f_1 \geq 0, \ldots, f_r \geq 0\}$$

is non-empty. Then, any affine function $f : N_{\mathbb{D}} \to \mathbb{D}$ that achieves its minimum in P can be written in the form

$$f - \min_P f = \lambda_1 f_1 + \cdots + \lambda_r f_r$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$.

Remark 1.3.2. One can state Farkas' lemma over $\mathbb{R}$ as follows. If $f, f_1, \ldots, f_r : N_{\mathbb{R}} \to \mathbb{R}$ is a family of affine functions such that

$$\{f \geq 0\} \supseteq \{f_1 \geq 0, \ldots, f_r \geq 0\}.$$

Then, there are $c, \lambda_1, \ldots, \lambda_r \in \mathbb{R}_{\geq 0}$ such that

$$f = \lambda_1 f_1 + \cdots + \lambda_r f_r + c.$$

The exact translation of this statement to $\mathbb{D}$ is false. For example, we can take $N = \mathbb{Z}$, $f(x) = x$, $f_1(x) = \langle \varepsilon\,, x\rangle + 1$ and $f_2(x) = \langle -\varepsilon\,, x\rangle - 1$. Then,

$$\{f \geq 0\} \supseteq \{f_1 \geq 0, f_2 \geq 0\} = \varnothing,$$

but there are no $c, \lambda_1, \lambda_2 \in \mathbb{D}_{\geq 0}$ such that

$$x = \lambda_1(\langle \varepsilon\,, x\rangle + 1) + \lambda_2(\langle -\varepsilon\,, x\rangle + 1) + c,\ \forall x \in \mathbb{D}.$$

In this way, we see that the hypothesis that P is not empty is unavoidable. Similarly, the hypothesis that f achieves its minimum over P is unavoidable as, we can take $f(x) = \langle \varepsilon\,, x\rangle + 1$ and $f_1(x) = \langle \varepsilon^2\,, x\rangle$. Then

$$\{\langle \varepsilon\,, x\rangle + 1 \geq 0\} = N_{\mathbb{R}} \supseteq \{\langle \varepsilon^2\,, x\rangle \geq 0\}.$$

But, there are no $c, \lambda_1 \in \mathbb{D}_{>0}$ such that

$$\langle \varepsilon\,, x\rangle = \lambda_1 \langle \varepsilon^2\,, x\rangle + c,\ \forall x \in \mathbb{D}.$$

We will deduce the theorem above from the following more general technical lemma.

Lemma 1.3.3. *Consider affine functions $f_1, \ldots, f_s, f_{s+1}, ..., f_t$ in $N_{\mathbb{D}}$ such that*

$$\{f_1 \geq 0, \ldots, f_s \geq 0, f_{s+1} \geq 0, \ldots, f_t \geq 0\} \neq \varnothing \tag{A}$$

$$\{f_1 \geq 0, \ldots, f_s \geq 0, f_{s+1} > 0, \ldots, f_t > 0\} = \varnothing. \tag{B}$$

Then, there are $\lambda_1, \ldots, \lambda_t \in \mathbb{D}_{\geq 0}$ such that

$$\lambda_1 f_1(x) + \cdots + \lambda_t f_t(x) = 0, \quad \forall\, x \in N_{\mathbb{D}}$$

and at least one element between $\lambda_{s+1}, \ldots, \lambda_t$ is invertible.

Proof of Theorem 1.3.1. We have

$$\{f_1 \geq 0, \dots, f_r \geq 0, -f + \min_P f \geq 0\} \neq \varnothing \text{ and,}$$
$$\{f_1 \geq 0, \dots, f_r \geq 0, -f + \min_P f > 0\} = \varnothing.$$

So, by Lemma 1.3.3, there are $\lambda_1, \dots, \lambda_r, \lambda \in \mathbb{D}_{\geq 0}$ with λ invertible such that

$$\lambda_1 f_1 + \cdots + \lambda_r f_r + \lambda \left(-f + \min_P f\right) = 0.$$

That is,

$$f - \min_P f = \frac{\lambda_1}{\lambda} f_1 + \cdots + \frac{\lambda_r}{\lambda} f_r$$

as we wanted. □

Proof of Lemma 1.3.3. After composing with an isomorphism we can suppose $N = \mathbb{Z}^n$. The proof is by induction on n.

Base case $n = 1$:

After multiplying by a positive invertible element in $\mathbb{D}_k$ if necessary, we can suppose that each f_i is of one of the forms

$$\varepsilon^{\alpha_i} x - a_i, \quad b_i - \varepsilon^{\beta_i} x, \qquad c_i \quad \text{for some } 0 \leq \alpha_i,\ \beta_i \leq k-1.$$

Then, by Lemma 1.2.2 the system of inequalities in (B) has no solutions iff at least one of the following conditions fail

1. For every i we have

   (1.1) $-\varepsilon^{k-\alpha_i} a \geq 0$ if $f_i = \varepsilon^{\alpha_i} x - a_i$

   (1.1) $\varepsilon^{k-\beta_i} b \geq 0$ if $f_i = b_i - \varepsilon^{\beta_i} x$.

   (1.2) Either

   $$c_i \geq 0 \text{ if } i \leq s \text{ or,}$$
   $$c_i > 0 \text{ if } i > s$$

   if $f_i = c_i$.

2. Whenever $f_i = \varepsilon^{\alpha_i} x - a_i$ and $f_j = b_j - \varepsilon^{\beta_j} x$ we have

(2.1) If $i, j \leq s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i \geq 0 \text{ if } \beta_j \geq \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i \geq 0 \text{ if } \alpha_i \geq \beta_j$$

(2.2) If $i > s$ and $j \leq s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i \geq 0 \text{ if } \beta_j > \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i > 0 \text{ if } \alpha_i \geq \beta_j$$

(2.3) If $i \leq s$ and $j > s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i > 0 \text{ if } \beta_j \geq \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i \geq 0 \text{ if } \alpha_i > \beta_j$$

(2.4) If $i, j > s$:

$$b_j - \varepsilon^{\beta_j - \alpha_i} a_i > 0 \text{ if } \beta_j \geq \alpha_i \text{ or,}$$
$$\varepsilon^{\alpha_i - \beta_j} b_j - a_i > 0 \text{ if } \alpha_i \geq \beta_j$$

As the system (A) does have a solution, the only conditions that can fail are the ones with strict inequalities. Moreover, this can fail only by getting an equality.

- If condition (1.2) fail then for some $i > s$ we have $f_i = c_i = 0$ so in this case we can take $\lambda_i = 1$ and $\lambda_j = 0$ for all $j \neq i$.
- If condition (2.2), (2.3), or (2.4) fail then we have either

  $$f_i + \varepsilon^{\alpha_i - \beta_j} f_j = \varepsilon^{\alpha_i - \beta_j} b_j - a_i = 0$$

  with $i > s$ or

  $$\varepsilon^{\beta_j - \alpha_i} f_i + f_j = b_j - \varepsilon^{\beta_j - \alpha_i} a_i = 0$$

  with $j > s$. In the former case we can take $\lambda_i = 1$, $\lambda_j = \varepsilon^{\alpha_i - \beta_j}$ and everything else 0. In the later case we can take $\lambda_j = 1$, $\lambda_i = \varepsilon^{\beta_j - \alpha_i}$ and everything else 0.

Induction step:

Assuming the result for $\mathbb{D}^n$ we will prove it for $\mathbb{D}^{n+1}$. For this let $x' = (x_1, \dots, x_n)$. As in the base case, after multiplying by a positive invertible scalar we can assume that each f_i is of one of the forms

$$x_{n+1} - a_i(x'), \quad b_i(x') - x_{n+1}, \quad c_i(x').$$

Now, for a given $x' \in \mathbb{D}^n$ fixed, there is an $x_{n+1} \in \mathbb{D}$ such that (x', x_{n+1}) belongs to the set (A) iff the same conditions that we use in the base case are satisfied. As the set (A) is empty, this cannot happen for any x'. Hence, the system of inequalities on the variable x' which is formed by all these conditions has an empty set of solutions. On the other hand, the same system but in which all the inequalities are closed does have a solution, because the set (B) is not empty. This allows us to use the induction hypothesis on this new system of inequalities.

Applying the induction hypothesis we get a positive linear combination of the new affine functions involved which is equal to zero. Now, by doing the following replacements

- $-\varepsilon^{k-\alpha_i} a_i(x') = \varepsilon^{k-\alpha_i} f_i(x)$
- $\varepsilon^{k-\alpha_j} b_j(x') = \varepsilon^{k-\alpha_j} f_j(x)$
- $c_l(x') = f_l(x)$
- $b_j(x') - \varepsilon^{\beta_j-\alpha_i} a_i(x') = f_j(x) + \varepsilon^{\beta_j-\alpha_i} f_i(x)$
- $\varepsilon^{\alpha_i-\beta_j} b_j(x') - a_i(x') = \varepsilon^{\beta_j-\alpha_i} f_j(x) + f_i(x)$,

we turn the linear combination into one involving the original affine functions. By the induction hypothesis we get that at least one of the coefficients of the linear combination in either $c_i(x')$ with $i > s$, $b_j(x') - \varepsilon^{\beta_j-\alpha_i} a_i(x')$ with $j > s$ or $\varepsilon^{\alpha_i-\beta_j} b_j(x') - a_i(x')$ with $i > s$ is invertible. Hence, at least one of the coefficients in f_i for $i > s$ is invertible. This finishes the induction step. □

1.4 The Relative Interior

In this section we introduce the relative interior of a polyhedron. We cannot introduce this concept as a topological interior, as we do not have appropriate topological tools over the ring $\mathbb{D}$. For this reason, we introduce it combinatorially by means of the structure of faces, and we show that, in the case of polyhedral cones, this coincides with an algebraic construction in terms of generators.

Definition 1.4.1. Let P be a polyhedron. The *relative interior* of P is the set

$$\operatorname{int}(P) := P \setminus \bigcup_{\substack{F \preceq P \\ F \neq P}} F$$

where the union goes over all the proper faces F of P.

The next proposition summarize some basic proprieties of this concept

Proposition 1.4.2. *For a given polyhedron $P \subseteq N_{\mathbb{D}}$:*

1. *There is a decomposition*
$$P = \bigsqcup_{F \leq P} \operatorname{int}(F),$$
where the disjoint union goes over all the different faces of P.

2. *For a face F of P and an element $x \in \operatorname{int}(F)$. A face G of P contains x iff $F \subseteq G$.*

3. *We have* $\operatorname{int}(P) \neq \varnothing$.

Proof.

1. Given $x \in P$. Let F be the smallest face of P containing x, this exists because of Corollary 1.6.3(2)[2] and the fact that there are only finitely many faces, a consequence of Proposition 1.6.1 part 1. Given this face, we have
$$x \in F \setminus \bigcup_{\substack{G \preceq F \\ G \neq F}} F$$
therefore $x \in \operatorname{int}(F)$. This shows that $\sigma = \bigcup_{\tau \leq \sigma} \operatorname{int}(\tau)$.

   Now, to see that the union is disjoint, notice that if $x \in \operatorname{int}(F) \cap \operatorname{int}(G)$, then $x \in F \cap G$, which is a face by Corollary 1.6.3. But as $x \in \operatorname{int}(F)$, this is only possible if $F \cap G = F$, that is $F \supseteq G$. Similarly $G \supseteq F$ so $F = G$.

2. If $F \not\subseteq G$, then $x \in F \cap G$ and $F \cap G$ is a face of P contained in F, hence it is a proper face of F. Then, $x \notin F \setminus F \cap G$ but as $\operatorname{int}(F) \subseteq F \setminus F \cap G$ we get a contradiction.

[2]This result is proved in Section 1.6 and it is based on Proposition 1.6.1, parts (1) and (3). It might be worth mentioning that this does not produce any loop in the logic.

3. For each polyhedron P consider

$$\operatorname{length}(P) = \max\left\{s \in \mathbb{N} \mid \exists\, F_1, \ldots, F_s \in \mathfrak{F}(P)^* \text{ s.t } \varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_s = P\right\}.$$

We show that $\operatorname{int}(P) \neq \varnothing$ by induction on $\operatorname{length}(P)$.

If $\operatorname{length}(P) = 1$, then P does not have any proper face, hence $\operatorname{int}(P) = P \neq \varnothing$. Now, suppose that $\operatorname{length}(P) = s + 1$ and the result is true whenever the length is smaller or equal to s. Consider a maximal chain of faces of P of the form

$$\varnothing \subsetneq F_1 \subsetneq \cdots \subsetneq F_s \subsetneq F_{s+1} = P.$$

Then, $\operatorname{length}(F_s) = s$, and so, by the induction hypothesis, we have $\operatorname{int}(F_s) \neq \varnothing$. Take $x \in \operatorname{int}(F_s)$ and $x' \in P \setminus F_s$. Then, we claim that $\frac{1}{2}(x + x') \in \operatorname{int}(P)$.

Indeed, if $\frac{1}{2}(x + x') \notin \operatorname{int}(P)$, then there is a $y \in M_{\mathbb{D}}$ such that

$$\frac{1}{2}(x + x') \in \operatorname{face}_y(P) \subsetneq P.$$

In particular, y achieves its minimum in $\frac{1}{2}(x + x')$. Moreover, as we have

$$\begin{aligned}\langle y\,, \frac{1}{2}(x + x')\rangle &= \frac{1}{2}(\langle y\,, x\rangle + \langle y\,, x'\rangle) \\ &\geq \min\{\langle y\,, x\rangle, \langle y\,, x'\rangle\}\end{aligned}$$

with equality iff $\langle y\,, \frac{1}{2}(x + x')\rangle = \langle y\,, x\rangle = \langle y\,, x'\rangle$. Therefore, we have that y also achieves its minimum in x and x', that is $x, x' \in \operatorname{face}_y(P)$. As $x \in \operatorname{int}(F_s)$, by part (2) of this proposition we have $F_s \subseteq \operatorname{face}_y(P)$, and as $x' \notin F_s$ we also have $F_s \neq \operatorname{face}_y(P)$. Hence, by the maximality of s, we get $\operatorname{face}_y(P) = P$ which is a contradiction. Therefore, $\frac{1}{2}(x + x') \in \operatorname{int}(P)$, so $\operatorname{int}(P) \neq \varnothing$ as we wanted.

□

In the context of finitely generated cones, the relative interior can be computed alternatively in terms of generators.

Proposition 1.4.3. *Given a finitely generated cone* $\sigma = \operatorname{cone}_{\mathbb{D}}(x_1, \ldots, x_r)$*, the relative* *interior can be computed as*

$$\operatorname{int}\sigma = \left\{ \sum_{i=1}^{r} \lambda_i x_i \;\middle|\; \lambda_1, \dots, \lambda_r \in \mathbb{D}_{>0}^{\times} \right\}. \tag{1.8}$$

Proof. We will prove the equality by a double inclusion.

First, take an x in the right-hand side of (1.8). For every face $\tau \subseteq \sigma$ there is $y \in M_{\mathbb{D}}$ attaining its minimum in σ such that $\tau = \operatorname{face}_y \sigma$. As σ is a polyhedral cone, y must be non-negative over σ to achieve its minimum. Now, suppose that $x = \sum_{i=1}^{r} \lambda_i x_i \in \tau$. Then, we must have

$$\langle y \,, x \rangle = \sum_{i=1}^{r} \lambda_i \langle y \,, x_i \rangle = 0.$$

As each term of the sum is non-negative, this happens iff $\lambda_i \langle y \,, x_i \rangle = 0$ for each i, and as all λ_i are invertible, this is the same as $\langle y \,, x_i \rangle = 0$ for each i. Then, we have $x_1, \dots, x_r \in \tau$ so $\sigma = \operatorname{cone}_D(x_1, \dots, x_r) \subseteq \tau$, and hence $\tau = \sigma$. This shows that

$$x \in \sigma \setminus \bigcup_{\substack{\tau \preceq \sigma \\ \tau \neq \sigma}} \tau = \operatorname{int}(\sigma).$$

On the other hand, take $x \in \operatorname{int}(\sigma)$ and fix some $1 \leq i \leq r$. We claim that there is a $\lambda \in \mathbb{D}_{>0}^{\times}$ such that

$$x - \lambda x_i \in \sigma.$$

If this is not the case, as σ is a polyhedral cone by Proposition 1.2.4 part 4, there are $y_1, \dots, y_s \in M_{\mathbb{D}}$ such that

$$\sigma = \{y_1 \geq 0, \dots, y_s \geq 0\}.$$

Then, there must be a y_j such that for every $\lambda \in \mathbb{D}_{>0}^{\times}$ small enough we have

$$\langle y_j \,, x - \lambda x_i \rangle < 0.$$

That is, $0 \leq \langle y_j \,, x \rangle < \lambda \langle y_j \,, x_i \rangle$ for every $\lambda \in \mathbb{D}_{>0}^{\times}$ small enough, which implies that $\langle y_j \,, x \rangle$ is *infinitesimally smaller* than $\langle y_j \,, x_i \rangle$, that is, $\langle y_j \,, x \rangle = \varepsilon \mu \langle y_j \,, x_i \rangle$ for some $\mu \in \mathbb{D}_{\geq 0}$. Then, for some $l \geq 0$ we have $\varepsilon^l \langle y_j \,, x \rangle = 0$ but $\varepsilon^l \langle y_j \,, x_i \rangle \neq 0$. Therefore, $y_j \varepsilon^l \in M_{\mathbb{D}}$ defines a face $\operatorname{face}_{y_j \varepsilon^l} \sigma$ containing x but not x_i which contradicts the fact that x is in the relative interior. This finishes the proof of the claim. Therefore, for each $1 \leq i \leq r$ there is a λ

such that

$$x = \lambda x_i + x'$$

for some $x' \in \sigma$. By writting x' in terms of $x_1, \ldots, x_r$ we get a representation of x in the form

$$x = \sum_{i=1}^{r} \lambda_i x_i$$

with $\lambda_i \in \mathbb{D}^{\times}_{>0}$ and $\lambda_j \geq 0$ for $j \neq i$. By taking an average of all this representations for all i, we get a representation with all $\lambda_i \in \mathbb{D}^{\times}_{>0}$. This finishes the proof. □

1.5 Cone Duality

In this section we will study polyhedral cones and their duals. After introducing the dual we show how to find generators for it from a representation of the original cone, or how to find a representation for it from generators of the original cone. In particular, this implies that finitely generated cones are the same as polyhedral cones. The main result of this section is the duality theorem, which gives an explicit order reversing bijection between the faces of a cone and the faces of its dual.

Definition 1.5.1. Let $\sigma \subseteq N_{\mathbb{D}}$ be a cone. Its *dual cone* is the set of all linear functionals non-negative over it, that is,

$$\sigma^{\vee} := \{y \in M_{\mathbb{D}} \mid \langle y\,, x\rangle \geq 0,\ \forall\, x \in \sigma\}\,.$$

Proposition 1.5.2.

1. *Given $y_1, \ldots, y_r \in M_{\mathbb{D}}$ we have*

$$\left(\bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i\,, x\rangle \geq 0\}\right)^{\vee} = \operatorname{conv}_{\mathbb{D}}(y_1, \ldots, y_r).$$

2. *Given $a_1, \ldots, a_r \in N_{\mathbb{D}}$, we have*

$$\operatorname{conv}_{\mathbb{D}}(a_1, \ldots, a_r)^{\vee} = \bigcap_{i=1}^{r} \{y \in M_{\mathbb{D}} \mid \langle y\,, a_i\rangle \geq 0\}\,.$$

3. *For any polyhedral cone σ, we have $\left(\sigma^{\vee}\right)^{\vee} = \sigma$.*

4. Polyhedral cones are the same as finitely generated cones.

Proof.

1. Let $\sigma = \bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i \, , x\rangle \geq 0\}$. We have that $\langle \lambda_1 y_1 + \cdots + \lambda_r y_r \, , \cdot \rangle$ is positive over σ, hence $\lambda_1 y_1 + \cdots + \lambda_r y_r \in \sigma^\vee$ for every $\lambda_1, \ldots, \lambda_r \geq 0$, then $\mathrm{cone}_{\mathbb{D}}(y_1, \ldots, y_r) \subseteq \sigma^\vee$. On the other hand, given $y \in \sigma^\vee$, we have that $\langle y, \cdot \rangle$ is positive over σ and its minimum is 0 on it. Hence, by Farkas' Lemma, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$\lambda_1 y_1 + \cdots + \lambda_r y_r = y.$$

   Therefore, $y \in \mathrm{conv}_{\mathbb{D}}(y_1, \ldots, y_r)$, so $\sigma^\vee \subseteq \mathrm{conv}_{\mathbb{D}}(y_1, \ldots, y_r)$.

2. We have

$$\begin{aligned} y \in \mathrm{cone}_{\mathbb{D}}(a_1, \ldots, a_r)^\vee &\iff \langle y \, , x \rangle \geq 0 \quad \forall x \in \mathrm{cone}_{\mathbb{D}}(a_1, \ldots, a_r) \\ &\iff \langle y \, , a_i \rangle \geq 0 \quad \forall 1 \leq i \leq r \\ &\iff y \in \bigcap_{i=1}^{r} \{y \in M_{\mathbb{D}} \mid \langle y \, , a_i \rangle \geq 0\}. \end{aligned}$$

3. By part (1) and (2), given a polyhedral cone as $\bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i \, , x \rangle \geq 0\}$ we have

$$\begin{aligned} \left(\left(\bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i \, , x \rangle \geq 0\} \right)^\vee \right)^\vee &= \mathrm{conv}_{\mathbb{D}}(y_1, \ldots, y_r)^\vee \\ &= \bigcap_{i=1}^{r} \{x \in N_{\mathbb{D}} \mid \langle y_i \, , x \rangle \geq 0\} \, . \end{aligned}$$

4. In Proposition 1.2.4, we saw that that finitely generated cones are polyhedral. On the other hand, let us suppose that a cone $\sigma \subseteq N_{\mathbb{D}}$ is polyhedral. By part (1) then $\sigma^\vee$ is finitely generated and hence polyhedral by Proposition 1.2.4. So by part (1) again $(\sigma^\vee)^\vee = \sigma$ is finitely generated.

□

Now, we will prove a duality result for cones, which states that the faces of a cone are in an order reversing correspondence with the faces of its dual cone.

Theorem 1.5.3 (Higher Rank Cone Duality)**.** *Given a polyhedral cone σ and its dual $\sigma^\vee$, there is an order reversing bijection between the reduced face poset of σ and the reduced face poset of its dual $\sigma^\vee$ given by*

$$\begin{aligned} \mathfrak{F}(\sigma)^* &\longrightarrow \mathfrak{F}(\sigma^\vee)^* \\ \tau &\longmapsto \tau^* := \tau^\perp \cap \sigma^\vee, \end{aligned}$$

where

$$\tau^\perp := \{y \in M_{\mathbb{D}} \mid \langle y\,, x\rangle = 0,\ \forall\, x \in \tau\}.$$

Proof. First, notice that, as τ is a polyhedral cone, it is finitely generated. Hence, $\tau = \text{cone}_{\mathbb{D}}(x_1, \ldots, x_r)$ for some $x_1, \ldots, x_r \in \sigma$. Then, we have

$$\tau^* = \{y \in \sigma^\vee \mid \langle y\,, x_i\rangle = 0,\ \text{for } i = 1, \ldots, r\} = \text{face}_{x_1+\cdots+x_r}(\sigma^\vee)$$

Therefore, τ^* is a face of $\sigma^\vee$ and the map is well defined.

Now, let us see that the map is surjective: An arbitrary face of $\sigma^\vee$ is of the form $\text{face}_{x_0}(\sigma^\vee)$ for some $x_0 \in \sigma$. By Proposition 1.4.2, there is a face τ of σ such that $x' \in \text{int}(\tau)$. If we consider generators $\tau = \text{cone}_D(x_1, \ldots, x_r)$ then, by Proposition 1.4.3, we have $x_0 = \sum_i \lambda_i x_i$ with $\lambda_i \in \mathbb{D}^\times_{>0}$. Hence, for $y \in \sigma^\vee$ we have

$$y \in \text{face}_{x_0}\, \sigma^\vee \Leftrightarrow \langle y\,, x_0\rangle = 0 \Leftrightarrow \langle y\,, x_i\rangle = 0\ \forall\, 0 \le i \le r \Leftrightarrow \langle y, x\rangle = 0\ \forall\, x \in \tau.$$

Therefore, $\text{face}_{x_0}\, \sigma^\vee = \tau^*$.

Finally, we will prove that the map is its own inverse: As the map is surjective it is enough to prove that $((\tau^*)^*)^* = \tau^*$. For this, we will prove that $(\tau^*)^* \subseteq \tau$. Suppose $\tau = \text{face}_{y'}\, \sigma$ for some $y' \in \sigma^\vee$. Then,

$$\begin{aligned} (\tau^*)^* &= \{x \in \sigma \mid \langle y, x\rangle = 0\ \forall y \in \tau^*\} \\ &\subseteq \{x \in \sigma \mid \langle y', x\rangle = 0\} \\ &= \tau. \end{aligned}$$

This finishes the proof. ∎

1.6 The Structure of Faces

In this section we develop tools to explicitly describe a given face of a polyhedron. These descriptions depend on the data used to present the polyhedron. In Proposition 1.6.1 we study the case in which the polyhedron is defined in terms of a representation by inequalities. After this, we introduce the concept of *weighted convex hull* which allows us to introduce any polyhedron in terms of generators. In Propositions 1.6.4 and Proposition 1.6.9 we describe faces in terms of these generators.

Proposition 1.6.1. *Let $P \subseteq N_{\mathbb{D}}$ be a polyhedron with a representation*

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}.$$

1. *For each $y \in M_{\mathbb{D}}$ achieving its minimum in P there are $\lambda_i \in \mathbb{D}_{\geq 0}$ for $1 \leq i \leq r$ such that*

$$y = \lambda_1 y_1 + \cdots + \lambda_r y_r \text{ and}$$
$$\min_{x \in P} \langle y\,, x\rangle = \lambda_1 a_1 + \cdots + \lambda_r a_r.$$

   Moreover, given such elements $\{\lambda_i\}_i$, we can write the face $F = \mathrm{face}_y(P)$ as

$$F = \bigcap_{i=1}^{r} \left\{x \in P \,\middle|\, \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i} a_i\right\} \tag{1.9}$$

   where $\alpha_i = \mathrm{ord}(\lambda_i)$ for each i.

2. *Similarly, if F is a face of P, given $x_0 \in \mathrm{int}(F)$ we have an equality of the form*

$$F = \bigcap_{i=1}^{r} \left\{x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i} a_i\right\} \tag{1.10}$$

   for $\beta_i = k - \mathrm{ord}(\langle y_i\,, x_0\rangle - a_i)$.

3. *Conversely, any choice of $0 \leq \alpha_i \leq k$ determines a set of the form (1.9) which is either empty or a face of P.*

Remark 1.6.2. Given a linear function $y \in M_{\mathbb{D}}$ with $\min_{x\in P}\langle y\,, x\rangle = a$. The set

$$\left\{x \in P \;\middle|\; \varepsilon^{k-\alpha}\langle y\,, x\rangle = \varepsilon^{k-\alpha} a\right\} = \left\{x \in P \;\middle|\; \operatorname{ord}\{\langle y\,, x\rangle - a\} \geq \alpha\right\}$$

should be interpreted as the set of all elements $x \in P$ such that y achieves the minimum *at least in the first* α *coordinates.*

As an example, if we take $y \in M_{\mathbb{R}}$ to be real and

$$\begin{aligned} x &= x^{(0)} + x^{(1)}\varepsilon + \cdots + x^{(k-1)}\varepsilon^{k-1} \in N_{\mathbb{D}}, \\ a &= a^{(0)} + a^{(1)}\varepsilon + \cdots + a^{(k-1)}\varepsilon^{k-1} \in \mathbb{D}. \end{aligned}$$

Then, $\langle y\,, x\rangle = \sum_{i=0}^{k-1}\langle y\,, x^{(i)}\rangle\varepsilon^i$. So, we have that $\langle y\,, x\rangle\varepsilon^{k-\alpha} = a\varepsilon^{k-\alpha}$ iff $\langle y\,, x^{(i)}\rangle = a^{(i)}$ for each $0 \leq i < \alpha$, and this happens iff x minimize the vector

$$(\langle y\,, x^{(0)}\rangle, \ldots, \langle y\,, x^{(\alpha)}\rangle)$$

among all $x \in P$ with respect to the lexicographic order.

In this way, the equality in (1.9) can be read as: $\operatorname{face}_y P$ is the set of all $x \in P$ for which y_i achieves its minimum at least in the first $k - \alpha_i$ coordinates for each $1 \leq i \leq r$.

Proof of Proposition 1.6.1.

1. If y achieves its minimum in P. By Farkas' Lemma, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that
$$\langle y\,, \cdot\rangle - \min_{x\in P}\langle y\,, x\rangle = \lambda_1(\langle y_1\,, \cdot\rangle - a_1) + \cdots + \lambda_r(\langle y_r\,, \cdot\rangle - a_r) \tag{1.11}$$
By evaluating this at $x = 0$ we get $\min_{x\in P}\langle y\,, x\rangle = \lambda_1 a_1 + \cdots + \lambda_r a_r$ and, if we add this equation to the previous one, we get $y = \lambda_1 y + \cdots + \lambda_r y_r$. This shows the first part.

   Now, if we evaluate (1.11) in an element $x \in F$, the left hand side vanishes and, as each term of the right hand side is positive, we get
$$\lambda_i\langle y_i\,, x\rangle = \lambda_i a_i \tag{1.12}$$
for each $1 \leq i \leq r$. After multiplication by an invertible element, this becomes

$\varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i}a_i$. Which shows that

$$F \subseteq \bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i}a_i \right\}.$$

Conversely, if x is in the right hand side of (1.9) for every $1 \leq i \leq r$ then, the right hand side of (1.11) vanishes at x. Hence, so does the left hand side which implies $x \in F$. This shows the equality we wanted.

2. Notice that $\varepsilon^{\mathrm{ord}(\langle y_i\,,x_0\rangle - a_i)}(\langle y_i\,, x_0\rangle - a_i) = 0$. Hence, by Proposition 1.4.2, as $x_0 \in \mathrm{int}(F)$ we have
$$F \subseteq \left\{ x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i}a_i \right\},$$
and therefore
$$F \subseteq \bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i}a_i \right\}.$$
On the other hand, by (1.9) and as $x_0 \in F$ we have
$$\varepsilon^{\alpha_i}\langle y_i\,, x_0\rangle = \varepsilon^{\alpha_i}a_i.$$
Hence, $\varepsilon^{\alpha_i}(\langle y_i\,, x_0\rangle - a_i) = 0$ from where $\alpha_i \geq \mathrm{ord}(\langle y_i\,, x_0\rangle - a_i) = \beta_i$. This implies
$$\bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\beta_i}\langle y_i\,, x\rangle = \varepsilon^{\beta_i}a_i \right\} \subseteq \bigcap_{i=1}^{r} \left\{ x \in P \;\middle|\; \varepsilon^{\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{\alpha_i}a_i \right\} = F.$$

3. Suppose now that F is a non-empty set of the form (1.9) and consider $y = \sum_{i=1}^{r} \varepsilon^{\alpha_i} y_i$. We will prove that $\mathrm{face}_y(P) = F$. For this, notice that as F is not empty, the function $\langle y, \cdot\rangle$ has as minimum over P the value $\sum_{i=1}^{r} \varepsilon^{\alpha_i} a_i$. Hence, we can consider $\mathrm{face}_y\,P$, and given $x \in P$, we have

$$\begin{aligned} x \in \mathrm{face}_y\,P &\iff \langle y\,, x\rangle = \sum_{i=1}^{r} \varepsilon^{\alpha_i}a_i \\ &\iff \sum_{i=1}^{r} \varepsilon^{\alpha_i}\left(\langle y_i\,, x\rangle - a_i\right) = 0 \\ &\iff \varepsilon^{\alpha_i}\langle y_i\,, x_i\rangle = \varepsilon^{\alpha_i}a_i \;\forall\, i \\ &\iff x \in F, \end{aligned}$$

as we wanted.

□

In particular, the proposition above implies that any polyhedron has finitely many faces. Another important consequence is the following.

Corollary 1.6.3. *Let P be a polyhedron.*

1. *If F is a face of P and G is a face of F, then G is a face of P.*
2. *If F and G are faces of P and $F \cap G$ is non-empty, then it is a face of P.*

Proof.

1. Take a representation $P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$ of P. If F is a face of P by Proposition 1.6.1 part 1, there are α_i such that

$$\begin{aligned} F &= \bigcap_{i=1}^{r} \left\{x \in P \;\middle|\; \varepsilon^{\alpha_i} \langle y_i \, , x \rangle = \varepsilon^{\alpha_i} a_i \right\} \\ &= \{y_1 \geq a_1, \ldots, y_r \geq a_r\} \cap \{\varepsilon^{\alpha_1} y_1 = \varepsilon^{\alpha_1} a_1, \ldots, \varepsilon^{\alpha_r} y_r = \varepsilon^{\alpha_r} a_r\}\,. \end{aligned}$$

Notice that this is expression gives a representation for F in terms of inequalities. Hence, if G is a face of F we can apply Proposition 1.6.1 part 1 again using this representation for F. In this way G can be written as

$$\begin{aligned} G &= \bigcap_{i=1}^{r} \left\{x \in F \mid \varepsilon^{\beta_i} \langle y_i \, , x \rangle = \varepsilon^{\beta_i} a_i \right\} \\ &= \bigcap_{i=1}^{r} \left\{x \in P \mid \varepsilon^{\max\{\alpha_i, \beta_i\}} \langle y_i \, , x \rangle = \varepsilon^{\max\{\alpha_i, \beta_i\}} a_i \right\} \end{aligned}$$

for some integers β_i. Which, by Proposition 1.6.1 part 3, is a face of P.

2. By Proposition 1.6.1 part 1 we have $F = \bigcap_{i=1}^{r} \left\{x \in P \mid \varepsilon^{\alpha_i} \langle y_i \, , x \rangle = \varepsilon^{\alpha_i} a_i \right\}$ and $G = \bigcap_{i=1}^{r} \left\{x \in P \mid \varepsilon^{\beta_i} \langle y_i \, , x \rangle = \varepsilon^{\beta_i} a_i \right\}$ for some α_i, β_i. Therefore,

$$F \cap G = \bigcap_{i=1}^{r} \left\{x \in P \mid \varepsilon^{\max\{\beta_i, \alpha_i\}} \langle y_i \, , x \rangle = \varepsilon^{\max\{\beta_i, \alpha_i\}} a_i \right\}$$

which is a face of P by Proposition 1.6.1 part 3.

□

We will now proceed to study the case in which the polyhedron is given in terms of finitely many generators. We start with the case of polyhedral cones which, by Proposition 1.5.2, are all given by the cone hull of finitely many elements.

Proposition 1.6.4. *For a polyhedral cone* $\sigma = \mathrm{cone}_{\mathbb{D}}(x_1, \ldots, x_r)$ *and an element* $y \in \sigma^\vee$*, the face of* σ *induced by* y *is given by*

$$\mathrm{face}_y\, \sigma = \mathrm{cone}_{\mathbb{D}}\left(\left\{x_i \varepsilon^{k-\beta_i}\right\}_{1 \leq i \leq r}\right)$$

where $\beta_i = \mathrm{ord}\langle y\,, x_i\rangle$*.*

Proof. Given $x = \sum_i \lambda_i x_i \in \sigma$ we have

$$x \in \mathrm{face}_y\, \sigma \iff \langle y, x\rangle = 0 \iff \sum_i \lambda_i \langle y, x_i\rangle = 0.$$

As $\lambda_i\langle y\,, x_i\rangle \geq 0$ for each i, this last thing happens iff $\lambda_i\langle y\,, x_i\rangle = 0$ for each i. Now, for $\beta_i = \mathrm{ord}\langle y\,, x_i\rangle$, this is equivalent to $\lambda_i \varepsilon^{\beta_i} = 0$ for each i, which correspond to the existence of elements $\lambda_i' \in \mathbb{D}_{\geq 0}$ such that $\lambda_i = \lambda_i' \varepsilon^{k-\beta_i}$. That is, $x \in \mathrm{cone}_{\mathbb{D}}(x_i \varepsilon^{k-\beta_i})$. □

To work out the general case, we need a new notion of finitely generatedness which allow us to understand every polyhedral cone as a finitely generated object. For this reason, we introduce the weighted convex hull of a family of vectors.

Definition 1.6.5. Consider elements $x_1, \ldots, x_r \in N_{\mathbb{D}}$ and integers $\alpha_1, \ldots, \alpha_r \in \{0, \ldots, k\}$. The *weighted convex hull* of the elements $x_1, \ldots, x_r$ with respect to the weights $\alpha_1, \ldots, \alpha_r$ is the set

$$\mathrm{wconv}_{\mathbb{D}}([x_1; \alpha_1], \ldots, [x_r; \alpha_r]) = \left\{ \sum_{i=1}^r \lambda_i x_i \;\middle|\; \lambda_1, \ldots, \lambda_r \geq 0, \sum_{i=1}^r \varepsilon^{\alpha_i} \lambda_i = 1 \right\}.$$

Remark 1.6.6.

1. If no α_i is equal to zero then the weighted convex hull is empty.

2. The weighted convex hull generalize the usual convex hull as we have

$$\mathrm{wconv}_{\mathbb{D}}([x_1; 0], \ldots, [x_r; 0]) = \mathrm{conv}_{\mathbb{D}}(x_1, \ldots, x_r).$$

3. If $\alpha_i = k$ for some i, then there is no restriction for the corresponding coefficient λ_i other that being non-negative. This implies the equality

$$\mathrm{wconv}_{\mathbb{D}}([x_1;\alpha_1],\ldots,[x_r;\alpha_r]) = \mathrm{wconv}_{\mathbb{D}}(\{[x_i;\alpha_i] \mid \alpha_i \neq k\}) + \mathrm{cone}_{\mathbb{D}}(\{x_i \mid \alpha_i = k\}).$$

In particular, for $x_1,\ldots,x_r \in N_{\mathbb{D}}$ we have the equality

$$\mathrm{wconv}_D\{[0;0],[x_1;k],\ldots,[x_r;k]\} = \mathrm{cone}_D(x_1,\ldots,x_r).$$

Proposition 1.6.7. *For any polyhedron $P \subseteq N_{\mathbb{D}}$, there are $x_1,\ldots,x_r \in N_{\mathbb{D}}$ and $0 \leq \alpha_1,\ldots,\alpha_r \leq k$ such that*

$$P = \mathrm{wconv}_{\mathbb{D}}([x_1;\alpha_1],\ldots,[x_r;\alpha_r]).$$

Conversely, any set of this form is a polyhedron.

Proof. Using the extended perfect pairing from Definition 1.1.12, if we have a representation

$$P = \{y_1 \geq a_1,\ldots,y_r \geq a_r\} \subseteq N_{\mathbb{D}}$$

we can consider the polyhedral cone

$$\widehat{P} = \{(y_1,-a_1) \geq 0,\ldots,(y_r,-a_r) \geq 0\} \subseteq N_{\mathbb{D}} \times \mathbb{D}.$$

Then, we have

$$\widehat{P} \cap N_{\mathbb{D}} \times \{1\} = P \times \{1\}. \tag{1.13}$$

As $\widehat{P}$ is a polyhedral cone, it is finitely generated, hence there are generators $(x_1,b_1),\ldots,(x_r,b_r) \in N_{\mathbb{D}} \times \mathbb{D}$ such that

$$\widehat{P} = \mathrm{cone}_{\mathbb{D}}\left((x_1,b_1),\ldots,(x_r,b_r)\right).$$

After multiplying by an invertible element, we can suppose that $b_i = \varepsilon^{\alpha_i}$ for each $i = 1,\ldots,r$. Hence, using (1.13) we get that

$$\begin{aligned} P \times \{1\} &= \mathrm{cone}_{\mathbb{D}}\left((x_1,\varepsilon^{\alpha_1}),\ldots,D(x_r,\varepsilon^{\alpha_r})\right) \cap N_{\mathbb{D}} \times \{1\} \\ &= \left\{ \sum_{i=1}^r \lambda_i x_i \in N_{\mathbb{D}} \,\middle|\, \lambda_1,\ldots,\lambda_r \geq 0, \sum_{i=1}^r \lambda_i \varepsilon^{\alpha_i} = 1 \right\} \times \{1\} \\ &= \mathrm{wconv}_{\mathbb{D}}([x_1;\alpha_1],\ldots,[x_r;\alpha_r]) \times \{1\}. \end{aligned}$$

as we wanted. On the other hand, to see that $\mathrm{wconv}_{\mathbb{D}}([x_1;\alpha_1],\dots,[x_r;\alpha_r])$ is a polyhedron, notice that

$$\left\{(x_1,\dots,x_r)\in\mathbb{D}^r \;\middle|\; x_1\geq 0,\dots,x_r\geq 0, \sum_{i=1}^{r}\lambda_i\varepsilon^{\alpha_i}=1\right\}$$

is a polyhedron in $\mathbb{D}^r$ and $\mathrm{wconv}_{\mathbb{D}}([x_1;\alpha_1],\dots,[x_r;\alpha_r])$ is the image of this polyhedron under the map

$$\begin{aligned}\mathbb{D}^r &\longrightarrow N_{\mathbb{D}}\\ e_i &\longmapsto x_i, \quad \forall\, i\in\{1,\dots,r\}.\end{aligned}$$

Where $\{e_1,\dots,e_r\}$ denotes the standard basis in $\mathbb{D}^r$. Hence, it is a polyhedron by Proposition 1.2.4. □

Remark 1.6.8. In the usual polyhedral geometry over $\mathbb{R}$, every polyhedron can be written as the sum of a polytope and a polyhedral cone, this is called a *Minkowski-Weil decomposition* for the polyhedron. The proposition above is the closest we can get to that statement for general polyhedra over $\mathbb{D}$. For a detailed study of when one can actually write a polyhedron over $\mathbb{D}$ as a sum of a polytope and a polyhedral cone we refer to Section 1.9.

We will proceed to study the faces of a polyhedron from this new description in terms of generators.

Proposition 1.6.9. *Let* $P=\mathrm{wconv}_{\mathbb{D}}\big([x_1,\alpha_1],\dots,[x_r;\alpha_r]\big)$ *be a polyhedron and* $y\in M_{\mathbb{D}}$ *a linear function achieving its minimum in* P*. Then, if* $a=\min_{x\in P}\langle y\,,x\rangle$*, we have*

$$\mathrm{face}_y(P)=\mathrm{wconv}\left([x_1\varepsilon^{k-\beta_1};k+\alpha_1-\beta_1],\dots,[x_r\varepsilon^{k-\beta_r};k+\alpha_r-\beta_r]\right)$$

where $\beta_i=\mathrm{ord}\big(\langle y\,,x_i\rangle-\varepsilon^{\alpha_i}a\big)$.

Proof. As in the proof of Proposition 1.6.7, if we consider

$$\widehat{P}=\mathrm{cone}_{\mathbb{D}}\left((x_1,\varepsilon^{\alpha_1}),\dots,(x_r,\varepsilon^{\alpha_r})\right)\subseteq N_{\mathbb{D}}\times\mathbb{D}$$

then we have

$$P=\widehat{P}\cap N_{\mathbb{D}}\times\{1\}. \tag{1.14}$$

Now, we claim that if y achieves its minimum a in P, then $(y,-a)\in\widehat{P}^{\vee}$. Indeed, as

$\langle y\,,x\rangle \geq a$ for any $x \in P$ we get

$$\langle (y,-a)\,,(x,1)\rangle \geq 0 \text{ for any } x \in P.$$

Then, given $(x,b) \in \widehat{P}$, with $b \in \mathbb{D}^{\times}_{>0}$ invertible, by the equality in (1.14), we have $x/b \in P$. Hence,

$$\langle (y,-a)\,,(x,b)\rangle = b\langle (y,-a)\,,(x/b,1)\rangle \geq 0.$$

Finally, let x' be an element in P achieving the minimum of y, that is $\langle (y,-a)\,,(x',1)\rangle = 0$. Then, for an element of the form $(x,b) \in \widehat{P}$ with b no invertible we can consider $(x',1)+(x,b) = (x'+x,1+b)$. Now $1+b$ is invertible, so from the previous step

$$0 \leq \langle (y,-a)\,,((x'+x,1+b))\rangle = \langle (y,-a)\,,(x',1)\rangle + \langle (y,-a)\,,(x,b)\rangle = \langle (y,-a)\,,(x,b)\rangle.$$

Hence, $(y,-a)$ is positive in (x,b) for any $(x,b) \in \widehat{P}$. Therefore, $(y,-a) \in \widehat{P}^{\vee}$, which proves the claim.

After this, we can consider $\mathrm{face}_{(y,-a)}\,\widehat{P}$ and, by Proposition 1.6.4 above, if $\beta_i = \mathrm{ord}(\langle (y,-a)\,,(x_i,\varepsilon^{\alpha_i})\rangle)$ $\mathrm{ord}(\langle y\,,x_i\rangle - \varepsilon^{\alpha_i}a)$, then we have that

$$\mathrm{face}_{(y,-a)}\left(\widehat{P}\right) = \mathrm{cone}_{\mathbb{D}}\left(\left\{(x_i,\varepsilon^{\alpha_i})\varepsilon^{k-\beta_i}\right\}_{1\leq i\leq r}\right).$$

Moreover, we have the equality

$$\mathrm{face}_{(y,-a)}\left(\widehat{P}\right) \cap N_{\mathbb{D}} \times \{1\} = \mathrm{face}_y(P) \times \{1\}. \tag{1.15}$$

This gives

$$\begin{aligned}
\mathrm{face}_y(P) &= \left\{x \in N_{\mathbb{D}} \;\middle|\; (x,1) \in \mathrm{cone}_{\mathbb{D}}\left(\left\{(x_i,\varepsilon^{\alpha_i})\varepsilon^{k-\beta_i}\right\}_{1\leq i\leq r}\right)\right\} \\
&= \left\{\sum_{i=1}^{r} \lambda_i \varepsilon^{k-\beta_i} x_i \;\middle|\; \lambda_1,\ldots,\lambda_r \geq 0, \sum_{i=1}^{r} \varepsilon^{k+\alpha_i-\beta_i}\lambda_i = 1,\right\} \\
&= \mathrm{wconv}\left([x_1\varepsilon^{k-\beta_1}; k+\alpha_1-\beta_1],\ldots,[x_r\varepsilon^{k-\beta_r}; k+\alpha_r-\beta_r]\right),
\end{aligned}$$

as we wanted. □

Corollary 1.6.10. *For a polytope $P = \mathrm{conv}_{\mathbb{D}}(x_1,\ldots,x_r)$ and any element $y \in M_{\mathbb{D}}$, the* *face of P induced by y is given by*

$$\mathrm{face}_y(P) = \mathrm{wconv}_{\mathbb{D}}\left([x_1\varepsilon^{k-\beta_1}; k-\beta_1], \ldots, [x_r\varepsilon^{k-\beta_r}; k-\beta_r]\right)$$

where $\beta_i = \mathrm{ord}(\langle y\,, x_i\rangle - a)$ *with* $a = \min_{x\in P}\langle y\,, x\rangle$.

Proof. It follows from the previous proposition by considering the equation

$$\mathrm{wconv}_{\mathbb{D}}([x_1, 0], \ldots, [x_r, 0]) = \mathrm{conv}_{\mathbb{D}}(x_1, \ldots, x_r).$$

□

Remark 1.6.11. In general, a face of a polytope is not necesarily a polytope. For example, consider the polytope $P = \mathrm{conv}_{\mathbb{D}}\{0, 1\} = [0, 1] \subseteq \mathbb{D}$ and the face

$$\mathrm{face}_{\varepsilon^{k-1}}(P) = \mathrm{cone}_{\mathbb{D}}(\varepsilon).$$

This is not a polytope as, on a polytope, any linear function always attains its minimum in at least one of its vertices, in fact, this characterize a polytope as we will see in Corollary 1.9.9. However, in $\mathrm{cone}_{\mathbb{D}}(\varepsilon)$ the linear function $y = -\varepsilon$ does not achieve a minimum at all.

1.7 The Support of the Normal Fan

In this section we introduce the *support of the normal fan* of a polyhedron P. This is the set of all elements $y \in M_{\mathbb{D}}$ for which $\mathrm{face}_y(P)$ is well defined. We regard it as a generalization of the dual cone of a polyhedral cone introduced in Section 1.5. The support of the normal fan for polyhedra over $\mathbb{D}$ happens to be a more subtle concept than its counterpart over $\mathbb{R}$, for instance, see Example 1.7.3 below. Moreover, for a polyhedron P we introduce its *recession cone* as the set of all directions for which, any point in the polyhedron that moves along this direction remain in the polyhedron. In Proposition 1.7.8 we show how the dual of the support of the normal cone coincides with its recession cone.

Definition 1.7.1. Let $P \subseteq N_{\mathbb{D}}$ be a non-empty polyhedron. The *support of the normal fan* of P, denoted by $|\,\mathrm{NF}(P)|$, is the set of all $y \in M_{\mathbb{D}}$ such that $\langle y\,, \cdot\rangle$ achieves its minimum over P. That is,

$$|\,\mathrm{NF}(P)| := \left\{ w \in M_{\mathbb{D}} \;\middle|\; \min_P \langle y\,, \cdot\rangle \;\text{ exists} \right\}.$$

Remark 1.7.2. If $\sigma \subseteq N_{\mathbb{D}}$ is a polyhedral cone, then $|\,\mathrm{NF}(\sigma)|$ recovers the dual cone $\sigma^{\vee}$.

Example 1.7.3. Given a polyhedron P, the set $|\operatorname{NF}(P)|$ is always closed under positive scalar multiplications, but is not convex in general. Indeed, consider $N = M = \mathbb{Z}^3$ together with the polyhedron

$$P = \{(x, y, z) \in \mathbb{D}^3 \mid x \geq 0,\ y \geq 0,\ z \geq 0,\ x + y + \varepsilon z = 1\}.$$

Then, we have $(1, 0, 0), (0, 1, 0) \in \operatorname{NF}(P)$ as both of these elements achieve 0 as their minimum over P. Nonetheless, their sum $(1, 1, 0)$ does not achieve its minimum over P, as for $(x, y, z) \in P$, we have

$$\langle (1, 1, 0)\,, (x, y, z) \rangle = x + y = 1 - \varepsilon z,$$

and the set $\{1 - \varepsilon z \in \mathbb{D} \mid z \geq 0\}$ does not have a minimum. Therefore, $|\operatorname{NF}(P)|$ is not a convex cone in this case.

Question 1.7.4. Is there a simple characterization for the sets of the form $|\operatorname{NF}(P)| \subseteq M_{\mathbb{D}}$ for some polyhedron $P \subseteq N_{\mathbb{D}}$?

Although we do not know the answer to the question above, in the following proposition we provide an understanding of $\operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)|$ in terms of a representation of P. Moreover, in Section 1.9 we give a characterization for the polyhedra P for which $|\operatorname{NF}(P)|$ is convex.

Proposition 1.7.5. *Let P be a non-empty polyhedron and suppose that*

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$$

is a non-redundant representation of P. Then,

$$\operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)| = \operatorname{cone}_{\mathbb{D}}(y_1, \ldots, y_r).$$

Remark 1.7.6.

1. As $|\operatorname{NF}(P)|$ is closed under positive scalar multiplications, we can replace $\operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)|$ by $\operatorname{conv}_{\mathbb{D}} |(\operatorname{NF}(P))|$ in the statement above.

2. The assumption that the representation is not-redundant is unavoidable in the hypothesis. For example, for $N = M = \mathbb{Z}$ consider

$$P = \{x \in \mathbb{D} \mid \langle \varepsilon\,, x \rangle \geq 0, \langle 1\,, x \rangle \geq -1\}.$$

Then, $1 \notin \operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)| = \operatorname{cone}_{\mathbb{D}}(\varepsilon)$ and this does not contradict the statement of the result as the inequality $\langle 1\,, x\rangle \geq -1$ is redundant in the representation.

Proof of Proposition 1.7.5. As the representation is non-redundant, by Proposition 1.1.8, each y_i attains its minimum in P. Hence, $y_i \in |\operatorname{NF}(P)|$ for each i. This shows $\operatorname{cone}_{\mathbb{D}}(y_1, \dots, y_r) \subseteq \operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)|$. On the other hand, given $y \in |\operatorname{NF}(P)|$, as y achieves its minimum in P, by Proposition 1.6.1 there are $\lambda_1 \dots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$y = \lambda_1 y_1 + \cdots + \lambda_r y_r$$

Hence, $y \in \operatorname{cone}_{\mathbb{D}}(y_1, \dots, y_r)$. This shows $\operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)| \subseteq \operatorname{cone}_{\mathbb{D}}(y_1, \dots, y_r)$. □

Definition 1.7.7. The recession cone of P is the set

$$\operatorname{recc}(P) \coloneqq \{x \in N_{\mathbb{D}} \mid P + x \subseteq P\}.$$

Proposition 1.7.8. *Let $P \subseteq N_{\mathbb{D}}$ be a non-empty polyhedron. Then,*

1. *given a non-redundant representation of P*

$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}$$

we have

$$\operatorname{recc}(P) = \{y_1 \geq 0, \dots, y_r \geq 0\}$$

2. *The dual of the cone hull of the support of the normal fan of P is the recession cone of P, that is,*

$$\operatorname{cone}_{\mathbb{D}}(|\operatorname{NF}(P)|)^\vee = \operatorname{recc}(P).$$

In particular, $\operatorname{recc}(P)$ *is a polyhedral cone. Moreover, by duality* $\operatorname{recc}(P)^\vee = \operatorname{cone}_{\mathbb{D}} |\operatorname{NF}(P)|$.

Proof. Notice that, by Proposition 1.7.5 together with Proposition 1.5.2 we have

$$\operatorname{cone}_D(|\operatorname{NF}(P)|)^\vee = \{y_1 \geq 0, \dots, y_r \geq 0\}\,.$$

Hence, it is enough to prove that $\operatorname{recc}(P)$ is equal to any of these sets. If $x' \in N_{\mathbb{D}}$ satisfies $\langle y_i\,, x'\rangle \geq 0$ for every $1 \leq i \leq r$, then for any $x \in P$, we have

$$\langle y_i\,, x + x'\rangle = \langle y_i\,, x\rangle + \langle y\,, x'\rangle \geq a_i \quad \forall\, 1 \leq i \leq r.$$

Thus, $P + x' \subseteq P$. This shows $\mathrm{cone}_{\mathbb{D}}(|\,\mathrm{NF}(P)|)^\vee \subseteq \mathrm{recc}(P)$. On the other hand, by Proposition 1.1.8, for any $1 \leq i \leq r$, there is an $x \in P$ such that $\langle y_i\,, x\rangle = a_i$. Then, given $x' \in \mathrm{recc}(P)$, we must have $x + x' \in P$. In particular, $\langle y_i\,, x + x'\rangle \geq a_i$, from which we infer that $\langle y_i\,, x'\rangle \geq 0$. As this happens for each $1 \leq i \leq r$, we must have $x' \in \mathrm{cone}_{\mathbb{D}}(|\,\mathrm{NF}(P)|)^\vee$, and so $\mathrm{recc}(P) \subseteq \mathrm{cone}_{\mathbb{D}}(|\,\mathrm{NF}(P)|)^\vee$. □

Corollary 1.7.9. *If P is simultaneously a polyhedron and a convex cone (in the sense of Definition 1.1.3), then P is a polyhedral cone.*

Proof. By the previous proposition, $\mathrm{recc}(P)$ is a polyhedral cone, so it is enough to prove that $P = \mathrm{recc}(P)$. As $0 \in P$ we have $0 + \mathrm{recc}(P) = \mathrm{recc}(P) \subseteq P$. On the other hand, as P is a convex cone, we have $P + P \subseteq P$, hence $P \subseteq \mathrm{recc}(P)$. □

1.8 Normal Fan Duality

In this section we introduce the *normal fan* of a polyhedron P. This is an arrangement of polyhedral cones in $M_{\mathbb{D}}$ encoding the behavior of the function $y \mapsto \mathrm{face}_y\, P$. Its construction provides a generalization of the cone duality in Theorem 1.5.3, and it gives us an important tool to study the combinatorial type of a polyhedron, as we do in Section 1.12 for $\mathbb{R}$-rational polyhedra.

Definition 1.8.1. Let $P \subseteq N_{\mathbb{D}}$ be a polyhedron. For each face F of P, its *normal cone* is the set

$$\mathrm{C}(F) \coloneqq \{y \in M_{\mathbb{D}} \mid \mathrm{face}_y\, P \supseteq F\}.$$

That is, the set of all $y \in M_{\mathbb{D}}$ such that $\mathrm{face}_y(P)$ exists and contains F.

Proposition 1.8.2. *Given a face F of a polyhedron P. The normal cone $\mathrm{C}(F)$ is a polyhedral cone. More precisely, given a non-redundant representation*

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$$

and an element $x \in \mathrm{int}\, F$, we have

$$\mathrm{C}(F) = \mathrm{cone}_{\mathbb{D}}\left(\varepsilon^{k-\alpha_1} y_1, \ldots, \varepsilon^{k-\alpha_r} y_r\right)$$

where $\alpha_i = \mathrm{ord}(\langle y_i\,, x\rangle - a_i)$.

Proof. Given $y \in \mathrm{C}(F)$, if $\min_{x\in P}\langle y\,, x\rangle = a$, then, by Farkas' Lemma, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$ such that

$$\langle y\,, \cdot\rangle - a = \lambda_1(\langle y_1\,, \cdot\rangle - a_1) + \cdots + \lambda_r(\langle y_r\,, \cdot\rangle - a_r).$$

By evaluating this equality in $x \in \mathrm{int}(F)$, the left hand side is 0 and the right hand side is a sum of non-negative terms. Hence, each term of the sum must be zero and we get

$$\lambda_i(\langle y_i\,, x\rangle - a_i) = 0 \quad \forall 1 \leq i \leq r.$$

So, if $\alpha_i = \mathrm{ord}(\langle y_i\,, x\rangle - a_i)$ then there are $\lambda_i' \in \mathbb{D}_{\geq 0}$ such that $\lambda_i = \varepsilon^{k-\alpha_i}\lambda_i'$. Therefore,

$$y = \varepsilon^{k-\alpha_1}\lambda_1' y_1 + \cdots + \varepsilon^{k-\alpha_r}\lambda_r' y_r. \tag{1.16}$$

On the other hand, if y is of the form (1.16) above, then $\langle y\,, x\rangle = a$. Therefore, $x \in \mathrm{face}_y(P)$. But as $x \in \mathrm{int}(F)$, by Proposition 1.4.2, we must have $F \subseteq \mathrm{face}_y(P)$. That is, $y \in \mathrm{C}(F)$.

With this we conclude that

$$\mathrm{C}(F) = \mathrm{cone}_{\mathbb{D}}\left(\varepsilon^{k-\alpha_1}y_1, \ldots, \varepsilon^{k-\alpha_r}y_r\right)$$

as we wanted. □

The normal cone $\mathrm{C}(F)$ encodes the local shape of P around F. To make this concrete we introduce the following notion.

Definition 1.8.3. Let $P \subseteq N_{\mathbb{D}}$ be a polyhedron and let F be a face of P. The *star* of F with respect to P is the set

$$\mathrm{Star}_P(F) := \{\lambda(x - x') \in N_{\mathbb{D}} \mid x \in P, x' \in F, \lambda \in \mathbb{D}^{\times}_{>0}\}.$$

Lemma 1.8.4. *Let P be a polyhedron and F a face of P, then*

$$\mathrm{C}(F)^{\vee} = \mathrm{Star}_P(F).$$

Proof. Fix elements $x \in P$, $x' \in F$ and $\lambda \in \mathbb{D}^{\times}_{>0}$. For any $y \in \mathrm{C}(F)$, as y achieves its minimum at x', we have $\langle y\,, x\rangle \geq \langle y\,, x'\rangle$. Hence, $\langle y\,, \lambda(x - x')\rangle \geq 0$, so $\lambda(x - x') \in \mathrm{C}(F)^{\vee}$. This shows that $\mathrm{Star}_P(F) \subseteq \mathrm{C}(F)^{\vee}$.

We will now prove the other direction, for this fix a representation

$$P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}.$$

Then, by Proposition 1.6.1, there are $0 \leq \alpha_i \leq k$ such that

$$F = \bigcap_{i=1}^{r} \{x \in P \mid \varepsilon^{\alpha_i} \langle y_i \,, x \rangle = \varepsilon^{\alpha_i} a_i\}. \tag{1.17}$$

Hence, an element $x' \in F$ satisfies

$$\varepsilon^{\alpha_i} \langle y_i \,, x' \rangle = \varepsilon^{\alpha_i} a_i \quad \forall\, 1 \leq i \leq r. \tag{1.18}$$

Moreover, we can take x' in such a way that

$$\varepsilon^{\alpha_i - 1} \langle y_i \,, x' \rangle > \varepsilon^{\alpha_i - 1} a_i \quad \forall\, 1 \leq i \leq r \text{ with } \alpha_i \geq 1 \tag{1.19}$$

Indeed, if for a certain i, we have

$$\varepsilon^{\alpha_i - 1} \langle y_i \,, x' \rangle = \varepsilon^{\alpha_i - 1} a_i, \quad \forall\, x' \in F.$$

Then, we can replace α_i with $\alpha_i - 1$ in (1.17) without altering the set F. We can proceed in this way and eventually there will be an $x'_i \in F$ such that

$$\varepsilon^{\alpha_i - 1} \langle y_i \,, x'_i \rangle > \varepsilon^{\alpha_i - 1} a_i.$$

After doing this for every i we can take $x' = \frac{1}{n} \sum_{i=1}^{r} x'_i$.

We will now prove that for every $w \in \mathrm{C}(F)^\vee$ there is a $\lambda \in \mathbb{D}^\times_{>0}$ such that $\lambda w + x' \in P$. This will finish the proof because then $w = \lambda^{-1}(x - x') \in \mathrm{Star}_P(F)$, so $\mathrm{C}(F)^\vee \subseteq \mathrm{Star}_P(F)$ as we needed. To prove this, notice that there is a $\lambda \in \mathbb{D}^\times_{>0}$ such that $\lambda w + x' \in P$ iff for each $1 \leq i \leq r$ there is a $\lambda_i \in \mathbb{D}^\times_{>0}$ such that

$$\langle y_i \,, \lambda_i w + x' \rangle \geq a_i$$

as then we can take $\lambda = \min_i\{\lambda_i\}$. We will work now with a fixed i and show that such a λ_i exist in all the possible cases in which the element $\langle y_i \,, w \rangle$ can be. Notice that as

$w \in \mathrm{C}(F)^\vee$ and $\varepsilon^{\alpha_i} y_i \in NC(F)$ (as it attains its minimum $\varepsilon^{\alpha_i} a_i$ on F) we have

$$\varepsilon^{\alpha_i}\langle y_i\,, w\rangle \geq 0.$$

- If $\varepsilon^{\alpha_i}\langle y_i\,, w\rangle > 0$ we are done, as this together with (1.18) give us $\varepsilon^{\alpha_i}\langle y_i\,, w+x'\rangle > \varepsilon^{\alpha_i} a_i$ which implies $\langle y_i\,, w + x'\rangle > a_i$ so we can take $\lambda_i = 1$.

- If $\varepsilon^{\alpha_i}\langle y_i\,, w\rangle = 0$ and $\alpha_i = 0$ we have $\langle y_i\,, \lambda_i w + x'\rangle = \langle y_i\,, x'\rangle \geq a_i$ so any $\lambda_i \in \mathbb{D}^\times_{>0}$ works.

- If $\varepsilon^{\alpha_i}\langle y_i\,, w\rangle = 0$ and $\alpha_i > 0$ then, it is enough to find λ_i small enough such that

  $$\varepsilon^{\alpha_i}\langle y_i\,, \lambda_i w + x'\rangle > \varepsilon^{\alpha_i} a_i \iff \lambda_i \varepsilon^{\alpha_i - 1}\langle y_i\,, w\rangle > \varepsilon^{\alpha_i-1} a_i - \varepsilon^{\alpha_i-1}\langle y_i\,, x'\rangle.$$

  In this last inequality, both $\varepsilon^{\alpha_i-1}\langle y_i\,, w\rangle$ and $\varepsilon^{\alpha_i-1} a_i - \varepsilon^{\alpha_i-1}\langle y_i\,, x'\rangle$ are of the form $\varepsilon^{k-1} A$ with $A \in \mathbb{R}$. As the right hand side is negative by (1.19), by taking $\lambda_i \in \mathbb{R}_{>0}$ small enough we can always make the left hand side bigger.

□

Theorem 1.8.5 (Normal Fan Duality). *Given a polyhedron $P \subseteq N_{\mathbb{D}}$, the family*

$$\mathrm{NF}(P) = \{\mathrm{C}(F) \subseteq M_{\mathbb{D}} \mid F \in \mathfrak{F}(P)^*\}$$

is a fan whose support is $|\,\mathrm{NF}(P)|$. *Moreover, for a polyhedral cone σ we have $\mathrm{C}(\tau) = \tau^*$ for each face τ of σ.*

Proof. By Proposition 1.8.2 each set $C(F)$ is a polyhedral cone. Also, for $F, G \in \mathfrak{F}(P)^*$ we have

$$y \in \mathrm{C}(F) \cap \mathrm{C}(G) \iff \mathrm{face}_y(P) \supseteq F \cup G \iff \mathrm{face}_y(P) \supseteq F \vee G \iff y \in \mathrm{C}(F \vee G).$$

Hence, $\mathrm{C}(F) \cap \mathrm{C}(G) = \mathrm{C}(F \vee G)$. Also, a face $\tau \preceq \mathrm{C}(F)$ is defined by an element $x_0 \in \mathrm{C}(F)^\vee$. By Lemma 1.8.4 we have $\mathrm{C}(F)^\vee = \mathrm{Star}_P(F)$, hence $x_0 = \lambda(x - x')$ for

$\lambda \in \mathbb{D}^{\times}_{>0}$, $x \in P$ and $x' \in F$. Therefore,

$$\begin{aligned}
\tau =& \operatorname{face}_{x_0}(C(F)) \\
=& \{y \in \mathrm{C}(F) \mid \langle y\,, x_0\rangle = 0\} \\
=& \{y \in \mathrm{C}(F) \mid \langle y\,, x\rangle = \langle y\,, x'\rangle\} \\
=& \{y \in \mathrm{C}(F) \mid \min_{w\in P}\langle y\,, w\rangle = \langle y\,, x\rangle\} \\
=& \{y \in P \mid \operatorname{face}_y(P) \supseteq F \cup \{x\}\} \\
=& \mathrm{C}(F \vee G)
\end{aligned}$$

where G is the only face of P such that $x \in \operatorname{int} G$. With this we have shown that $\mathrm{NF}(P)$ is a fan. Finally, for a polyhedral cone σ and a face τ of σ we have that

$$\begin{aligned}
C(\tau) =& \{y \in |\mathrm{NF}(\sigma)| \mid \operatorname{face}_y \supseteq \tau\} \\
=& \{y \in \sigma^{\vee} \mid \langle y\,, x\rangle = 0\ \forall x \in \tau\} \\
=& \sigma^{\vee} \cap \tau^{\perp} \\
=& \tau^{*}.
\end{aligned}$$

□

Remark 1.8.6.

1. The name of the theorem comes from the fact that the normal fan gives us an order reversing bijection
$$\mathfrak{F}(P)^{*} \xrightarrow{\sim} \mathrm{NF}(P)$$
in which each face $F \preceq P$ is orthogonal to its corresponding face $\mathrm{C}(F) \in \mathrm{NF}(P)$.

2. If σ is a polyhedral cone then, the bijection above is given by
$$\begin{aligned}
\mathfrak{F}(\sigma)^{*} &\xrightarrow{\sim} \mathrm{NF}(\sigma) \\
\tau &\longmapsto \tau^{*}.
\end{aligned}$$
Therefore, this maps correspond to the one from the cone duality in Theorem 1.5.3. In this way, we see that the normal fan duality in Theorem 1.8.5 is a strict generalization of the dual cone duality in Theorem 1.5.3.

We finalize with the following concept.

Definition 1.8.7. A fan Σ in $M_{\mathbb{D}}$ is said to be *regular* if there is a polyhedron P such that $\Sigma = \mathrm{NF}(P)$.

1.9 The Support Function

In this section we go one step further in our dual understanding of a polyhedron and consider the map $y \mapsto \min_{x \in P} \langle y\,, x\rangle$. This is a piecewise linear concave function called the *support function* of the polyhedron P. Under mild hypothesis, in Theorem 1.9.4 we use the support function to obtain an alternative description of the normal fan, and in Theorem 1.9.6 we show how the support function gives us a bijection between polyhedra and piecewise linear concave functions. We use this in Corollary 1.9.9 to understand when a given polyhedron has a *Minkowski-Weyl decomposition*. That is, an equation of the form $P = Q + \sigma$, where Q is a polytope and σ is a polyhedral cone. In particular, this characterization allow us to show that polytopes are exactly the polyhedra in which any linear function achieves its minimum

Definition 1.9.1. Given a polyhedron P, we define its *support function* as the map

$$\begin{aligned} h_P : |\,\mathrm{NF}(P)| &\longrightarrow \mathbb{D} \\ y &\longmapsto \min_{x \in P} \langle y\,, x\rangle. \end{aligned}$$

Remark 1.9.2.

1. The function h_P is positive homogeneous in the sense that, for any $\lambda \in \mathbb{D}_{\geq 0}$, if $y \in |\,\mathrm{NF}(P)|$ then $h_P(\lambda y) = \lambda h_P(y)$.

2. For each face $F \preceq P$, if we take a point $x_F \in \mathrm{int}(F)$, then

$$h_P(y) = \min_{x \in P} \langle y\,, x\rangle = \langle y\,, x_F\rangle$$

for each $y \in C(F)$. In particular, h_P is linear along $C(F)$.

3. The minimum in the definition of h_P can be taken to be finite because, as above, if we take for each face $F \preceq P$ a point $x_F \in \mathrm{int}(P)$ then,

$$h_P(y) = \min_{F \in \mathfrak{F}(P)^*} \langle y\,, x_F\rangle.$$

4. If $P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}$ is a non-redundant representation of P, then by Proposition 1.1.8 we have $h_P(y_i) = a_i$.

5. From the support of the normal fan and the support function, we can recover the polyhedron as
$$P = \bigcap_{y \in |\,\mathrm{NF}(P)|} \{x \in N_{\mathbb{D}} \mid \langle y\,, x\rangle \geq h_P(y)\}.$$

We can use the support function to give new characterizations of the normal fan. For this, we will use the concepts from Definition 1.1.12 together with the following one.

Definition 1.9.3. Given a polyhedron P, the lifted normal fan is the set

$$|\,\mathrm{NF}(P)|^h := \mathrm{cone}_{\mathbb{D}} \left\{(y, h_P(y)) \in M_{\mathbb{D}} \times \mathbb{D} \;\middle|\; y \in |\,\mathrm{NF}(P)|\right\}.$$

Theorem 1.9.4. *Let P be a polyhedron with a non-redundant representation $P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}$. The lifted normal fan can be computed as*

$$|\,\mathrm{NF}(P)|^h = \mathrm{cone}_{\mathbb{D}} \big((y_1, a_1), \dots, (y_r, a_r)\big).$$

Moreover, $\mathrm{NF}(P)$ can be obtained as the family of all projections of the upper faces of $|\,\mathrm{NF}(P)|^h$ from $M_{\mathbb{D}} \times \mathbb{D}$ to $M_{\mathbb{D}}$.

Proof. The proof goes in three steps.

1. $|\,\mathrm{NF}(P)|^h = \mathrm{cone}_{\mathbb{D}}((y_1, a_1), \dots, (y_r, a_r))$:

   As $y_i \in |\,\mathrm{NF}(P)|$ for each $i = 1, \dots, r$, we have $(y_i, h(y_i)) = (y_i, a_i) \in |\,\mathrm{NF}(P)|^h$, hence
   $$|\,\mathrm{NF}(P)|^h \supseteq \mathrm{cone}_{\mathbb{D}}((y_1, a_1), \dots, (y_r, a_r))$$

   For the other inclusion, by Proposition 1.8.2, for any face F of P we have

   $$\mathrm{C}(F) = \mathrm{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} y_1, \dots, \varepsilon^{k-\alpha_r} y_r\right)$$

   where $\alpha_i = \mathrm{ord}(\langle y_i\,, x\rangle - a_i)$. Hence, for $y \in \mathrm{C}(F)$ there are $\lambda_1, \dots, \lambda_r \in \mathbb{D}_{\geq}0$ such that
   $$y = \lambda_1 \varepsilon^{k-\alpha_1} y_1 + \cdots + \lambda_r \varepsilon^{k-\alpha_r} y_r.$$

Moreover, as h_P is positive homogeneous and linear over $C(F)$ we have

$$\begin{aligned} h_P(y) &= \lambda_1 h_P(\varepsilon^{k-\alpha_1} y_1) + \cdots + \lambda_r h_P(\varepsilon^{k-\alpha_r} y_r) \\ &= \lambda_1 \varepsilon^{k-\alpha_1} h_P(y_1) + \cdots + \lambda_r \varepsilon^{k-\alpha_r} h_P(y_r) \\ &= \lambda_1 \varepsilon^{k-\alpha_1} a_1 + \cdots + \lambda_r \varepsilon^{k-\alpha_r} a_r. \end{aligned}$$

Hence, $(y, h_P(y)) \in \text{cone}_{\mathbb{D}}((y_1, a_1), \ldots, (y_r, a_r))$ for any $y \in C(F)$. As $|\,\text{NF}(P)| = \bigcup_F C(F)$ we conclude that $|\,\text{NF}(P)|^h = \text{cone}_{\mathbb{D}}((y_1, a_1), \ldots, (y_r, a_r))$.

2. $\text{face}_{(x,-1)}(Q)$ can be considered iff $x \in P$:

   We have that $\text{face}_{(x,-1)}(Q)$ exists iff $(x, 1) \in Q^\vee$. Moreover, for $x \in N_{\mathbb{D}}$

$$\begin{aligned} (x, 1) \in Q^\vee &\iff \langle (y, h_P(y)) \,, (x, -1) \rangle \geq 0 \text{ for every } y \in |\,\text{NF}(P)| \\ &\iff \langle y \,, x \rangle \geq h(y) \text{ for every } y \in \text{NF}(P) \\ &\iff x \in P. \end{aligned}$$

3. For $x \in P$, if $x \in \text{int}(F)$ then $\text{face}_{(x,-1)}(Q) = F$:

   By Proposition 1.6.4, using the generators from part (1) we get

$$\text{face}_{(x,-1)}(Q) = \text{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} (y_1, a_1) \,, \ldots, \varepsilon^{k-\alpha_s} (y_r, a_r) \right),$$

   where

$$\alpha_i = \text{ord}\langle (y_i, h_P(y_i)) \,, (x, -1) \rangle = \text{ord}(\langle w_i \,, x \rangle - h_P(y_i)) = \text{ord}(\langle w_i \,, x \rangle - a_i).$$

   Hence, if π denotes the projection from $M_{\mathbb{D}}$ to $\mathbb{D}$ we have

$$\pi \left(\text{face}_{(x,-1)}(Q) \right) = \text{cone}_{\mathbb{D}} \left(\varepsilon^{k-\alpha_1} y_1, \ldots, \varepsilon^{k-\alpha_s} y_r \right),$$

   which is exactly equal to $C(F)$ by Proposition 1.8.2.

□

Definition 1.9.5. Given a polyhedral cone $\sigma \subseteq M_{\mathbb{D}}$, a function $l : \sigma \to \mathbb{D}$ is called *piecewise linear concave* if there is a finite subset $A \subseteq N_{\mathbb{D}}$ such that

$$l(y) = \min_{x \in A} \langle y \, , x \rangle, \quad \forall\, y \in \sigma.$$

Theorem 1.9.6 (Higher Rank Minkowki Theorem)**.** *There is a bijection between polyhedra with convex normal fan and polyhedral cones endowed with concave linear functions. Explicitly:*

1. *We associate to a polyhedron P with convex normal fan the pair*

$$\Psi(P) = (|\operatorname{NF}(P)|, h_P).$$

2. *We associate to a pair (σ, h) the polyhedron*

$$\Phi(\sigma, h) = \operatorname{conv}_{\mathbb{D}}(A) + \sigma^{\vee}$$

where $A \subseteq N_{\mathbb{D}}$ is a finite subset such that $h = \min_{x \in A} \langle \cdot \, , x \rangle$.

Proof. The map in Ψ is well define because, as $|\operatorname{NF}(P)|$ is convex, it is a polyhedral cone by Proposition 1.7.5. Moreover, the support function is pieciewise linear and concave as mentioned in part (3) of Remark 1.9.2.

Let us see now that the map Φ is well defined as well. For this, notice that an element $y \in M_{\mathbb{D}}$ achieves the minimum in $\operatorname{conv}_{\mathbb{D}}(A) + \sigma^{\vee}$ iff it achieves the minimum independently in $\operatorname{conv}_{\mathbb{D}}(A)$ and in $\sigma^{\vee}$. Moreover, y always achieves the minimum in $\operatorname{conv}_{\mathbb{D}}(A)$ in one element of A, and it achieves the minimum in $\sigma^{\vee}$ iff $y \in (\sigma^{\vee})^{\vee} = \sigma$. Hence, the support of the normal fan of $\Phi(\sigma, h) = \operatorname{conv}_{\mathbb{D}}(A) + \sigma^{\vee}$ is σ, which is convex. Moreover, the support function of this polyhedron is

$$\begin{aligned} \sigma &\longrightarrow \mathbb{D} \\ y &\longmapsto \min_{x \in \Phi(\sigma, h)} \langle y \, , x \rangle = \min_{x \in A} \langle y \, , x \rangle \end{aligned}$$

which is exactly h. As, mentioned in Remark 1.9.2 part (5), the support function of a polyhedron determines the polyhedron. Hence, $\operatorname{conv}_{\mathbb{D}}(A) + \sigma^{\vee}$ does not depend on A and then the map is well defined.

Moreover, the maps Ψ and Φ are mutually inverse: If we start with a pair (σ, h), we get a polyhedron $\Phi(\sigma, h) = \operatorname{conv}_{\mathbb{D}}(A) + \sigma^{\vee}$ which, as we saw above, has σ as normal fan and h as support function. Hence,

$$\Psi \circ \Phi(\sigma, h) = (\sigma, h).$$

This shows that Ψ is surjective. Moreover, the map Ψ is already injective by Remark 1.9.2 part (5). Hence, it is bijective and then Ψ and Φ are mutually inverse. □

Definition 1.9.7. A Minkowski-Weyl decomposition for a polyhedron, is an equality of the form $P = Q + \sigma$ with Q a polytope and σ a polyhedral cone.

Remark 1.9.8. If $P = \operatorname{conv}_{\mathbb{D}}(A) + \sigma$ is a Minkowski-Weyl decomposition we get

$$(\sigma^{\vee}, \min_{x\in A}\langle\cdot\,, x\rangle) = \Psi \circ \Phi(\sigma^{\vee}, \min_{x\in A}\langle\cdot\,, x\rangle) = \Psi(\operatorname{conv}_{\mathbb{D}}(A) + \sigma) = \Psi(P) = (|\operatorname{NF}(P)|, h_P).$$

Hence, $\sigma^{\vee} = |\operatorname{NF}(P)|$ and then, by Proposition 1.7.8, we get $\sigma = |\operatorname{NF}(P)|^{\vee} = \operatorname{recc}(P)$. In particular, the polyhedral cone in the decomposition is uniquely determined. On the other hand, the polytope on the decomposition is not uniquely determine. For example,

$$\{0\} + \sigma = \operatorname{conv}_{\mathbb{D}}(A) + \sigma$$

for any polyhedron σ and any finite set $A \subseteq \sigma$, but $\{0\} \neq \operatorname{conv}_{\mathbb{D}}(A)$ in general.

Corollary 1.9.9.

1. *A polyhedron admits a Minkowski-Weyl decomposition iff the support of its normal fan is convex.*

2. *A polyhedron is a polytope iff any linear function attains its minimum over it.*

Proof.

1. If a polyhedron admits a Minkowski-Weyl decomposition $P = Q + \sigma$, then as in Remark 1.9.8 we get $|\operatorname{NF}(P)| = \sigma^{\vee}$ which is a convex set. On the other hand, if the support of the normal fan is convex then we can apply Theorem 1.9.6 and we get

$$P = \Phi \circ \Psi(P) = \Phi(|\operatorname{NF}(P)|, h_P) = \operatorname{conv}_{\mathbb{D}}(A) + |\operatorname{NF}(P)|^{\vee}$$

which is a Minkowski-Weyl decomposition for P.

2. If $P = \operatorname{conv}_{\mathbb{D}}(A)$ is a polytope then, any linear function achieves its minimum in one its generators from A. In particular the minimum exists.

   On the other hand, if $|\operatorname{NF}(P)| = M_{\mathbb{D}}$, then $|\operatorname{NF}(P)|$ is convex. By the previous part then P admits a Minkowski-Weyl decomposition $P = Q + \sigma$. As in Remark 1.9.8 we have

$$\sigma = \operatorname{recc}(P) = |\operatorname{NF}(P)|^{\vee} = M_{\mathbb{D}}^{\vee} = \{0\}.$$

   Hence, $P = Q + \{0\} = Q$ is a polytope.

□

Remark 1.9.10.

1. In Example 1.7.3 we saw a polyhedron whose normal fan is not convex. Hence, by Corollary 1.9.9 this also gives an example of a polyhedron which does not accept a Minkowski-Weyl decomposition.

2. Using part (2) of Corollary 1.9.9 together with Theorem 1.9.6 we get a bijection between piecewise linear concave functions $h : M_{\mathbb{D}} \to \mathbb{D}$ and polytopes in $N_{\mathbb{D}}$.

1.10 The Fibration Point of View

Notice that for positive integers $i < j$, there is an order preserving surjective ring morphism

$$\mathbb{D}_j = \mathbb{R}[\varepsilon]/(\varepsilon^j) \to \mathbb{D}_i = \mathbb{R}[\varepsilon]/(\varepsilon^i)$$

given by modding out by the ideal (ε^i). For a given rank k, we can fit all the projections to the lower rank rings together in the sequence

$$\mathbb{D} := \mathbb{D}_k \to \mathbb{D}_{k-1} \to \cdots \to \mathbb{D}_1 = \mathbb{R}. \tag{1.20}$$

We propose to study this sequence, and many different sequences that can be deduced from it, geometrically. To do this we introduce the following concept.

Definition 1.10.1. For a given lattice N, an *iterated fibration of subsets of* $N_{\mathbb{R}}$ or simply, an *iterated fibration*, is a diagram of sets of the form

$$X^{[r]} \overset{\pi_{r-1}}{\to} X^{[r-1]} \overset{\pi_{r-2}}{\to} \ldots \overset{\pi_1}{\to} X^{[0]}$$

where each map is surjective, $X^{[0]} \subseteq N_{\mathbb{R}}$ and for each $x \in X^{[i]}$ the fiber $\pi_i^{-1}(x)$ can be identified with a subset of $N_{\mathbb{R}}$, denoted by $X_x^{[i+1]}$.

In this sense, the sequence in (1.20) is an iterated fibration of subsets of $\mathbb{R}$ in which each fiber is equal to $\mathbb{R}$ itself.

More generally, by extension of scalars, the diagram in (1.20) induce the sequence of projections

$$N_{\mathbb{D}} = N_{\mathbb{D}_k} \to N_{\mathbb{D}_{k-1}} \to \cdots \to N_{\mathbb{D}_1} = N_{\mathbb{R}} \tag{1.21}$$

Given a subset $X \subseteq N_{\mathbb{D}}$ and an integer $0 \le r \le k$, we define the set $X^{[r-1]}$ as the image of X under the projection to $N_{\mathbb{D}_r}$. In this way, there is a sequence of projections

$$X = X^{[k-1]} \to X^{[k-2]} \to \cdots \to X^{[0]}$$

which allows us to regard X as an iterated fibration of subsets of $N_{\mathbb{R}}$. Given $x \in X^{[i]}$ its fiber at x is the set

$$X_x^{[i+1]} = \{y \in N_{\mathbb{R}} \mid x + \varepsilon^i y \in X^{[i+1]}\}.$$

In order to get an idea of the objects involved, let us start with a small example.

Example 1.10.2. Consider $k = 2$, $N = \mathbb{Z}^2$ and the polyhedral cone

$$\sigma = \{(x_1, x_2) \in \mathbb{D}^2 \mid x_1, x_2 \ge 0\}.$$

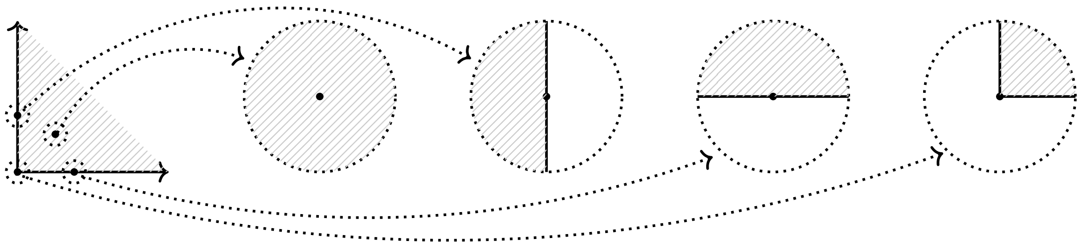
Notice that in order for $x = x^{(0)} + \varepsilon x^{(1)}$ to be positive we should have either $x^{(0)} > 0$ or $x^{(0)} = 0$ and $x^{(1)} \ge 0$. Therefore, if we regard σ as a fibration, its base is

$$\sigma^{[0]} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \ge 0\}$$

and the possible fibers are

$$\mathbb{R}^2,\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \ge 0\},\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 \ge 0\},\ \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \ge 0\}$$

depending on the position of the base-point as the following picture represents.



Another way to describe this fibration is as follows: The point $(x_1^{(0)} + \varepsilon x_1^{(1)}, x_2^{(0)} + \varepsilon x_2^{(1)})$ belongs to σ iff $(x_1^{(0)}, x_2^{(0)})$ belongs to $\sigma^{[0]}$ and $(x_1^{(1)}, x_2^{(1)})$ is a tangent point at $(x_1^{(0)}, x_2^{(0)})$ *pointing inside* to $\sigma^{[0]}$.

The example above is a special case of the notion of tangent cone bundle which we now introduce. This object has been already defined for polyhedral cone complexes in [AI21] where it plays a major role.

Definition 1.10.3. Given a set $A \subseteq N_\mathbb{R}$ and a point $x \in A$ we define the *tangent cone* of A at x as the set $\mathcal{TC}_x\, A$ of all vectors y in $N_\mathbb{R}$ such that $x + \delta y \in A$ for each $\delta \in \mathbb{R}_{>0}$ small enough. The *tangent cone bundle* of A is then the disjoint union

$$\mathcal{TC}\, A := \bigsqcup_{x \in A} \{x\} \times \mathcal{TC}_x\, A$$

together with the projection $\mathcal{TC}\, A \to A$ given by $(x_0, x_1) \mapsto (x_0)$.

We can extend this definition inductively to an iterated fibration

$$\mathcal{TC}^r\, A \to \mathcal{TC}^{r-1}\, A \to \cdots \to \mathcal{TC}^1\, A \to A,$$

by fixing $\mathcal{TC}^1\, A := \mathcal{TC}\, A$, and for $r \geq 1$ and $(x_0, \ldots, x_r) \in \mathcal{TC}^r A$

$$\mathcal{TC}^{r+1}_{(x_0 \ldots, x_r)}\, A := \mathcal{TC}_{x_r}(\mathcal{TC}^r_{(x_0, \ldots, x_{r-1})}\, A).$$

Then, we have

$$\mathcal{TC}^{r+1}\, A := \bigsqcup_{(x_0, \ldots, x_r) \in \mathcal{TC}^r A} \{(x_0, \ldots, x_r)\} \times \mathcal{TC}^{r+1}_{(x_0, \ldots, x_r)}\, A$$

together with the map $\mathcal{TC}^{r+1} A \to \mathcal{TC}^r A$ given by $(x_1, \dots, x_{r+1}) \mapsto (x_1, \dots, x_r)$.

Proposition 1.10.4. *If $A \subseteq N_{\mathbb{R}}$ is a convex set then a point $(x_0, \dots, x_r) \in (N_{\mathbb{R}})^{r+1}$ belongs to $\mathcal{TC}^r A$ iff*

1. *For every $1 \le i \le r$ and for every $\delta > 0$ small enough we have*

$$x_0 + \delta x_1 + \cdots + \delta^i x_i \in A.$$

2. *For every $1 \le i \le r$ and for every sequence of positive numbers $\{\delta_j\}_{j=1}^i$ small enough we have*

$$x_0 + \delta_1 x_1 + \cdots + \delta_1 \cdots \delta_i x_i \in A.$$

Proof. If $(x_0, \dots, x_{r-1}) \in \mathcal{TC}^{r-1} A$ then we have

$$\begin{aligned}
(x_0, \dots, x_r) \in \mathcal{TC}^r A &\iff x_r \in \mathcal{TC}_{x_{r-1}}(\mathcal{TC}_{x_{r-2}}(\dots(\mathcal{TC}_{x_0} A)\dots)) \\
&\iff x_{r-1} + \delta_r x_r \in \mathcal{TC}_{x_{r-2}}(\dots(\mathcal{TC}_{x_0} A)\dots) \text{ for } \delta_r > 0 \text{ small} \\
&\ \ \vdots \\
&\iff x_0 + \delta_1 x_1 + \cdots + \delta_1 \cdots \delta_r x_r \in A \text{ for } \delta_1, \dots, \delta_r > 0 \text{ small.}
\end{aligned}$$

We can transform $(x_0, \dots, x_{r-1}) \in \mathcal{TC}^{r-1} A$ in a similar statement, and in this way we can show that $(x_0, \dots, x_r) \in \mathcal{TC}^r A$ is equivalent to condition (2) above. Moreover, it is clear than (2) implies (1) by taking $\delta_j = \min\{\delta_i\}$ for all j. To see that (1) implies (2) take $\delta = \max\{\delta_1, \dots, \delta_i\}$, by the convexity assumption for $t_1, \dots, t_i > 0$ small we have

$$\begin{aligned}
&(1 - t_1 - \cdots - t_i)x_0 + t_1(x_0 + \delta x_1) + \cdots + t_i(x_0 + \delta x_1 + \dots \delta^i x_i) \\
&= x_0 + (t_1 + \cdots + t_i)\delta x_1 + \cdots + t_i \delta^i x_i.
\end{aligned}$$

Then, by taking $t_1 + \cdots + t_i = \delta_1/\delta$, $t_2 + \cdots + t_i = \delta_1\delta_2/\delta^2, \dots, t_i = \delta_1 \cdots \delta_i/\delta^i$ we are done. ☐

We will identify $\mathcal{TC}^{k-1} A$ with a subset of $N_{\mathbb{D}}$ using the map

$$\begin{aligned}
\mathcal{TC}^{k-1} A &\longrightarrow N_{\mathbb{D}} \\
(x_0, \dots, x_{k-1}) &\longmapsto x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1} x_{k-1}.
\end{aligned}$$

In this map, ε is a formal variable which we regard as an infinitesimal, nonetheless by Proposition 1.10.4 above we can think of $x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1} x_{k-1}$ as morally lying on A.

Remark 1.10.5. In Example 1.10.2 we can consider $\sigma = \{(x_1, x_2) \in \mathbb{D}^2 \mid x_1, x_2 \geq 0\}$ as the extension of scalars of $\sigma^{[0]} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0\}$ from $\mathbb{R}$ to the dual numbers $\mathbb{D}$. In this regard the equation $\sigma = T\mathcal{C}\, \sigma^{[0]}$ should be considered as a polyhedral version of the equality $X(k[\varepsilon]/(\varepsilon^2)) = TX(k)$, for a variety X over a field k, from algebraic geometry. In Corollary 1.11.3 below we extend this statement to a general real polyhedron. Moreover, in Section 14 we give another manifestation on the extension of scalars and we discuss how the elements of $T\mathcal{C}^r\, A$ can be seen dually as tangent derivative operators.

Using Proposition 1.10.4 we can generalize the notion of tangent cone to flag of subsets.

Definition 1.10.6. Let us consider a flag of convex subsets in $N_{\mathbb{D}}$ of the form

$$\mathcal{A} : A_0 \subseteq A_1 \subseteq \cdots \subseteq A_r.$$

We define the *tangent cone of* $\mathcal{A}$ as the set $T\mathcal{C}\, \mathcal{A}$ of all tuples $(x_0, x_1, \ldots, x_r) \in (N_{\mathbb{R}})^{r+1}$ such that $x_0 \in A_0$ and for each $1 \leq i \leq r$,

$$x_0 + \delta x_1 + \cdots + \delta^i x_i \in A_i$$

for each $\delta > 0$ small enough. If for $0 \leq i \leq r$ we denote by $\mathcal{A}|_i$ the restriction flag given by $A_0 \subseteq A_1 \subseteq \cdots \subseteq A_i$. Then, we have an iterated fibration

$$T\mathcal{C}\, \mathcal{A} = T\mathcal{C}\, \mathcal{A}|_r \to T\mathcal{C}\, \mathcal{A}|_{r-1} \to \cdots \to T\mathcal{C}\, \mathcal{A}|_1 \to A_0.$$

Again, for a flag of length k we identify $T\mathcal{C}\, \mathcal{A}$ with a subset of $N_{\mathbb{D}}$ with the map

$$\begin{aligned} T\mathcal{C}\, \mathcal{A} &\longrightarrow N_{\mathbb{D}} \\ (x_0, \ldots, x_{k-1}) &\longmapsto x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1} x_{k-1} \end{aligned}$$

Remark 1.10.7.

1. For the constant flag $\mathcal{A}$ equals to A we recover the iterated fibration of $T\mathcal{C}^r\, A$ as $T\mathcal{C}\, \mathcal{A}$.

2. The tangent cone behaves well with intersections: If $\mathcal{A} = (A_i)_{i=0}^r$ and $\mathcal{B} = (B_i)_{i=0}^r$ are

flags of the same length, then $\mathcal{TC}\,\mathcal{A}\cap\mathcal{TC}\,\mathcal{B}=\mathcal{TC}(\mathcal{A}\cap\mathcal{B})$, where $\mathcal{A}\cap\mathcal{B}=(A_i\cap B_i)_{i=0}^r$. In particular, $\mathcal{TC}^r\,A\cap\mathcal{TC}^r\,B=\mathcal{TC}^r\,A\cap B$.

3. The tangent cone behaves well with subdivisions: Given a flag of polyhedra

$$\mathcal{P}:P_0\subseteq P_1\subseteq\cdots\subseteq P_r$$

consider polyhedral complexes $\Sigma_0,\Sigma_1,\ldots,\Sigma_r$ with supports $P_0,P_1,\ldots,P_r$ respectively, and such that each cell of Σ_i is also a cell of Σ_{i+1}. Then

$$\mathcal{TC}\,\mathcal{P}=\bigcup_{\mathcal{Q}}\mathcal{TC}\,\mathcal{Q}$$

where the union goes over all flags

$$\mathcal{Q}:Q_0\subseteq Q_1\subseteq\cdots\subseteq Q_r$$

where $Q_i\in\Sigma_i$ and Q_i is a face of Q_{i+1} for each i.

1.11 Tangent Cones of real Polyhedra and Flags of real Polyhedra

As we have seen, polyhedra over the generalized dual numbers $\mathbb{D}$ give rise to iterated fibrations. In general, these fibrations may be difficult to understand, but in some particular cases, it may be possible to give a complete description of them. We will study two situations in which this happens, the one given by strongly $\mathbb{R}$-rational polyhedra and the one given by Strongly $\varepsilon\mathbb{R}$-rational polyhedra.

Let us recall that, from Definition 1.1.6, a polyhedron P is called strongly $\mathbb{R}$-rational if it admits a representation of the form

$$P=\{y_1\geq a_1,\ldots,y_r\geq a_r\}$$

with $y_i\in M_{\mathbb{R}}$ and $a_i\in\mathbb{R}$ for each $1\leq i\leq r$. For the other concept we have the following definition.

Definition 1.11.1. A polyhedron is called *strongly $\varepsilon\mathbb{R}$-rational* if it is an intersection of semispaces of the form

$$H=\{x\in N_{\mathbb{D}}\mid\varepsilon^\alpha\langle y\,,x\rangle\geq\varepsilon^\alpha a\}$$

for some $y \in M_{\mathbb{R}}$, $a \in \mathbb{R}$ and $0 \leq \alpha \leq k-1$. That is, it admits a representation of the form

$$P = \{\varepsilon^{\alpha_1} y_1 \geq \varepsilon^{\alpha_1} a_1, \ldots, \varepsilon^{\alpha_r} y_r \geq \varepsilon^{\alpha_r} a_r\}$$

with $y_i \in M_{\mathbb{R}}$, $a_i \in \mathbb{R}$ and $0 \leq \alpha_i \leq k-1$. If we can take $a_i = 0$ for each i we say that P is an *strongly* $\varepsilon\mathbb{R}$*-rational polyhedral cone.*

We will start with the following results. Which in particular shows that the tangent cone of polyhedra produces *strongly* $\mathbb{R}$*-rational* polyhedra and, more generally, the tangent cone of a flag of polyhedra produces *strongly* $\varepsilon\mathbb{R}$*-rational* polyhedra.

Theorem 1.11.2.

1. *Given a flag of polyhedra in* $N_{\mathbb{R}}$ *of the form*

$$\mathcal{P} : P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{k-1},$$

the tangent cone $TC\,\mathcal{P}$ *is a polyhedra in* $N_{\mathbb{D}}$*.*

In concrete terms,

(a) *if for each* $0 \leq i \leq r$ *we have* $P_i = \operatorname{conv}_{\mathbb{R}}(\{x_{ij}\}_j)$*. Then*

$$TC\,\mathcal{P} = \operatorname{wconv}_{\mathbb{D}}\left(\left\{[\varepsilon^i x_{ij}; i]\right\}_{ij}\right)$$

(b) *if for each* $0 \leq i \leq r$ *we have*

$$P_i = \bigcap_j \{x \in N_{\mathbb{R}} \mid \langle x\,, y_{ij}\rangle \geq a_{ij}\}.$$

Then

$$TC\,\mathcal{P} = \bigcap_{i,j} \{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq \varepsilon^{k-i} a_{ij}\}.$$

2. *Let*

$$\mathcal{S} : \sigma_0 \subseteq \sigma_1 \subseteq \cdots \subseteq \sigma_{k-1}$$

be a flag of polyhedral cones in $N_{\mathbb{R}}$*. Then* $TC\,\mathcal{S}$ *is a finitely generated polyhedral cone in* $N_{\mathbb{D}}$*. In concrete terms,* *(a) if for each $0 \leq i \leq r$ we have $\sigma_i = \mathrm{cone}_{\mathbb{R}}(\{x_{ij}\}_j)$. Then*

$$\mathcal{TC}\,\mathcal{S} = \mathrm{cone}_{\mathbb{D}}\left(\{\varepsilon^i x_{ij}\}_{ij}\right).$$

(b) If for each $0 \leq i \leq r$ we have $\sigma_i = \bigcap_j \{x \in N_{\mathbb{R}} \mid \langle x\,, y_{ij}\rangle \geq 0\}$. Then

$$\mathcal{TC}\,\mathcal{S} = \bigcap_{i,j}\{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq 0\}.$$

Proof. Let us start with the proof of (1) part (b). For this pick $x \in N_{\mathbb{D}}$ and $y \in M_{\mathbb{R}}$. If we write $x = x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1}x^{(k-1)}$, then we have

$$\varepsilon^{k-i}\langle x\,, y\rangle = \varepsilon^{k-i}\langle x^{(0)}\,, y\rangle + \varepsilon^{k-i+1}\langle x^{(1)}\,, y\rangle + \cdots + \varepsilon^{k}\langle x^{(i)}\,, y\rangle.$$

Hence, for $a \in \mathbb{R}$, $\varepsilon^{k-i}\langle x\,, y\rangle \geq \varepsilon^{k-i}a$ happens in $\mathbb{D}$ iff for each $\delta \in \mathbb{R}_{>0}$ small enough we have

$$\begin{aligned}&\delta^{k-i}\langle x^{(0)}\,, y\rangle + \delta^{k-i+1}\langle x^{(1)}\,, y\rangle + \cdots + \delta^{k}\langle x^{(i)}\,, y\rangle \geq \delta^{k-i}a\\ \iff\ &\langle x^{(0)}\,, y\rangle + \delta^{1}\langle x^{(1)}\,, y\rangle + \cdots + \delta^{i}\langle x^{(i)}\,, y\rangle \geq a\end{aligned}$$

which is equivalent to $x \in \mathcal{TC}\,\mathcal{A}^{i,y,a}$, where $\mathcal{A}^{i,y,a}$ is the flag

$$\mathcal{A}^{i,y,a} : A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{k-1}$$

given by $A_j = \{x \in N_{\mathbb{R}} \mid \langle x\,, y\rangle \geq a\}$ for $j \leq i$ and $A_j = N_{\mathbb{R}}$ for $j \geq i+1$ This shows that

$$\{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y\rangle \geq \varepsilon^{k-i}a\} = \mathcal{TC}\,\mathcal{A}^{i,y,a}.$$

Then, by Remark 1.10.7 part (2) we have that

$$\mathcal{TC}\,\mathcal{P} = \bigcap_{i,j}\mathcal{TC}\,\mathcal{A}^{i,y_{ij},a_{ij}} = \bigcap_{i,j}\{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i}\langle x\,, y_{ij}\rangle \geq \varepsilon^{k-i}a_{ij}\}.$$

This finishes the proof. By taking $a_{ij} = 0$ for each pair i, j we obtain (2) part (b).

Let us now prove (2) part (a). First, we can write each σ_i in the form

$$\sigma_i = \bigcap_{j}\{x \in N_{\mathbb{R}} \mid \langle x\,, y_{ij}\rangle \geq 0\}.$$

Then, if $i \leq i'$ we have $\sigma_i \subseteq \sigma_{i'}$ and hence $x_{ij} \in \sigma_{i'}$ for each j, so we get $\langle x_{ij}, y_{i'j'} \rangle \geq 0$ for each j', and we conclude that

$$\varepsilon^{k-i'} \langle \varepsilon^i x_{ij}, y_{i'j'} \rangle \geq 0 \quad \forall\, i, i', j, j'.$$

So, applying (2) part (b) we have

$$\varepsilon^i x_{ij} \in \bigcap_{i,j} \{x \in N_{\mathbb{D}} \mid \varepsilon^{k-i} \langle x, y_{ij} \rangle \geq 0\} = T\mathcal{C}\,\mathcal{S}$$

which implies $\operatorname{cone}_{\mathbb{D}}(\{\varepsilon^i x_{ij}\}_{ij}) \subseteq T\mathcal{C}\,\mathcal{S}$. We prove now the other inclusion. For this take $x \in T\mathcal{C}\,\mathcal{S}$. We have to construct $\lambda_{ij} = \lambda_{ij}^{(0)} + \cdots + \varepsilon^{(k-1)} \lambda_{ij}^{(k-1)} \in \mathbb{D}_{>0}$ such that

$$x = \sum_{i,j} \lambda_{ij} \varepsilon^i x_{ij}.$$

Without loss of generality we can assume

$$\{x_{ij}\}_j \subseteq \{x_{i'j}\}_j \text{ for } i \leq i' \tag{1.22}$$

otherwise we add the generators of σ_i to $\sigma_{i'}$. Write $x = x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1} x^{(k-1)}$. For $\delta > 0$ small we have

$$x^{(0)} \in \sigma_0, \quad x^{(0)} + \delta x^{(1)} \in \sigma_1, \quad x^{(0)} + \cdots + \delta^{k-1} x^{(k-1)} \in \sigma_{k-1}.$$

Denote by $\tau_0, \tau_1, \ldots, \tau_{k-1}$ the faces of $\sigma_0, \sigma_1, \ldots, \sigma_{k-1}$ respectively containing $x^{(0)}, x^{(0)} + \delta x^{(1)}, x^{(0)} + \cdots + \delta^{k-1} x^{(k-1)} \in \sigma_{k-1}$ in their relative interior. As each vertex of τ_i is a vertex of σ_i we have

$$\tau_i = \operatorname{cone}_{\mathbb{R}}(\{x_{ij}\}_j \cap \tau_i).$$

Hence, as $x^{(0)} \in \mathring{\tau}_0$ and

$$\mathring{\tau}_0 = \left\{ \sum_{x_{0j} \in \tau_0} \lambda_{0j} x_{0j} \mid \lambda_{0j} \in \mathbb{R}_{>0} \right\}$$

there are $\lambda_{0j}^{(0)} \in \mathbb{R}_{>0}$ such that $x^{(0)} = \sum_{x_{0j} \in \tau_0} \lambda_{0j}^{(0)} x_{0j}$. Now as $\tau_0 \subseteq \tau_1$ we can consider $\tau_1 / \tau_0 := (\tau_1 + \operatorname{span} \tau_0) / \operatorname{span} \tau_0$ as a cone in $N_{\mathbb{R}} / \operatorname{span} \tau_0$. Then, as $x^{(0)} + \delta x^{(1)} \in \mathring{\tau}_1$ we get

$[x^{(0)} + \delta x^{(1)}] \in (\tau_1/\tau_0)^\circ$ so $[x^{(1)}] \in (\tau_1/\tau_0)^\circ$ and as

$$(\tau_1/\tau_0)^\circ = \left\{ \sum_{x_{1j}\in\tau_1} \lambda_{1j}[x_{1j}] \mid \lambda_{0j} \in \mathbb{R}_{>0} \right\}$$

there are $\lambda_{1j}^{(0)} \in \mathbb{R}_{>0}$ such that $[x^{(1)}] = \sum_{x_{1j}\in\tau_1} \lambda_{1j}^{(0)}[x_{1j}]$. Lifting this equation to τ_1 there are $\lambda_{0j}^{(1)} \in \mathbb{R}$ such that

$$x^{(1)} = \sum_{x_{1j}\in\tau_1} \lambda_{1j}^{(0)} x_{1j} + \sum_{x_{0j}\in\tau_0} \lambda_{0j}^{(1)} x_{0j}.$$

In a similar way, $\tau_1 \subseteq \tau_2$ so we can consider τ_1/τ_2. As $x^{(0)} + \delta x^{(1)} + \delta^2 x^{(2)} \in \mathring{\tau}_2$ and $x^{(0)} + \delta x^{(1)} \in \tau_1$ we get $[x^{(2)}] \in (\tau_1/\tau_2)^\circ$ from where there are $\lambda_{2j}^{(0)} \in \mathbb{R}_{>0}$ such that $[x^{(2)}] = \sum_{x_{2j}\in\tau_2} \lambda_{2j}^{(2)}[x_{2j}]$. Lifting this equation to τ_2 we get $\lambda_{1j}^{(1)} \in \mathbb{R}$ and $\lambda_{0j}^{(2)} \in \mathbb{R}$ such that

$$x^{(2)} = \sum_{x_{2j}\in\tau_2} \lambda_{2j}^{(0)} x_{2j} + \sum_{x_{1j}\in\tau_1} \lambda_{1j}^{(1)} x_{1j} + \sum_{x_{0j}\in\tau_0} \lambda_{0j}^{(2)} x_{0j}.$$

Continuing in this way we have constructed $\lambda_{ij} = \lambda_{ij}^{(0)} + \cdots + \varepsilon^{(k-1)}\lambda_{ij}^{(k-1)} \in \mathbb{D}_{>0}$ such that

$$x = \sum_{i,j} \lambda_{ij}\varepsilon^i x_{ij}$$

as we wanted. This finishes the proof of (2) part (a).

Now, (1) part (a) follows from (2) part (a). For this, given the polytope $P_i = \operatorname{conv}_{\mathbb{R}}(\{x_{ij}\}_i) \subseteq N_{\mathbb{R}}$ consider the cone

$$\widehat{P}_i = \operatorname{cone}_{\mathbb{D}}(\{(x_{ij}, 1)\}_i) \subseteq N_{\mathbb{R}} \times \mathbb{R}.$$

In this way, we obtain a flag of polyhedral cones

$$\widehat{\mathcal{P}} : \widehat{P}_0 \subseteq \widehat{P}_1 \subseteq \cdots \subseteq \widehat{P}_{k-1}.$$

By (2) part (a) we have $TC\,\widehat{\mathcal{P}} = \operatorname{cone}_{\mathbb{D}}(\{\varepsilon^i(x_{ij}, 1)\}_{ij})$, hence

$$
\begin{aligned}
T\mathcal{C}\,\mathcal{P} \times \{1\} &= T\mathcal{C}\,\widehat{\mathcal{P}} \cap N_{\mathbb{D}} \times \{1\} \\
&= \operatorname{cone}_{\mathbb{D}}\left(\{\varepsilon^i(x_{ij},1)\}_{ij}\right) \cap N_{\mathbb{D}} \times \{1\} \\
&= \left\{x \in N_{\mathbb{D}} \;\middle|\; (x,1) \in \operatorname{cone}_{\mathbb{D}}\left(\{\varepsilon^i(x_{ij},1)\}_{ij}\right)\right\} \times \{1\} \\
&= \left\{\sum_{ij} \lambda_{ij} x_{ij} \varepsilon^i \in N_{\mathbb{D}} \;\middle|\; \lambda_{ij} \geq 0 \text{ for all } i,j \text{ and}, \sum_{i,j} \lambda_{ij}\varepsilon^i = 1\right\} \times \{1\} \\
&= \operatorname{wconv}_{\mathbb{D}}\left(\left\{[\varepsilon^i x_{ij}; i]\right\}_{ij}\right)
\end{aligned}
$$

□

Two immediate corollaries are the following.

Corollary 1.11.3 (Base change principle)**.** *The real polyhedra (resp. real polyhedral cones) in* $N_{\mathbb{D}}$ *correspond exactly to the tangent cone of polyhedra (resp. polyheral cones) in* $N_{\mathbb{R}}$*. In explicit terms.*

1. *Given a finite subset* $X \subseteq N_{\mathbb{R}}$ *we have*

$$\operatorname{cone}_{\mathbb{D}} X = T\mathcal{C}^{k-1} \operatorname{cone}_{\mathbb{R}} X.$$

2. *Given* $y_1, \ldots, y_r \in M_{\mathbb{R}}$ *and* $a_1, \ldots a_r \in \mathbb{R}$ *we have*

$$
\begin{aligned}
&\{x \in N_{\mathbb{D}} \mid \langle y_1\,, x\rangle \geq a_1, \ldots, \langle y_r\,, x\rangle \geq a_r\} \\
&= T\mathcal{C}^{k-1} \{x \in N_{\mathbb{R}} \mid \langle y_1\,, x\rangle \geq a_1, \ldots, \langle y_r\,, x\rangle \geq a_r\}.
\end{aligned}
$$

Which in particular if $a_i = 0$ *for all* i*, gives us an equality between polyhedral cones.*

Proof. This is simply the case in which we take a constant flag in Theorem 1.11.2 above. □

Corollary 1.11.4. *Given a polyhedral* P *in* $N_{\mathbb{D}}$*, the following are equivalent.*

1. P *is strongly* $\varepsilon\mathbb{R}$*-rational.*

2. $P = T\mathcal{C}^{k-1}\mathcal{P}$ *for* $\mathcal{P} : P_1 \subseteq \cdots \subseteq P_k$ *a sequence of real polyhedra in* $N_{\mathbb{R}}$*.*

Moreover, if P *is a polyhedral cone then, this are also also equivalent to the fact that* P *is finitely generated by elements of the form* $\varepsilon^i x$ *with* $x \in N_{\mathbb{R}}$*.*

Proof. This is a restatement of Theorem 1.11.2 above. ☐

Semi-real polyhedra appear naturally as faces of real polyhedra as the next proposition shows.

Proposition 1.11.5. *Let P be a polyhedron in $N_{\mathbb{R}}$. Then, the faces of $T\mathcal{C}^{k-1}\,P$ in $N_{\mathbb{D}}$ are given exactly by the sets of the form $T\mathcal{C}\,\mathcal{F}$ for a flag*

$$\mathcal{F} : F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1}$$

where each F_i is a face of P in $N_{\mathbb{R}}$.

Proof. We start proving that each set of the form $T\mathcal{C}\,\mathcal{F}$ is a face of $T\mathcal{C}^{k-1}\,P$. For this let $\mathrm{NF}_{\mathbb{R}}\,P$ be the normal fan of P. The flag of faces $\mathcal{F}$ correspond to a flag of cones

$$\sigma_0 \supseteq \sigma_1 \supseteq \cdots \supseteq \sigma_{k-1}$$

in $\mathrm{NF}_{\mathbb{R}}\,P$, in which σ_{i+1} is a face of σ_i for each i. Now, take $y^{(0)}, y^{(1)}, \ldots, y^{(k-1)} \in M_{\mathbb{R}}$ such that for each $\delta > 0$ small enough we have

$$y^{(0)} \in \sigma_{k-1}, \quad y^{(0)} + \delta y^{(1)} \in \sigma_{k-2}, \quad \ldots \quad y^{(0)} + \cdots + \delta^{k-1} y^{(k-1)} \in \sigma_0 \tag{1.23}$$

and consider $y = y^{(0)} + \cdots + \varepsilon^{(k-1)} y^{(k-1)} \in N_{\mathbb{D}}$. We claim that y defines $T\mathcal{C}\,F$ as a face. For this given $x = x^{(0)} + \cdots + \varepsilon^{(k-1)} x^{(k-1)} \in N_{\mathbb{D}}$ consider

$$\langle y \,, x\rangle = \langle y^{(0)} \,, x^{(0)}\rangle + \varepsilon \left(\langle y^{(0)} \,, x^{(1)}\rangle + \langle y^{(1)} \,, x^{(0)}\rangle\right) + \cdots + \varepsilon^{k-1} \left(\sum_{i+j=k-1} \langle y^{(i)} \,, x^{(j)}\rangle \right).$$

In order to minimize this expression for $x \in T\mathcal{C}^{k-1}\,P$ we need to first find the $x^{(0)}$ which minimize $\langle y^{(0)} \,, x^{(0)}\rangle$, then between those $x^{(0)}$ we need to minimize $\langle y^{(0)} \,, x^{(1)}\rangle + \langle y^{(1)} \,, x^{(0)}\rangle$, and so on.

To minimize $\langle y^{(0)} \,, x^{(0)}\rangle$, as we have $y^{(0)} \in \sigma_{k-1}$ we have to take $x^{(0)} \in F_{k-1}$.

To minimize $\langle y^{(0)} \,, x^{(1)}\rangle + \langle y^{(1)} \,, x^{(0)}\rangle$, we will minimize simultaneously $\langle y^{(0)} \,, x^{(1)}\rangle$ and $\langle y^{(1)} \,, x^{(0)}\rangle$. Notice that we already minimized $\langle y^{(0)} \,, x^{(0)}\rangle$, so $\langle y^{(0)} \,, x^{(1)}\rangle$ attach its minimum iff

$$\delta\langle y^{(0)} \,, x^{(1)}\rangle + \langle y^{(0)} \,, x^{(0)}\rangle = \langle y^{(0)} \,, x^{(0)} + \delta x^{(1)}\rangle$$

achieves its minimum, which happens iff $x^{(0)} + \delta x^{(1)} \in \sigma_{k-1}$. In the same way $\langle y^{(1)}, x^{(0)}\rangle$ is minimized exactly when

$$\delta\langle y^{(1)}, x^{(0)}\rangle + \langle y^{(0)}, x^{(0)}\rangle = \langle y^{(0)} + \delta y^{(1)}, x^{(0)}\rangle$$

achieves its minimum, which happens iff $x^{(0)} \in \sigma_{k-2}$. Therefore, $\langle y^{(0)}, x^{(1)}\rangle + \langle y^{(1)}, x^{(0)}\rangle$ is minimized when $x^{(0)} + \delta x^{(1)} \in \sigma_{k-1}$ and $x^{(0)} \in \sigma_{k-2}$ simultaneously.

In general, we want to minimize $\sum_{i+j=r}\langle y^{(i)}, x^{(j)}\rangle$ given that we have minimized $\sum_{i+j=s}\langle y^{(i)}, x^{(j)}\rangle$ for every $s < r$, and even more, we know that the minimum in $\sum_{i+j=s}\langle y^{(i)}, x^{(j)}\rangle$ is achieved exactly when each term has been independently minimized. Let us prove that, under this conditions, $\sum_{i+j=r}\langle y^{(i)}, x^{(j)}\rangle$ is also minimized when each tearm is independently minimized, and the minimum in the term $\langle y^{(i)}, x^{(j)}\rangle$ is achieved exactly when

$$x^{(0)} + \delta x^{(1)} + \cdots + \delta^{(j)} \in F_{k-1-i} \text{ for every small enough } \delta > 0$$

For this, notice that $\langle y^{(i)}, x^{(j)}\rangle$ is minimized iff

$$\langle y^{(0)} + \delta y^{(0)} + \cdots + \delta^i y^{(i)}, x^{(0)} + \delta x^{(1)}\rangle + \cdots + \delta^j x^{(j)}$$

is minimized, because if one expand this, then each term is constant except the term $\delta^{i+j}\langle y^{(i)}, x^{(j)}\rangle$. As $y^{(0)} + \delta y^{(0)} + \cdots + \delta^i y^{(i)} \in \sigma_{k-1-i}$, we have that the minimum is achieved when $x^{(0)} + \delta x^{(1)} + \cdots + \delta^j x^{(j)} \in F_{k-1-i}$ as we wanted.

In conclusion, $\langle y, x\rangle$ is minimized iff we have $x^{(0)} + \delta x^{(1)} + \cdots + \delta^j x^{(j)} \in F_{k-1-i}$ for every i, j with $i + j \leq k - 1$ and $\delta > 0$ small enough, which happens iff

$$x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1} x^{(k-1)} \in \mathcal{TC}\,\mathcal{F}.$$

Hence, $\mathcal{TC}\,\mathcal{F}$ is the face defined by y as we wanted.

Conversely, take an element $y = y^{(0)} + \varepsilon y^{(1)} + \cdots + \varepsilon^{k-1} y^{(k-1)} \in M_{\mathbb{D}}$. Then, it needs to defined a flag of cones in the normal fan of P as in equation (1.23). Then, the argument above shows that y defines the face $\mathcal{TC}\,\mathcal{F}$. Hence, every face of $\mathcal{TC}^{k-1} P$ is of the form $\mathcal{TC}\,\mathcal{F}$ for some flag of faces $\mathcal{F}$. This finishes the proof. □

This gives us an understanding of the combinatorial type of a real polyhedron: Its lattice of faces is the chain poset of length k (see [Joh18] for the definition) of the lattice of face of the underlying rank 1 polyhedron.

1.12 ℝ-Rational Polyhedra

Recall that an ℝ-Rational polyhedron P is a polyhedron for which there are $y_1, \dots, y_r \in M_{\mathbb{R}}$ and $a_1, \dots, a_r \in \mathbb{D}$ such that

$$P = \{y_1 \geq a_1, \dots, y_r \geq a_r\}.$$

The objective of this section is to give a new description for the normal fan of an ℝ-rational polyhedron, and use it to understand the combinatorial behavior of the iterated fibration determined by the polyhedron.

First, we will start by introducing the concept of a *layered polyhedral complex*, these are sequences of real polyhedral complexes in which each term subdivide the previous one. The first example of such a layered polyhedral complex we will present is the *layered normal fan* of an ℝ-rational polyhedron, which we introduce in Proposition 1.12.2. Later, in Theorem 1.12.4 we show how it is possible to recover the usuan normal fan of the polyhedron from its layered normal fan by a tangent cone construction.

Definition 1.12.1. A *layered polyhedral complex* is a sequence of real polyhedral complex of the form

$$\underline{\Sigma} : \Sigma_0 \preceq \Sigma_1 \preceq \cdots \preceq \Sigma_{k-1}$$

where all Σ_i are polyhedral complexes in $N_{\mathbb{R}}$ of the same support and Σ_{i+1} is a subdivision of Σ_i for each $0 \leq i \leq k-2$. A *layered face* of $\underline{\Sigma}$ is a flag of faces

$$\underline{F} : F_{k-1} \subseteq F_{k-2} \subseteq \cdots \subseteq F_0$$

with $F_i \in \Sigma_i$ for each i. The *support* of $\underline{\Sigma}$, denote by $|\underline{\Sigma}|$, is defined as the support of $|\Sigma_i|$ for any i. A layered polyhedral complex in which each term is a flag is called a *layered fan.*

Proposition 1.12.2 (Layered Normal Fan)**.** *Let P be an ℝ-rational polyhedron with a fixed non-redundant representation*

$$P = \{x \in N_{\mathbb{D}} \mid y_1 \geq a_1, \dots, y_r \geq a_r\}.$$

We can construct a sequence of fans in $M_{\mathbb{R}}$, which we call the layered normal fan of P *and denote by*

$$\underline{\Delta}(P) := \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1}, \tag{1.24}$$

in the following equivalent ways:

1. - Δ_0 *is the normal fan of the real polyhedron* $P^{[0]}$.

   - Δ_1 *is constructed by the following process. Given a cell* $\sigma \in \Delta_0$*, there is a face* F *of* $P^{[0]}$ *such that* σ *is the normal cone* $\mathrm{C}(F)$*. Given a point* $x_0 \in \mathrm{int}(F)$*, the fiber* $P^{[1]}_{x_0}$ *is a real polyhedron such that* $|\,\mathrm{NF}(P^{[1]}_{x_0})| = \mathrm{C}(F)$*. Then,* $\mathrm{NF}(P^{[1]}_{x_0})$ *is independent of the* x_0 *chosen and* Δ_1 *is obtained by replacing* $C(F)$ *by* $\mathrm{NF}(P^{[0]}_{x_0})$ *for every face* $\mathrm{C}(F) \in \Delta_0$.

   - *Similarly,* Δ_2 *is constructed as follows. Given a cell* $\sigma \in \Delta_1$*, there is a point* $x_0 \in P^{[0]}$ *such that* σ *is the normal cone* $\mathrm{C}(F)$ *of a face* F *of* $P^{[1]}_{x_0}$*. Given a point* $x_1 \in \mathrm{int}(F)$*, the fiber* $P^{[1]}_{x_0+\varepsilon x_1}$ *is a real polyhedron such that* $|\,\mathrm{NF}(P^{[1]}_{x_0+\varepsilon x_1})| = \mathrm{C}(F)$*. Then,* $\mathrm{NF}(P^{[1]}_{x_0+\varepsilon x_1})$ *is independent of the* x_1 *chosen and* Δ_1 *is obtained by replacing* $\mathrm{C}(F)$ *by* $\mathrm{NF}(P^{[1]}_{x_0+\varepsilon x_1})$ *for every face* $\mathrm{C}(F) \in \Delta_1$.

   Continuing in this way we construct Δ_i *for every integer* $0 \le i \le k-1$.

2. *For* $\delta \in \mathbb{R}_{>0}$*, the normal fan of the polyhedron*

$$P_i(\delta) := \left\{ x \in N_{\mathbb{R}} \;\middle|\; \langle y\,, x\rangle \ge a_j^{(0)} + \delta a_j^{(1)} + \cdots + \delta^i a_j^{(i)}, \quad \forall\, 1 \le j \le r \right\}$$

   is independent of δ *if it is small enough. Then, we let* Δ_i *to be this fan.*

3. Δ_i *is the fan in* $N_{\mathbb{R}}$ *whose faces are the sets of the form* $\mathrm{cone}_{\mathbb{R}}(S_i(x'))$ *as* x' *moves along* $P^{[i]}$*, where*

$$S_i(x') = \left\{ y_j \in M_{\mathbb{R}} \;\middle|\; 1 \le j \le r \text{ and } \langle y_j\,, x'\rangle = a_j^{[i]} \right\}.$$

In particular, from (1) we see that Δ_i *is independent of the representation of* P *and that* Δ_{i+1} *is a subdivision of* Δ_i.

Moreover, given a sequence of normal fans in $N_{\mathbb{R}}$ *of the form*

$$\underline{\Delta} : \Delta_0 \preceq \cdots \preceq \Delta_{k-1},$$

all of them with the same support, there is an $\mathbb{R}$*-rational polyhedron* $P \subseteq N_{\mathbb{D}}$ *such that* $\underline{\Delta} = \underline{\Delta}(P)$.

Remark 1.12.3. Notice that, by the first definition that we present for the layered normal fan, we have an explicit algorithm to understand the combinatorial structure of the fibers $P_x^{[i]}$ from the layered normal fan of P.

Proof of Proposition 1.12.2. Let us denote by $\Delta_i^{(1)}$, $\Delta_i^{(2)}$ and $\Delta_i^{(3)}$ the fans constructed in (1), (2) and (3) respectively. We need to prove that all of them are equal. Let us start by showing that $\Delta_i^{(2)}$ equals $\Delta_i^{(3)}$. For this, given $\delta \in \mathbb{R}_{>0}$ we consider the map

$$\begin{aligned} \psi_\delta : \mathbb{D} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^{(0)} + \delta x^{(1)} + \cdots + \delta^{k-1} x^{(k-1)}. \end{aligned}$$

As this is an $\mathbb{R}$-linear map, by extension of scalars and composition, this map naturally extends to a map $N_{\mathbb{D}_i} \to N_{\mathbb{R}}$ which we still denote by ψ_δ. We can now write

$$P_i(\delta) = \{x \in N_{\mathbb{R}} \mid \langle y_j\, , x\rangle \geq \psi_\delta\big(a_j^{[i]}\big),\ \forall\, 1 \leq j \leq r\},$$

and we have

$$\begin{aligned} x \in P^{[i]} &\iff \langle y_j\, , x^{[i]}\rangle \geq a^{[i]}, \quad \forall\, 1 \leq j \leq r \\ &\iff \psi_\delta\big(\langle y\, , x^{[i]}\rangle\big) \geq \psi_\delta\big(a^{[i]}\big), \quad \forall\, 1 \leq j \leq r,\ \forall\, \delta > 0 \text{ small enough} \quad \text{(By Remark 1.1.1)} \\ &\iff \langle y\, , \psi_\delta\big(x^{[i]}\big)\rangle \geq \psi_\delta\big(a^{[i]}\big), \quad \forall\, 1 \leq j \leq r,\ \forall\, \delta > 0 \text{ small enough} \quad \text{(By } \mathbb{R}\text{-linearity)} \\ &\iff \psi_\delta(x) \in P_i(\delta), \quad \forall\, \delta > 0 \text{ small enough.} \end{aligned}$$

Thus, for all $\delta \in \mathbb{R}_{>0}$ small enough we have $\psi_\delta\big(P^{[i]}\big) \subseteq P_i(\delta)$. Now, given a point $x \in P^{[i]}$, as $\psi_\delta(x) \in P_i(\delta)$ we can consider the cell of $\Delta_i^{(2)}$ of the form $\mathrm{C}(F)$, where F is the face of $P_i(\delta)$ such that $\psi_\delta(x) \in \mathrm{int}(F)$. By Proposition 1.8.2, if we consider

$$S_{i,\delta}(x) = \big\{y \in A \;\big|\; \langle y\, , x\rangle = \psi_\delta\big(h(y)^{[i]}\big)\big\}$$

then $\mathrm{C}(F) = \mathrm{conv}_{\mathbb{R}}\big(S_{i,\delta}(\psi_\delta(x))\big)$. Moreover, by Remark 1.1.1, for $\delta \in \mathbb{R}_{>0}$ small enough we have

$$S_{i,\delta}(\psi_\delta(x)) = \big\{y \in A \;\big|\; \langle y\, , x\rangle = \psi_\delta\big(h(y)^{[i]}\big)\big\} = \big\{y \in A \;\big|\; \langle y\, , x\rangle = h(y)^{[i]}\big\} = S_i(x).$$

This shows that each cell of $\Delta_i^{(3)}$ belongs to $\Delta_i^{(2)}$, as both fans have the same support we conclude that they are equal. In particular, $\Delta_i^{(2)}$ does not depend on δ when it is small enough.

Now, let us see that $\Delta_i^{(3)}$ is also equal to $\Delta_i^{(1)}$ and $\Delta_i^{(2)}$.

1. If $i = 0$ then $\Delta_0^{(1)}$ equals $\Delta_0^{(2)}$ by definition.

2. If $i = 1$ then, to construct $\Delta_1^{(3)}$ we need to take for each cell $\mathrm{C}(F) \in \Delta_0^{(3)}$ a point $x_0 \in \mathrm{int}(F)$ and consider

$$\begin{aligned} P_{x_0}^{[1]} &= \left\{x \in N_{\mathbb{R}} \mid x_0 + \varepsilon x \in P^{[1]}\right\} \\ &= \left\{x \in N_{\mathbb{R}} \mid \langle y_j \,, x_0\rangle + \varepsilon\langle y_j \,, x\rangle \geq a^{(1)} + \varepsilon a^{(1)} \ \ \forall\, 1 \leq j \leq r\right\} \\ &= \left\{x \in N_{\mathbb{R}} \mid \langle y_j \,, x\rangle \geq a_j^{(1)} \ \ \forall\, 1 \leq j \leq r \text{ such that } \langle y_j \,, x_0\rangle = a_j^{(0)}\right\} \\ &= \left\{x \in N_{\mathbb{R}} \mid \langle y_j \,, x\rangle \geq a_j^{(1)} \ \ \forall j \text{ such that } y_j \in S_0(x_0)\right\}. \end{aligned}$$

Then, by Proposition 1.7.5 we have $|\,\mathrm{NF}(P_{x_0}^{[1]})| = \mathrm{cone}_{\mathbb{R}}(S_0(x_0)) = \mathrm{C}(F)$. Moreover, for $x_1 \in P_{x_0}^{[1]}$, if $x_1 \in \mathrm{int}(G)$ for a face G then the normal cone $\mathrm{C}(G)$ with respect to $P_{x_0}^{[1]}$ is a cell of $\mathrm{NF}(P_{x_0}^{[1]})$. By Proposition 1.8.2 using the representation for $P_{x_0}^{[1]}$ we have found above we have

$$\begin{aligned} \mathrm{C}(G) &= \mathrm{cone}_{\mathbb{R}}\left\{y_j \in M_{\mathbb{R}} \mid \langle y_j \,, x_1\rangle = a^{(1)} \text{ with } y_j \in S_0(x_0)\right\} \\ &= \mathrm{cone}_{\mathbb{R}}(S_1(x_0 + \varepsilon x_1)). \end{aligned}$$

Hence, each face of $\Delta_1^{(3)}$ is a face of $\Delta_1^{(2)}$ and as they have the same support they must be equal.

3. The general case is similar. Suppose the result is true for i and let us check it is true for $i+1$. By the induction hypothesis, a cell of $\Delta_i^{(3)}$ is of the form $\mathrm{cone}_{\mathbb{R}}(S_i(x_0 + \varepsilon x_1 + \cdots + \varepsilon^i x_i))$ for some $x_0, \ldots, x_i \in N_{\mathbb{R}}$ such that $x_0 + \cdots + \varepsilon^i x_i \in P^{[i]}$. Then,

$$P_{x_0+\cdots+\varepsilon^i x_i}^{[i+1]} = \left\{x \in N_{\mathbb{R}} \mid \langle y_j \,, x\rangle \geq a_j^{i+1} \ \ \forall\, j \text{ such that } y_j \in S_i(x_0 + \cdots + \varepsilon^i x_i)\right\}.$$

Hence, $|\,\mathrm{NF}(P_{x_0+\cdots+\varepsilon^i x_i}^{[i]})| = \mathrm{cone}_{\mathbb{R}}(S_i(x_0+\cdots+\varepsilon^i x_i))$ and for a point x_{i+1} in the fiber, if $x_{i+1} \in \mathrm{int}(G)$ for a face G of the normal cone $C(G)$ with respect to $P_{x_0+\cdots+\varepsilon^i x_i}^{[i]}$, then by Proposition 1.8.2 we have

$$\begin{aligned} \mathrm{C}(G) &= \mathrm{cone}_{\mathbb{R}}\left\{y_j \in M_{\mathbb{R}} \mid \langle y_j \,, x_{i+1}\rangle = a^{(i+1)} \text{ with } y_j \in S_i(x_0 + \cdots + \varepsilon^i x_i)\right\} \\ &= \mathrm{cone}_{\mathbb{R}}(S_1(x_0 + \cdots + \varepsilon^i x_i + \varepsilon^{i+1} x_{i+1})). \end{aligned}$$

Which is a face of $\Delta^{(3)}_{i+1}$. Hence, every face of $\Delta^{(1)}_{i+1}$ is a face of $\Delta^{(3)}_{i+1}$ and as they have the same support they are equal.

Finally, given a sequence of normal fans in $N_{\mathbb{R}}$ of the form

$$\underline{\Delta} : \Delta_0 \preceq \cdots \preceq \Delta_{k-1},$$

for each $0 \leq i \leq k-1$ we can consider a polyhedron P_i in $N_{\mathbb{R}}$ such that $\mathrm{NF}(P_i) = \Delta_i$. If

$$P_i = \{x \in N_{\mathbb{R}} \mid y_j \geq a_j^{(i)},\ \forall\, 1 \leq j \leq r\},$$

for some $y_j \in M_{\mathbb{R}}$, $a_j^{(i)} \in M_{\mathbb{R}}$. Without loss of generality, we can suppose that $\{y_1, \ldots, y_r\}$ is independent of i. Then, we can consider

$$P = \left\{x \in N_{\mathbb{D}} \,\middle|\, y_j \geq a_j^{(0)} + \varepsilon a_j^{(1)} + \cdots + \varepsilon^{(k-1)} a_j^{(k-1)},\ \forall\, 1 \leq j \leq r\right\}.$$

Then, we have that $\Delta_i = \Delta_i(P)$ for each i. Indeed, it is enough to prove that if

$$\underline{\delta} : \delta_0 \subseteq \delta_1 \subseteq \cdots \subseteq \delta_{k-1}$$

is a sequence of faces with $\delta_i \in \Delta_i$, and $\delta_i = \mathrm{conv}_{\mathbb{R}}(S_{P_i}(x_i))$ for some $x_i \in P_i$. Then, for $x := x_0 + \varepsilon x_1 + \cdots + \varepsilon^{k-1} x_{k-1} \in P$ we have

$$\delta_i = \mathrm{conv}_{\mathbb{R}}\left(S_{i,P}(x^{[i]})\right) \tag{1.25}$$

for each $0 \leq i \leq k-1$. Because, if we prove this, then each face of Δ_i is a face of $\Delta_i(P)$ and as they have the same support we are done.

We will prove the equality in (1.25) by induction on i. If $i = 0$ this is trivial. For $i > 0$, as

$$S_{P_{i+1}}(\psi_\delta(x^{[i+1]})) = \left\{y_j \in S_{P_i}(\psi_\delta(x^{[i]})) \,\middle|\, \langle y_j\,, \psi_\delta(x^{[i+1]})\rangle = a_j^{(i+1)}\right\}$$

for $\delta \in \mathbb{R}_{>0}$ small enough, we have

$$\begin{aligned} S_{i+1,P}(x) &= \left\{ y_j \in M_{\mathbb{R}} \,\middle|\, 1 \leq j \leq r \text{ and } \langle y_j\,, x\rangle = a_j^{[i+1]} \right\} \\ &= \left\{ y_j \in M_{\mathbb{R}} \,\middle|\, 1 \leq j \leq r,\ y_j \in S_{i,P} \ \text{ and } \langle y_j\,, x\rangle^{(i+1)} = a_j^{(i+1)} \right\} \\ &= \left\{ y_j \in M_{\mathbb{R}} \,\middle|\, 1 \leq j \leq r,\ y_j \in S_{P_i}(\psi_\delta(x^{[i]})) \ \text{ and } \langle y_j\,, x\rangle^{(i+1)} = a_j^{(i+1)} \right\} \\ &= \left\{ y_j \in M_{\mathbb{R}} \,\middle|\, 1 \leq j \leq r,\ y_j \in S_{P_i}(\psi_\delta(x^{[i]})) \ \text{ and } \langle y_j\,, \psi_\delta(x^{[i+1]})\rangle = a_j^{(i+1)} \right\} \\ &= S_{P_{i+1}}(\psi_\delta(x^{[i+1]})) \end{aligned}$$

□

Theorem 1.12.4 (Local Duality)**.** *Given an $\mathbb{R}$-rational polyhedron P, we can recover the normal fan of P from the layered normal fan as*

$$\mathrm{NF}(P) = T\mathcal{C}\,\underline{\Delta}(P).$$

In the sense that, $\mathrm{NF}(P)$ *is the fan consisting of all the polyhedral cones of the form* $T\mathcal{C}\,\underline{\delta}$ *where*

$$\underline{\delta}\colon \sigma_{k-1} \subseteq \sigma_{k-2} \cdots \subseteq \sigma_0$$

is a layered face of $\underline{\Delta}$*.*

Proof. Fix a point $x \in P$ and a non-reduced representation $P = \{y_1 \geq a_1, \ldots, y_r \geq a_r\}$.

Using x, we can construct a face of $\mathrm{NF}(P)$ by considering the normal cone $\mathrm{C}(F)$ of the face F of P such that $x \in \mathrm{int}(F)$.

On the other hand, using the same x, by the definition in part (3) of Proposition 1.12.2 we can construct a layered face $\underline{\delta}(x)$ of $\underline{\Delta}(P)$ by

$$\underline{\delta}(x)\colon\ \mathrm{cone}_{\mathbb{R}}\left(S_{k-1}(x)\right) \subseteq \cdots \subseteq \mathrm{cone}_{\mathbb{R}}\left(S_0(x)\right).$$

To prove the theorem, it will be enough to show that

$$\mathrm{C}(F) = T\mathcal{C}\,\underline{\delta}(x).$$

In order to do this, notice that by Proposition 1.8.2 we can write $\mathrm{C}(F)$ as

$$\mathrm{C}(F) = \mathrm{cone}_{\mathbb{D}}\left(\varepsilon^{k-\alpha_1} y_1, \ldots, \varepsilon^{k-\alpha_r} y_r\right)$$

where $\alpha_i = \operatorname{ord}\big(\langle y_i\,, x\rangle - a_i\big)$, i.e, α_i is the biggest integer in $\{0,\ldots,k\}$ such that

$$\varepsilon^{k-\alpha_i}\langle y_i\,, x\rangle = \varepsilon^{k-\alpha_i}a_i \iff \langle y_j\,, x\rangle^{[\alpha_i-1]} = a_j^{[\alpha_i-1]} \iff y_i \in S_{\alpha_i-1}(x).$$

Hence, we can write this normal cone as

$$\begin{aligned} \mathrm{C}(F) &= \operatorname{cone}_{\mathbb{D}}\left(\bigcup_{i=0}^{k-1}\left\{\varepsilon^{k-1-i}y_j \;\middle|\; y_j \in S_i(x)\right\}\right) \\ &= \operatorname{cone}_{\mathbb{D}}\left(\bigcup_{i=0}^{k-1}\left\{\varepsilon^{i}y_j \;\middle|\; y_j \in S_{k-1-i}(x)\right\}\right) \end{aligned}$$

which is exactly equal to $\mathcal{TC}\,\underline{\delta}(x)$ by Theorem 1.11.2 part (2). This finishes the proof. □

Remark 1.12.5. In particular, we see that the normal type of an $\mathbb{R}$-rational polyhedron, that is, the information encoded in its normal fan, is equivalent to the data of a sequence of length k normal types of real polyhedra each of them refining the previous one.

1.13 Regular Subdivisions

In this section we will extend the notion of regular subdivision of a polytope to the polyhedral geometry over $\mathbb{D}$. Moreover, in a similar way as we did in the previous section, we will study how this concept relates to *layered regular subdivisions*, which are defined in analogy to the *layered normal fans* of the previous section.

Using the extended perfect pairing of Definition 1.1.12 we can introduce the following concept.

Definition 1.13.1 (Regular subdivisions over $\mathbb{D}$)**.** Consider a finite subset $A \subseteq M_{\mathbb{D}}$, a function $h : A \to \mathbb{D}$, which we refer to as a *height function* on A, and the polytope $P = \operatorname{conv}_{\mathbb{D}}(A)$.

1. The *lifted convex hull* of A is the set

$$\operatorname{conv}^h_{\mathbb{D}}(A) := \operatorname{conv}_{\mathbb{D}}\{(a, h(a)) \in M_{\mathbb{D}} \times \mathbb{D} \mid a \in A\} \subseteq M_{\mathbb{D}} \times \mathbb{D}.$$

2. The *regular subdivision of P with respect to h*, denoted by $\Delta^h(P)$, is the family

$$\begin{aligned}\Delta^h(P) &:= \left\{ \pi\left(\mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(P))\right) \,\middle|\, x \in N_{\mathbb{D}} \right\} \\ &= \left\{ \pi(F) \,\middle|\, F \textit{ is a lower face of } \mathrm{conv}^h_{\mathbb{D}}(A) \right\},\end{aligned}$$

   where $\pi : M_{\mathbb{D}} \times \mathbb{D} \to M_{\mathbb{D}}$ denotes the projection to the second coordinate. That is, $\Delta^h(P)$ is the projection of all the lower faces of $\mathrm{conv}^h_{\mathbb{D}}(A)$ to $M_{\mathbb{D}}$.

Proposition 1.13.2. *The regular subdivision $\Delta^h(P)$ is a polyhedral complex and the restriction of $M_{\mathbb{D}} \times \mathbb{D} \to M_{\mathbb{D}}$ to the set of lower faces of $\Delta^h(P)$ is injective and has P as image.*

Proof. We start by proving that the set of lower faces of $\mathrm{conv}^h_{\mathbb{D}}(A)$ is a polyhedral complex. For this, consider two lower faces F and G of $\mathrm{conv}^h_{\mathbb{D}}(A)$. They are of the form

$$F = \mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A)) \text{ and } G = \mathrm{face}_{(x',1)}(\mathrm{conv}^h_{\mathbb{D}}(A))$$

for some $x, x' \in N_{\mathbb{D}}$. If $F \cap G \neq \varnothing$ then every element $(y, y_0) \in F \cap G$ minimize simultaneously $\langle (y, y_0), (x, 1)\rangle$ and $\langle (y, y_0), (x', 1)\rangle$. Hence, the minimum of $\langle \cdot, (x, 1) + (x', 1)\rangle$ is achieved if and only if both $\langle \cdot, (x, 1)\rangle$ and $\langle \cdot, (x', 1)\rangle$ achieve their minimum simultaneously. This shows that

$$F \cap G = \mathrm{face}_{(x,1)+(x',1)}(\mathrm{conv}^h_{\mathbb{D}}(A)) = \mathrm{face}_{\left(\frac{x+x'}{2},1\right)}(\mathrm{conv}^h_{\mathbb{D}}(A))$$

which is a lower face.

Similarly, if F is a lower face of $\mathrm{conv}^h_D(A)$ and G is a face of F, then

$$F = \mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A)) \text{ and } G = \mathrm{face}_{(x',x_0)}(\mathrm{conv}^h_{\mathbb{D}}(A))$$

for some $x', x \in M_{\mathbb{D}}$ and $x_0 \in \mathbb{D}$. If x_0 is invertible then $G = \mathrm{face}_{(x'/x_0,1)}(\mathrm{conv}^h_{\mathbb{D}}(A))$, so it is a lower face. If x_0 is not invertible then $1 + x_0$ is invertible and then

$$G = G \cap F = \mathrm{face}_{(x,1)+(x',x_0)}(\mathrm{conv}^h_{\mathbb{D}}(A)) = \mathrm{face}_{(\frac{x+x'}{1+x_0},1)}(\mathrm{conv}^h_{\mathbb{D}}(A)).$$

Hence, G is a lower face as well in this case. This finishes the proof that the set of lower faces defines a polyhedral complex.

Now, notice that the restriction of

$$\begin{aligned} \pi : M_{\mathbb{D}} \times \mathbb{D} &\longrightarrow M_{\mathbb{D}} \\ (x, a) &\longmapsto x \end{aligned}$$

to the set of lower faces gives a bijection onto $\mathrm{conv}^h_{\mathbb{D}}(A)$. Indeed, if we have a lower face $F = \mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A))$ containing an element $(y, y_0) \in F$, then

$$\langle (y, y_0) , (x, 1) \rangle = \langle y , x \rangle + y_0$$

should be minimized among all $(y, y_0) \in \mathrm{conv}^h_{\mathbb{D}}(A)$, in particular we should have

$$y_0 = \min \left\{ y' \in \mathbb{D} \;\middle|\; (y, y') \in \mathrm{conv}^h_{\mathbb{D}}(A) \right\},$$

hence y_0 is uniquely determined in terms of y and the map is injective.

This shows that $\Delta^h(\mathrm{conv}_{\mathbb{D}}(A))$ is a polyhedral complex, as it is the injective image of another polyhedral complex and by Proposition 1.2.4 the image of a polyhedron is a polyhedron. □

In this way, we have introduced the concept of regular subdivisions for a polytope over $\mathbb{D}$. In the next proposition, we introduce the concept of a layered regular subdivision in several equivalent ways.

Proposition 1.13.3 (Layered Regular Subdivisions)**.** *Let $A \subseteq M_{\mathbb{R}}$ be a finite subset of real vectors and consider a height function*

$$\begin{aligned} h : A &\longrightarrow \mathbb{D} \\ a &\longmapsto h(y) = h^{(0)}(y) + \cdots + \varepsilon^{k-1} h^{(k-1)}(y). \end{aligned}$$

We can construct a sequence of subdivisions of $\mathrm{conv}_{\mathbb{R}}(A)$

$$\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A)) \colon \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1} \tag{1.26}$$

in the following equivalent ways:

1. *Δ_0 is the regular subdivision of $\mathrm{conv}_{\mathbb{R}}(A)$ induced by $h^{(0)}$ and, for $0 < i \leq k-1$, Δ_i is the subdivision of Δ_{i-1} obtained by subdividing each cell $\delta \in \Delta_i$ by the regular*

subdivision induced by the height function

$$\begin{aligned} h^{(i)}\mid_{\delta}: \delta \cap A &\longrightarrow \mathbb{R} \\ y &\longmapsto h^{(i)}(y) \end{aligned}$$

2. Δ_i *is the regular subdivision defined by the height function*

$$h^{(0)} + \delta h^{(1)} + \cdots + \delta^i h^{(i)} : A \longrightarrow \mathbb{R}$$

for $\delta \in \mathbb{R}_{>0}$ *small enough.*

3. *Given an element* $x \in N_{\mathbb{D}}$, *for each* $0 \leq i \leq k-1$ *consider the set*

$$S_i^h(x) \coloneqq \arg.\min_{a\in A}\{\langle y\,, x^{[i]}\rangle + h^{[i]}(y)\} \subseteq A.$$

That is, $S_i^h(x)$ *is the set of all* $y \in A$ *for which the expression*

$$\langle y\,, x^{[i]}\rangle + h^{[i]}(y)$$

is minimal among all $y \in A$. *The subdivision* Δ_i *is the one whose cells are the real polyhedra of the form* $\operatorname{conv}_{\mathbb{R}}(S_i^h(x))$ *for some* $x \in N_{\mathbb{D}}$.

Notice that, by item (3) above, Δ_{i+1} *is a refinement of* Δ_i *and, by item (2) above,* Δ_i *is a regular subdivision for each* i*. Conversely, any sequence of regular subdivisions in which each term is a refinement of the previous one is a layered regular subdivision for some height function.*

Proof. Let us call Δ_i^1, Δ_i^2 and Δ_i^3 the subdivisions defined by (1), (2) and (3) respectively.

First, let us see that Δ_i^2 and Δ_i^3 coincide. For this, given $\delta \in \mathbb{R}_{>0}$ consider the map

$$\begin{aligned} \psi_\delta : \mathbb{D} &\longrightarrow \mathbb{R} \\ x &\longmapsto x^{(0)} + \delta x^{(1)} + \cdots + \delta^{k-1} x^{(k-1)} \end{aligned}$$

This map is $\mathbb{R}$-linear and extend to a map $\psi_\delta : N_{\mathbb{D}} \to N_{\mathbb{R}}$. Moreover, for a given $x \in N_{\mathbb{D}}$ it satisfies

$$S_i^h(x) = S_1^{\psi_\varepsilon \circ h}(\psi_\delta(x)) \tag{1.27}$$

for $\delta \in \mathbb{R}_{>0}$ small enough. As we need only finitely many x to cover all the sets of the form $S_i^h(x)$, we can take $\delta > 0$ small enough so (1.27) holds for every set in Δ_i^2. As Δ_i^2 is exactly the regular subdivision induced by the height function $\psi_\delta \circ h$ we get $\Delta_i^1 = \Delta_i^2$.

We will now see that $\Delta_i^1 = \Delta_i^3$. By definition $\Delta_0^1 = \Delta_0^2$ so we are done. This is true by definition if $i = 0$. Now, if $i = 1$ we have that

$$\langle y\,, x^{[1]}\rangle + h(y)^{[1]}$$

is minimal with the lexicographic order among all $y \in A$ iff we have that

$$\langle y\,, x\rangle^{(0)} + h^{(0)}(y)$$

is minimal and, among the ones which are minimal, that is, among $S_h^{(0)}(x^{(0)})$, we have that

$$\langle y\,, x\rangle^{(1)} + h^{(1)}(y)$$

is also minimal. Hence, we get that

$$S_1^{h^{[1]}}(x^{[1]}) = S_0^{h^{(1)}} \mid_{S_0^{h^{(0)}(x^{(0)})}} (x^{(1)})$$

which shows $\Delta_2^1 = \Delta_2^3$. In a similar way, we have

$$S_{i+1}^{h^{[i+1]}}(x^{[i+1]}) = S_1^{h^{(i+1)}} \mid_{S_i^{h^{[i]}(x^{[i]})}} (x^{(i+1)})$$

so $\Delta_i^1 = \Delta_i^3$ follows by induction.

Finally, take a sequence $\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))$ of regular subdivisions of $\mathrm{conv}_{\mathbb{R}}\, A$ in which each term is a refinement of the previous one. By Theorem 2.4 in [GKZ08], if we consider $\Lambda = \{\sigma\}_\sigma$ to be the normal fan of the secondary polytope of A, then, the sequence of subdivision $\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))$ in (1.26) correspond to a flag of cones

$$\underline{\sigma} = \sigma_0 \succeq \sigma_2 \succeq \cdots \geq \sigma_k - 1$$

where each σ_{i+1} is a face of σ_i, and a height function $h^{(i)}$ defines Δ_i iff it is in the relative interior of σ_i. Hence, by taking $h^{(i)}$ in the relative interior of σ_i for each i, we get that

$$h := h^{(0)} + \varepsilon h^{(1)} + \cdots + \varepsilon^{k-1} h^{(k-1)}$$

defines the layered regular subdivision $\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A))$. □

Now, given a set $A \subseteq M_{\mathbb{R}}$ of real vectors and a height function $h : A \to \mathbb{D}$ we can construct two different objects: A regular subdivision for $\mathrm{conv}_{\mathbb{D}}(A)$ and a layered regular subdivision for $\mathrm{conv}_{\mathbb{R}}(A)$. The exact connection between these two objects is given in the following theorem.

Theorem 1.13.4. *Consider a finite set of real points $A \subseteq M_{\mathbb{R}}$ and a height function $h : A \to \mathbb{D}$. Then, we have an equality of the form*

$$\Delta^h(\mathrm{conv}_{\mathbb{D}}(A)) = T\mathcal{C}\,\underline{\Delta}^h(\mathrm{conv}_{\mathbb{R}}(A)).$$

In the sense that, the elements of $\Delta^h(\mathrm{conv}_{\mathbb{D}}(A))$ are exactly the polyhedra of the form $T\mathcal{C}(\underline{F})$ for

$$\underline{F}\colon F_{k-1} \subseteq F_{k-2} \subseteq \cdots \subseteq F_0$$

where F_i is a face of Δ_i for each i.

Lemma 1.13.5. *Given an element $x \in N_{\mathbb{D}}$ and $a \in A$. The integer*

$$\beta = \mathrm{ord}\left(\langle a\,, x\rangle + h(a) - \min_{b\in A}\left(\langle b\,, x\rangle + h(b)\right)\right)$$

is the maximal integer in $\{0, 1\ldots, k\}$ for which $a \in S^h_{\beta-1}(x)$.

Proof. If $\beta = \mathrm{ord}\left(\langle a\,, x\rangle + h(a) - \min_{b\in A}\left(\langle b\,, x\rangle + h(b)\right)\right)$ then β is the maximal element in $\{0, 1, \ldots, k\}$ such that

$$\begin{aligned}
&\varepsilon^{k-\beta}\left(\langle a\,, x\rangle + h(a)\right) = \varepsilon^{k-\beta}\Big(\min_{b\in A}\left(\langle b\,, x\rangle + h(b)\right)\Big)\\
\iff &\left(\langle a\,, x\rangle + h(a)\right)^{[\beta-1]} = \left(\min_{b\in A}\left(\langle b\,, x\rangle + h(b)\right)\right)^{[\beta-1]} = \min_{b\in A}\left(\left(\langle b\,, x\rangle + h(b)\right)^{[\beta-1]}\right)\\
\iff &\langle a\,, x\rangle^{[\beta-1]} + h(a)^{[\beta-1]} \text{ is minimal among all } a \in A\\
\iff &a \in S^h_{\beta-1}(x)
\end{aligned}$$

□

Proof of Theorem 1.13.4. Given $x \in N_{\mathbb{D}}$, we can define a face of $\Delta^h(\mathrm{conv}_{\mathbb{D}}(A))$ by

$$\pi\left(\mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A))\right),$$

on the other hand, the same x defines a layered face in $\underline{\Delta}^h(\mathrm{cone}_{\mathbb{R}}(A)$ by

$$\underline{F}(x) : \mathrm{conv}_{\mathbb{R}}(S^h_{k-1}(x)) \subseteq \cdots \subseteq \mathrm{conv}_{\mathbb{R}}(S^h_0(x)).$$

In this way, it is enough to prove that

$$\pi\left(\mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A))\right) = \mathcal{TC}\,\underline{F}(x) \tag{1.28}$$

For this, from Proposition 1.6.10 we have

$$\begin{aligned} \mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A)) &= \mathrm{wconv}_{\mathbb{D}}\left([\varepsilon^{k-\beta_a}(a, h(a)); k-\beta_a] \,\middle|\, a \in A\right) \\ \Longrightarrow \pi\left(\mathrm{face}_{(x,1)}(\mathrm{conv}^h_{\mathbb{D}}(A))\right) &= \mathrm{wconv}_{\mathbb{D}}\left([\varepsilon^{k-\beta_a} a; k-\beta_a] \,\middle|\, a \in A\right) \end{aligned}$$

where

$$\beta_a = \mathrm{ord}(\langle (a, h(a))\,, (x,1)\rangle - c) = \mathrm{ord}(\langle a\,, x\rangle + h(a) - c)$$

with c the minimum of $\langle \cdot\,, (x,1)\rangle$ on $\mathrm{conv}^h_{\mathbb{D}}(A)$. As this minimum must be achieved in one of the generators of $\mathrm{conv}^h_{\mathbb{D}}(A)$, we have $c = \min_{b\in A}\left(\langle b\,, x\rangle + h(b)\right)$. Then, by Lemma 1.13.5, β_a is equal to the maximum integer such that $a \in S^h_{\beta_a - 1}(x)$.

On the other hand, by Theorem 1.11.2 we can write explicitly $\mathcal{TC}\,\underline{F}(x)$ in term of the generators of $\underline{F}(x)$ and we get

$$\begin{aligned} \mathcal{TC}\,\underline{F}(x) &= \mathrm{wconv}\left([\varepsilon^i a\,;\, i] \,\middle|\, i \in \{0, \dots, k\}, a \in S^h_{k-1-i}(x)\right) \\ &= \left\{ \sum_{i=0}^{k-1} \sum_{a \in S^h_i(x)} \lambda_{a,i} a \varepsilon^{k-1-i} \,\middle|\, \lambda_{a,i} \geq 0\ \forall a\, \forall i, \text{ and } \sum_{i=0}^{k-1} \sum_{a \in S^h_i(x)} \lambda_{a,i} \varepsilon^{k-1-i} = 1 \right\} \\ &= \Bigg\{ \sum_{a\in A} a(\lambda_{a,\beta_a}\varepsilon^{k-\beta_a} + \lambda_{a,\beta_a-1}\varepsilon^{k-(\beta_a-1)} + \dots \lambda_{a,0}\varepsilon^{k-1}) \\ &\qquad \Bigg|\ \lambda_{a,i} \geq 0\ \forall a \forall i, \text{ and } \sum_{a\in A} \lambda_{a,\beta_a}\varepsilon^{k-\beta_a} + \lambda_{a,\beta_a-1}\varepsilon^{k-(\beta_a-1)} + \dots \lambda_{a,0}\varepsilon^{k-1} = 1 \Bigg\} \\ &= \left\{ \sum_{a\in A} \mu_a a \varepsilon^{k-\beta_a} \,\middle|\, \mu_a \geq 0\ \forall a, \text{ and } \sum_{a\in A} \mu_a \varepsilon^{k-\beta_a} = 1 \right\} \\ &= \mathrm{wconv}_{\mathbb{D}}\left([\varepsilon^{k-\beta_a} a; k-\beta_a] \,\middle|\, a \in A\right) \end{aligned}$$

Where, for the third equality we factorized by a and for the fourth equality we did the change of variable

$$\mu_a = \lambda_{a,\beta_a} + \lambda_{a,\beta_a-1}\varepsilon + \dots \lambda_{a,0}\varepsilon^{k-1}.$$

In this way, we have shown the equality in (1.28) as we wrote both sets in the same way.

□

1.14 Higher Rank Tropical Hypersurfaces.

This section gives a first set of applications of the theory of polyhedral geometry of higher rank to tropical geometry of higher rank. After introducing the basics objects of the theory, in Proposition 1.14.5 we show that higher rank tropical hypersurfaces can naturally be regarded as iterated fibrations. This fibration is studied in the Hypersurface Duality (Theorem 1.14.12). Where we show that the base and each fiber of a higher rank tropical hypersurface consist of tropical hypersurfaces of rank one, moreover, the normal type of these tropical hypersurfaces is encoded in a layered regular subdivision of the Newton polytope. Finally, in Theorem 1.14.17 we put a polyhedral structure over $\mathbb{D}$ on higher rank tropical hypersurfaces which is compatible with the Hypersurface Duality previously presented.

Definition 1.14.1. The *tropical semifield of rank k* or *min-plus algebra of rank k*, is the semifield

$$\mathbb{T}_k = \big(\mathbb{D} \cup \{\infty\},\ \min,\ +\big),$$

were we consider by addition the map $(a, b) \mapsto \min\{a, b\}$ and by multiplication the map $(a, b) \mapsto a + b$.

An expression in $\mathbb{T}_k$ will be written between quotation marks and with the usual symbols $+$ and $\cdot$, for example,

$$\text{“}\sum_{i=1}^{n} x_i y_i\text{”} = \min\{x_i + y_i \mid i = 1, \dots, n\}.$$

Remark 1.14.2.

1. In general, for any ordered abelian group $(\Gamma, +)$ one can consider its associated tropical semifield

$$\mathbb{T}_\Gamma = \big(\Gamma \cup \{\infty\},\ \min,\ +\big).$$

In this way, $\mathbb{T}_k$ corresponds to the case in which the ordered group is $(\mathbb{D},+)$ or, equivalently, $(\mathbb{R}^k,+)$ with its lexicographic order.

2. In $\mathbb{T}_k$ the element ∞ becomes the additive identity as we have $\min\{\infty, a\} = a$ for every $a \in \mathbb{T}_k$. Similarly, 0 becomes the multiplicative identity in $\mathbb{T}_k$. For these reasons we have equalities of the form

$$\text{“}\, x + y\, \text{”} = \text{“}\, 0x + 0y + \infty\, \text{”}.$$

In particular, the coefficient of x in “ $x + y$ ” is 0 and not 1.

Definition 1.14.3. Given a lattice M, the ring of *Laurent tropical polynomials* on M is the set $\mathbb{T}_k[M]$ of all formal sums of the form

$$f = \text{“} \sum_{m \in M} a_m T^m \text{”}$$

whose *support*

$$\mathrm{Supp}(f) \coloneqq \{m \in M \mid a_m \neq \infty\}$$

is a finite set. We endow $\mathbb{T}_k[M]$ with the semiring structure induced by $\mathbb{T}_k$.

Let N be the dual lattice of M. A non-zero tropical polynomial $f = \text{“} \sum_{m \in M} a_m T^m \text{”}$ defines a map from $N_{\mathbb{D}}$, the *tropical torus of rank k*, to $\mathbb{D}$ by

$$\begin{aligned} f \colon N_{\mathbb{D}} &\longrightarrow \mathbb{D} \\ x &\longmapsto f(x) = \min\{\langle m\,, x\rangle + a_m \mid m \in M\}\,. \end{aligned}$$

Definition 1.14.4. Consider a tropical polynomial $f = \text{“} \sum_{m \in M} a_m T^m \text{”} \in \mathbb{T}_k[M]$.

1. A point $x \in N_{\mathbb{D}}$ is said to be a *zero* of f if the minimum in

$$f(x) = \min\{\langle m\,, x\rangle + a_m \mid m \in M\}$$

is achieved at least twice. The set of all zeros of f is denoted by $V(f)$ and is called the *vanishing set* of f.

2. A *tropical hypersurface of rank k* is a set of the form $V(f) \subseteq N_{\mathbb{R}^k}$ for a nonzero tropical polynomial $f \in \mathbb{T}_k[M]$.

Notice that the projections $\mathbb{D} = \mathbb{D}_k \xrightarrow{\pi} \mathbb{D}_k \xrightarrow{\pi} \dots \xrightarrow{\pi} \mathbb{D}_1$ induce, by applying them in each coefficient, the projections

$$\begin{array}{c} \mathbb{T}_k[M] \xrightarrow{\pi} \mathbb{T}_{k-1}[M] \xrightarrow{\pi} \dots \xrightarrow{\pi} \mathbb{T}_1[M] \\ f =: f^{[k-1]} \mapsto \quad f^{[k-2]} \quad \mapsto \dots \mapsto f^{[0]}. \end{array}$$

Using these projection we can get a natural fibered structure on the tropical hypersurface $V(f)$.

Proposition 1.14.5. *For each tropical polynomial $f =$ " $\sum_{m\in M} a_m T^m$ " $\in \mathbb{T}[M]$, the image of $V\big(f^{[i]}\big)$ under $N_{\mathbb{D}_{i+1}} \to N_{\mathbb{D}_i}$ goes inside $V\big(f^{[i-1]}\big)$. In this way, we get an iterated fibration*

$$V(f) = V\big(f^{[k-1]}\big) \to V\big(f^{[k-2]}\big) \to \dots \to V\big(f^{[1]}\big).$$

Given a point $x \in V\big(f^{[i]}\big)$ we denote by $V_x\big(f^{[i+1]}\big)$ the fiber of $V(f)$ at x in this fibration.

Proof. Notice that for a given $x \in N_{\mathbb{D}_{i+1}}$, if $x \in V\big(f^{[i]}\big)$ then $f^{[i]}(x)$ achieves its minimum in two elements $\langle m\,, x\rangle + a_m$ and $\langle n\,, x\rangle + a_n$. That is,

$$f^{[i]}(x) = \langle m\,, x\rangle + a_m^{[i]} = \langle n\,, x\rangle + a_n^{[i]}$$

from which

$$f^{[i-1]}(x^{[i-1]}) = \langle m\,, x^{[i-1]}\rangle + a_m^{[i-1]} = \langle n\,, x^{[i-1]}\rangle + a_n^{[i-1]}.$$

Hence, $f^{[i-1]}(x^{[i-1]})$ also achieves its minimum at least two times, so $x^{[i-1]} \in V\big(f^{[i-1]}\big)$. □

In order to understand this fibration we introduce the following elements.

Definition 1.14.6. Consider a tropical polynomial $f =$ " $\sum_{m\in M} a_m T^m$ " $\in \mathbb{T}_k[M]$.

1. The $\mathbb{R}$-*Newton polytope* of f is the real polytope defined by

$$\mathrm{New}_{\mathbb{R}}(f) \coloneqq \mathrm{conv}_{\mathbb{R}}(\mathrm{Supp}(f)) \subseteq M_{\mathbb{R}}$$

   Similarly, we introduce the $\mathbb{D}$-*Newton polytope* of f by

$$\mathrm{New}_{\mathbb{D}}(f) \coloneqq \mathrm{conv}_{\mathbb{D}}(\mathrm{Supp}(f)) \subseteq M_{\mathbb{D}}$$

2. Given an integer $0 \leq i \leq k-1$ and an element $x \in N_{\mathbb{D}_i}$, the *i-initial part of f with respect to x* is the tropical polynomial

$$\mathrm{in}_x^i(f) = \text{“} \sum_{\substack{m \in M \\ \langle m, x \rangle + a_m^{[i]} = f^{[i]}(x)}} a_m^{(i+1)} T^m \text{”} \in \mathbb{T}[M].$$

Where, for $i+1=k$ we will use the convention

$$a_n^{(k)} := \begin{cases} 0 & \text{if } a_m \neq \infty \\ \infty & \text{if } a_m = \infty. \end{cases}$$

3. The height function

$$h : \mathrm{Supp}(f) \longrightarrow \mathbb{D}$$
$$m \longmapsto a_m$$

naturally induce a layered regular subdivision on $\mathrm{New}_{\mathbb{R}}(f)$ which we denote by $\underline{\Delta}(f)$ and a regular subdivision on $\mathrm{New}_{\mathbb{D}}(f)$ which we denote by $\Delta(f)$.

Remark 1.14.7.

1. By definition (3) on Proposition 1.13.3, the layered regular subdivision $\underline{\Delta}(f)$ is defined to be the one whose layered faces are of the form

$$\underline{F}(x) := \mathrm{conv}_{\mathbb{R}}(\mathrm{Supp}(\mathrm{in}_x^{k-1}(f))) \subseteq \cdots \subseteq \mathrm{conv}_{\mathbb{R}}(\mathrm{Supp}(\mathrm{in}_x^0(f))).$$

where x moves over all elements $x \in N_{\mathbb{D}}$. Hence, $\underline{\Delta}(f)$ encodes all the possible values for the vector

$$(\mathrm{in}_x^0(f), \mathrm{in}_x^1(f), \ldots, \mathrm{in}_x^{k-1}(f))$$

as x moves around $N_{\mathbb{D}}$.

2. By Corollary 1.11.3 we have

$$\mathrm{New}_{\mathbb{D}}(f) = T\mathcal{C}^{k-1}\,\mathrm{New}_{\mathbb{R}}(f).$$

Moreover, by Theorem 1.13.4 this equality can be lifted to an equality of subdivisions of the form

$$\Delta(f) = T\mathcal{C}\,\underline{\Delta}(f).$$

The first important result of this section is the *Higher Rank Hypersurface Duality Theorem* below, which states that the layered regular subdivision $\underline{\Delta}(f)$ obtained by using as height function the coefficients of f, allow us to obtain the normal type of both the base and the fibers in the iterated fibration of Proposition 1.14.5.

In order to introduce this, let us recall the following concept.

Definition 1.14.8. Given polyhedral complexes Σ in $M_{\mathbb{D}}$ and Σ' in $N_{\mathbb{D}}$. A duality between Σ and Σ' is a map $\Lambda : \Sigma \to \Sigma'$ such that

1. The map Λ is a bijection.

2. Given faces $F, G \in \Sigma$, whenever $F \vee G$ exists we have that $\Lambda(F) \wedge \Lambda(G)$ exists and

$$\Lambda(F \vee G) = \Lambda(F) \wedge \Lambda(G).$$

3. Similarly, given faces $F, G \in \Sigma$, whenever $F \wedge G$ exists we have that $\Lambda(F) \vee \Lambda(G)$ exists and

$$\Lambda(F \wedge G) = \Lambda(F) \vee \Lambda(G).$$

4. For each $F \in \Sigma$, $\Lambda(F)$ is orthogonal to F in the sense that

$$\langle x\,, y\rangle = 0,\ \forall\, x \in F, y \in F'$$

Because of properties (2) and (3) we say that Λ preserves *incidences*.

Remark 1.14.9.

1. Given $F, G \in \Sigma$ we have $F \preceq G$ iff $\Lambda(G) \preceq \Lambda(F)$. Indeed,

$$F \preceq G \iff F \wedge G = G \iff \Lambda(G) \vee \Lambda(F) = \Lambda(G) \iff \Lambda(G) \preceq \Lambda(F).$$

2. In the case in which Σ and Σ' are real polyhedral complex, i.e, in rank 1. We have that

$$\dim(F) = \operatorname{codim}(\Lambda(F)).$$

Indeed, a maximal flag

$$F_0 \preceq \cdots \preceq F_{dim(F)} \coloneqq F$$

gives rise to a maximal flag

$$\Lambda(F) = \Lambda(F_{dim(F)}) \preceq \cdots \preceq \Lambda(F_0).$$

Let us recall the following fact from the usual theory of tropical geometry. For a proof of this result, we refer to [MS15] Theorem .

Theorem 1.14.10 (Hypersurface Duality)**.** *Given $f \in \mathbb{T}[M]$, if we denote by $\Delta(f)$ the regular subdivision of* New(f) *induced by the coefficients of f. Then, there is a polyhedral complex* GC(f)*, called its Gröbner complex, whose support is $N_{\mathbb{D}}$ and whose cells are parametrized by the faces $F \in \Delta(f)$. Explicitly they are given by*

$$\mathrm{GC}(F) = \{x \in N_{\mathbb{R}} \mid \mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x(f))) \supseteq F\}.$$

Moreover, the map

$$\begin{aligned} \Lambda : \Delta(f) &\longrightarrow \mathrm{GC}(f) \\ F &\longmapsto GC(F) \end{aligned}$$

is a duality in the sense of Definition 1.14.8. Furthermore, if we restrict Λ to the elements of $\Delta(f)$ that are not points we obtain a subcomplex $\Sigma(f)$ of GC(f) *whose support is $V(f)$.*

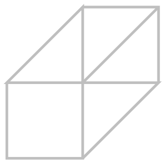
In a explicit way, to obtain the shape of the tropical hypersurface we have to

1. Do a point reflection of $\Delta(f)$.
2. Consider one point x_F for each facet F of the reflected $\Delta(f)$.
3. Join the different points according to the incidence of $\Delta(f)$.
4. Draw a cone pointed at x_F perpendicular to each face of F laying in the boundary of the Newton polytope.

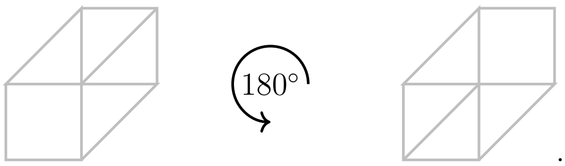
Example 1.14.11. If $f =$ "$7x^2y^2 + 5x^2y + 5xy^2 + 4xy + 2x + 2y + 0$" then, the subdivision

$\Delta(f)$ of $\text{New}(f)$ looks like

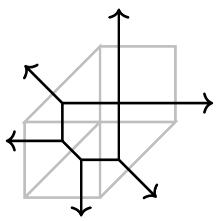
$$\underline{\Delta}(f) =$$



If we do a point reflection of it we get



Hence, the shape of the tropical hypersurface in this case is



Theorem 1.14.12 (Higher Rank Hypersurface Duality)**.** *Let $f \in \mathbb{T}_k[M]$ be a non-zero polynomial and consider*

$$\underline{\Delta}(f) = \Delta_0 \preceq \Delta_1 \preceq \cdots \preceq \Delta_{k-1}$$

the layered regular subdivision induce by f over New f*. Then, we have that:*

1. *The base $V\left(f^{[0]}\right)$ is a rank one tropical hypersurface with the structure of a polyhedral complex dual to the first subdivision Δ_0.*

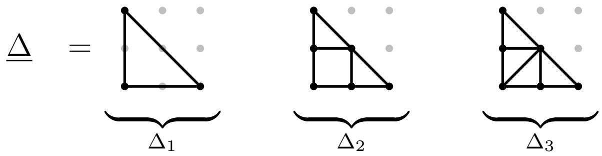
2. *For each $x \in V\left(f^{[0]}\right)$, the fiber $V_x\left(f^{[1]}\right)$ is also a rank one tropical hypersurface. Moreover, $V_x\left(f^{[1]}\right)$ remains constant as x varies over the interior of a cell* $\text{GC}(F) \subseteq V\left(f^{[0]}\right)$ *for some $F \in \Delta_1$ and the normal type of $V_x\left(f^{[1]}\right)$ is dual to the subdivision Δ_1 restricted to F.*

3. *More generally, for each $x \in V\left(f^{[i]}\right)$, the fiber $V_x\left(f^{[i+1]}\right)$ is also a rank one tropical hypersurface. It remains constant as x varies over the interior of a cell* $\text{GC}(F) \subseteq V_{x^{[i-1]}}\left(f^{[i]}\right)$ *for some $F \in \Delta_{i-1}$ and the normal type of $V_x\left(f^{[i+1]}\right)$ is dual to the subdivision Δ_{i+1} restricted to F.*

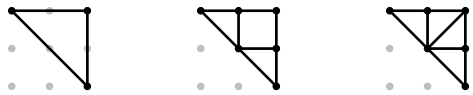
Example 1.14.13. Consider $k = 3$, $M = \mathbb{Z}^2$ and the polynomial

$$f(x, y) = (0, 1, 2) + (0, 1, 1)x + (0, 1, 1)y + (0, 1, 2)xy + (0, 0, 0)x^2 + (0, 0, 0)y^2$$

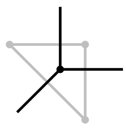
The Newton polytope of f is $\operatorname{New} f = \operatorname{conv}_{\mathbb{R}}((0, 0), (2, 0), (0, 2))$ and its associated layered subdivision is the following:



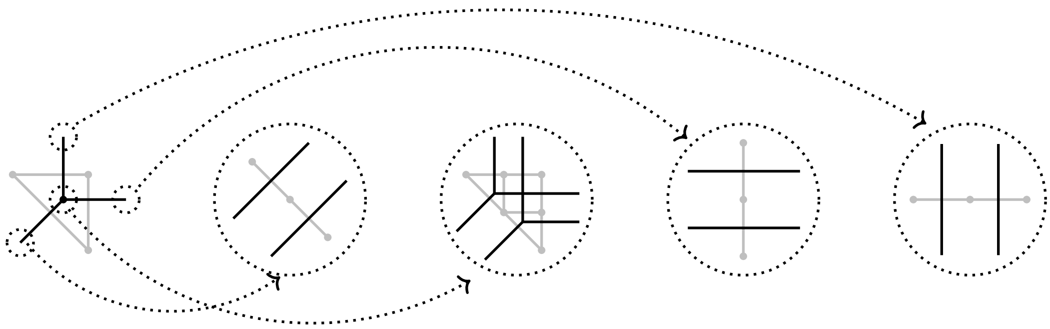
After a point reflection it becomes



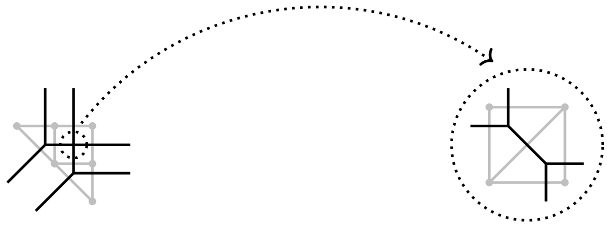
Therefore, the base of the fibration $V(f^{[1]})$ has the shape



And over each point of the base, there are 4 possible shapes for the fibers of $V(f^{[2]})$, represented in the following diagram:



Moreover, each of these fibers is the base for a fibration determined by $V(f^{[3]})$. All the fibers of this fibrations will have the shape of the corresponding tangent cone, with the exception of one fiber, the one corresponding to the subdivision of the square, which we sketch as follows:



Proof of Theorem 1.14.12. The subdivision Δ_0 corresponds to the regular subdivision induced by the coordinates of $f^{[0]}$. Hence, part (1) of the theorem follows directly from the hypersurface duality of rank one (Theorem 1.14.10).

In order to prove (2), let $x^{(0)} \in V(f^{[0]})$, then $V_{x^{(0)}}(f^{[1]})$ is the set of all $x^{(1)} \in N_{\mathbb{R}}$ such that

$$x^{(0)} + \varepsilon x^{(1)} \in N_{\mathbb{D}_2}.$$

Therefore, if we consider the polynomial

$$\mathrm{in}^0_{x^{(0)}}(f) = \text{``} \sum_{\substack{m \in M \\ f^{[0]}(x) = \langle m, x^{(0)} \rangle + a^{[0]}_m}} a^{(1)}_m T^m \text{''}$$

we see that $x^{(0)} + \varepsilon x^{(1)}$ is a zero of $f^{[1]}$ if and only if $x^{(0)}$ is a zero of $f^{[0]}$ and $x^{(1)}$ is a zero of $\mathrm{in}^1_{x^{(0)}}(f)$. Hence, we obtain

$$V_x(f^{[1]}) = V(\mathrm{in}^0_{x^{(0)}}(f)).$$

As $\mathrm{New}(\mathrm{in}^0_{x^{(0)}} f) = F$, again by the original hypersurface duality, we get that $V(\mathrm{in}^0_{x^{(0)}} f)$ is dual to the regular subdivision induced by the height function $m \mapsto a^{(1)}_m$, which, by Proposition 1.13.3, is exactly Δ_1.

The general case follows similarly as one can show that

$$V_{x^{[i]}}(f^{[i+1]}) = V(\mathrm{in}^i_{x^{[i]}} f).$$

□

The objective now is to put a polyhedral structure on $V(f)$ which is dual to the layered regular subdivision of its Newton polytope in a natural way. Generalizing to higher rank the polyhedral part of Theorem 1.14.10. For this, we will introduce the analog of the Gröbner complex in higher rank.

Definition 1.14.14. Given a layered face $\underline{F} \in \underline{\Delta}(f)$ of the form

$$\underline{F}\colon F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k-1}$$

where F_i is a face of Δ_i for each i. We consider its corresponding *Gröbner cell* as

$$\mathrm{GC}(\underline{F}) \coloneqq \left\{ x \in N_{\mathbb{D}} \;\middle|\; \mathrm{conv}_{\mathbb{R}}(\mathrm{in}^i_x(f)) \supseteq F_i,\ \forall 1 \leq i \leq k \right\}.$$

We will prove that the family of all Gröner cells is a polyhedral complex. For this, the idea is to consider the *lifted Newton polytope*

$$\mathrm{New}_{\mathbb{D}}(f)^h \coloneqq \mathrm{conv}^h_{\mathbb{D}}(\mathrm{Supp}(f)) \subseteq M_{\mathbb{D}} \times \mathbb{D}$$

used to define the regular subdivision as in Definition 1.13.1. We will show that the normal fan of this polytope intersected with $N_{\mathbb{D}} \times \{1\}$ is again a polyhedral complex and its cells are the Gröbner cells.

We start with the following lemmas.

Lemma 1.14.15. *Given a layered face $\underline{F} \in \underline{\Delta}(f)$, we have that*

$$x \in \mathrm{GC}(\underline{F}) \iff (x, 1) \in \mathrm{C}\left((\mathcal{TC}\,\underline{F})^h\right).$$

Where $(\mathcal{TC}\,\underline{F})^h$ represents the only lower face of $\mathrm{New}_{\mathbb{D}}(f)^h$ which projects to $\mathcal{TC}\,\underline{F}$ under $M_{\mathbb{D}} \times \mathbb{D} \to \mathbb{D}$, and $\mathrm{C}\left((\mathcal{TC}\,\underline{F})^h\right) \subseteq N_{\mathbb{D}} \times \mathbb{D}$ is the normal cone of this face. In other words, we have the equality

$$\mathrm{GC}(\underline{F}) \times \{1\} = \mathrm{C}\left((\mathcal{TC}\,\underline{F})^h\right) \cap N_{\mathbb{D}} \times \{1\}.$$

Proof. A point $x \in N_{\mathbb{D}}$ belongs to $\mathrm{GC}(\underline{F})$ if and only if the flag

$$\underline{F}(x) : F_0(x) \subseteq F_2(x) \subseteq \ldots F_{k-1}(x)$$

where $F_i(x) = \mathrm{conv}_{\mathbb{R}}\left(\mathrm{Supp}\left(\mathrm{in}^i_x(f)\right)\right)$ satisfies $F_i(x) \supseteq F_i$ for all $i = 0, \ldots, k-1$, and this happens iff

$$\mathcal{TC}\,\underline{F}(x) \supseteq \mathcal{TC}\,\underline{F}.$$

Moreover, if we consider the projection $\pi : M_{\mathbb{D}} \times \mathbb{D} \to M_{\mathbb{D}}$ then we have that

$$\mathcal{TC}\,\underline{F} = \pi\left((\mathcal{TC}\,F)^h\right)$$

and, by Theorem 1.13.4,

$$\mathcal{TC}\,\underline{F}(x) = \pi\left(\mathrm{face}_{(x,1)}\,\mathrm{New}_{\mathbb{D}}(f)^h\right).$$

Hence, as by Proposition 1.13.2 the map π restricted to the set of lower faces of $\mathrm{New}(f)^h$ is injective, we conclude that $\mathcal{TC}\,\underline{F}(x) \supseteq \mathcal{TC}\,\underline{F}$ happens iff

$$\mathrm{face}_{(x,1)}\,\mathrm{New}_{\mathbb{D}}(f)^h \supseteq (\mathcal{TC}\,\underline{F})^h.$$

Which by definition means $(x,1) \in \mathrm{C}\left((\mathcal{TC}\,\underline{F})^h\right)$. In this way we have seen that

$$x \in \mathrm{GC}(\underline{F}) \iff (x,1) \in \mathrm{C}\left((\mathcal{TC}\,\underline{F})^h\right)$$

as we wanted. ☐

Lemma 1.14.16. *Let σ be a polyhedral cone in $N_{\mathbb{D}} \times \mathbb{D}$ and consider P to be the projection of $\sigma \cap N_{\mathbb{D}} \times \{1\}$ to $N_{\mathbb{D}}$. Then, the map*

$$\begin{aligned} \mathfrak{F}_\sigma &\longrightarrow \mathfrak{F}_P \\ \tau &\longmapsto \pi\left(\tau \cap N_{\mathbb{D}} \times \{1\}\right) \end{aligned}$$

is surjective where $\pi : N_{\mathbb{D}} \times \mathbb{D} \to N_{\mathbb{D}}$ is the usual projection.

Proof. A face of P is of the form $\mathrm{face}_y\,P$ for some $y \in M_{\mathbb{D}}$. Let us consider $a = \min_{x\in P}\langle y\,,x\rangle$. If we show that $(y,-a) \in \sigma^\vee$ we are done, as then we can consider $\mathrm{face}_{(y,-a)}\,\sigma$ and this satisfies

$$\mathrm{face}_{(y,-a)}\,\sigma \cap N_{\mathbb{D}} \times \{1\} = \mathrm{face}_y\,P \times \{1\}.$$

Let us see now that $(y,-a) \in \sigma^\vee$. For this, notice that

$$\sigma \cap N_{\mathbb{D}} \times \{1\} = P \times \{1\}. \tag{1.29}$$

Moreover, as $\langle y\,,x\rangle \geq a$ for any $x \in P$ we get

$$\langle (y,-a)\,,(x,1)\rangle \geq 0 \text{ for any } x \in P \times \{1\}.$$

Now, if we take $(x, b) \in \sigma$ with $b \in \mathbb{D}^{\times}_{>0}$ invertible, by the equality in (1.29), we have $x/b \in P$. Hence,

$$\langle (y, -a), (x, b) \rangle = b\langle (y, -a), (x/b, 1) \rangle \geq 0.$$

On the other hand, take an element of the form $(x, b) \in \sigma$ with b not invertible and consider an element x' in P achieving the minimum of y, that is $\langle (y, -a), (x', 1) \rangle = 0$. Then, we can consider $(x', 1) + (x, b) = (x' + x, 1 + b)$. Now $1 + b$ is invertible, so from the previous step

$$0 \leq \langle (y, -a), ((x' + x, 1 + b)) \rangle = \langle (y, -a), (x', 1) \rangle + \langle (y, -a), (x, b) \rangle = \langle (y, -a), (x, b) \rangle.$$

Hence, $(y, -a)$ is positive in (x, b) for any $(x, b) \in \sigma$. Therefore, $(y, -a) \in \sigma^{\vee}$. □

Theorem 1.14.17 (Polyhedral Structure)**.** *The family*

$$\mathrm{GC}(f) = \{\mathrm{GC}(\underline{F}) \mid \underline{F} \in \underline{\Delta}(f)\}$$

is a polyhedral complex with support $N_{\mathbb{D}}$ called the Gröbner complex *of f.*

Moreover, if we consider only the layered faces of $\underline{\Delta}(f)$ in which F_{k-1} is not a point, that is,

$$\Sigma(f) = \{\mathrm{GC}(\underline{F}) \mid \underline{F} \in \underline{\Delta}(f) \text{ and } F_{k-1} \text{ is not a point}\},$$

we obtain a polyhedral complex with support $V(f)$.

Remark 1.14.18. The Gröbner complex $\mathrm{GC}(f)$ is exactly the subdivision of $N_{\mathbb{D}}$ under which the map

$$x \mapsto \left(\mathrm{in}^0_x(f), \ldots, \mathrm{in}^{k-1}_x(f)\right)$$

is constant over the interior of each cell.

Proof of Theorem 1.14.17. We will start by showing that $\mathrm{GC}(f)$ is a polyhedral complex. First notice that, by Lemma 1.14.15, we have $\mathrm{GC}(\underline{F}) = \pi\left(\mathrm{C}\left((\mathcal{TC}\,\underline{\delta})^h\right) \cap N_{\mathbb{D}} \times \{1\}\right)$, which in particular implies that $\mathrm{GC}(\underline{F})$ is a polyhedron for each $\underline{F} \in \underline{\Delta}$. Moreover, given $\underline{F}, \underline{F}' \in$

$\underline{\Delta}(f)$, we can consider $\underline{F} \vee \underline{F}'$ the layered face given by $F_i \vee F_i'$. Then,

$$\begin{aligned}
\mathrm{GC}(\underline{F}) \cap \mathrm{GC}(\underline{F}') &= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h\right)\right) \cap \pi\left(\mathrm{C}\left(\mathcal{TC}(\underline{F}')^h\right)\right) \\
&= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h\right) \cap \mathrm{C}\left(\mathcal{TC}(\underline{F}')^h\right)\right) \\
&= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h \vee \mathcal{TC}(\underline{F}')^h\right)\right) \\
&= \pi\left(\mathrm{C}\left(\left(\mathcal{TC}\left(\underline{F}\right) \vee \mathcal{TC}(\underline{F}')\right)^h\right)\right) \\
&= \pi\left(\mathrm{C}\left(\mathcal{TC}\left(\underline{F} \vee \underline{F}'\right)^h\right)\right) \\
&= \mathrm{GC}(\underline{F} \vee \underline{F}').
\end{aligned}$$

Therefore $\mathrm{GC}(\underline{F}) \cap \mathrm{GC}(\underline{F}') = \mathrm{GC}(\underline{F} \vee \underline{F}') \in \mathrm{GC}(f)$ and it is a face of both $\mathrm{GC}(F)$ and $\mathrm{GC}(F')$ because $\mathrm{C}\left(\mathcal{TC}\left(\underline{F} \vee \underline{F}'\right)^h\right)$ is a face of both $\mathrm{C}\left(\mathcal{TC}\left(\underline{F}\right)^h\right)$ and $\mathrm{C}\left(\mathcal{TC}\left(\underline{F}'\right)^h\right)$.

Finally, if H is a face of $\mathrm{GC}(\underline{F})$ we will show that $H = \mathrm{GC}(\underline{F}')$ for some $\underline{F}' \in \underline{\Delta}(f)$. For this, notice that by Lemma 1.14.16 there is a face τ of $\mathrm{C}(\mathcal{TC}(\underline{F})^h)$ such that $\tau \cap N_{\mathbb{D}} \times \{1\} = H \times \{1\}$. Now, given $x \in \mathrm{int}(H)$ we have that $(x, 1) \in \mathrm{int}(\tau)$. Then, $\mathrm{face}_{(x,1)} \mathrm{New}(f)^h$ is a lower face of $\mathrm{New}(f)^h$ with

$$C\left(\mathrm{face}_{(x,1)} \mathrm{New}(f)^h\right) = \tau.$$

By Theorem 1.13.4, the projection of $\mathrm{face}_{(x,1)} \mathrm{New}(f)^h$ to $\mathrm{New}(f)$ is of the form $\mathcal{TC}\left(\underline{F}'\right)$ for some $\underline{F}' \in \underline{\Delta}(f)$. This $\underline{F}'$ satisfies $\mathrm{GC}(\underline{F}') = H$.

With this we have prove that $\mathrm{GC}(f)$ is a polyhedral complex with support

$$\pi\left(\left|\mathrm{NF}\left(\mathrm{New}(f)^h\right)\right| \cap N_{\mathbb{D}} \times \{1\}\right) = N_{\mathbb{D}}.$$

In order to see that $\Sigma(f)$ is a polyhedral complex, it is enough to notice that if $\mathrm{GC}(\underline{F}), \mathrm{GC}(\underline{F}) \in \mathrm{GC}(f)$ then F_{k-1} and F'_{k-1} are not points, hence $F_{k-1} \vee F'_{k-1}$ is not a point, so

$$\mathrm{GC}(\underline{F}) \cap \mathrm{GC}(\underline{F}) = \mathrm{GC}(\underline{F} \vee \underline{F}) \in \mathrm{GC}(f).$$

Similarly, if $\mathrm{GC}(\underline{F}) \in \mathrm{GC}(f)$ and $\mathrm{GC}(\underline{F}')$ is a face it, then $F'_{k-1} \supseteq F_{k-1}$. Hence, F'_{k-1} is also not a point, that is, $\mathrm{GC}(\underline{F}') \in \Sigma(f)$. This shows that $\Sigma(f)$ is a polyhedral complex.

Moreover, let us see that the support of $\Sigma(f)$ is $V(f)$. If $x \in V(f)$ then we can consider

$$\underline{F}(x) : \mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x^{k-1}(f)) \subseteq \cdots \subseteq \mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x^0(f))$$

and, as the minimum in $f(x)$ is attained at least twice, we have that $\mathrm{conv}(\mathrm{Supp}(\mathrm{in}_x^{k-1}(f))$ is not a point. Hence,

$$x \in \mathrm{GC}(\underline{F}(x)) \subseteq |\Sigma(f)|,$$

and we conclude that $V(f) \subseteq |\Sigma(f)|$. On the other hand, if $x \in |\Sigma(f)|$ then there is a face $\underline{F} \in \underline{\Delta}(f)$ such that $x \in \mathrm{GC}(\underline{F})$. Hence,

$$\mathrm{Supp}\left(\mathrm{in}_x^{k-1}(f)\right) \supseteq F_{k-1},$$

and as F_{k-1} is not a point, the minimum in $f(x)$ is attained at least twice, so $x \in V(f)$. $\square$

Chapter 2

Geometry of Higher Rank Valuations

In this chapter, we study geometric aspects of higher rank valuations and the geometry of analytified spaces in this setting. In particular, we provide a higher rank notion of skeleton.

Basic notations

Along the text we work with varieties over an algebraically closed field κ, that is, integral schemes of finite type over κ. Points on varieties are not necessarily closed. Moreover, we use the notations $\mathbb{R}_+ = \{a \in \mathbb{R} \mid a \geq 0\}$ and $\mathbb{Z}_+ = \{a \in \mathbb{Z} \mid a \geq 0\}$.

In the following, we will denote by $\leq_{\text{cw}}$ the coordinate-wise partial order on $\mathbb{Z}^I$, that is, given elements $\beta, \beta' \in \mathbb{Z}^I$, we have $\beta \leq_{\text{cw}} \beta'$ if and only if $\beta_i \leq_{\text{cw}} \beta'_i$ for each $i \in I$. Sometimes, we only use $\leq$ if the partial order is understood from the context.

We write the symbol $a \gg b$ to indicate that a is large enough compared to b.

For a ring R, we denote by $R^\times$ the set of invertible elements of R.

2.1 Cone complexes and tangent cones

In this section, we introduce the polyhedral geometry notions used along the document. This includes the notion of cone complexes and their tangent cones, as well as dual cone complexes associated to simple normal crossing divisors. We endow cone complexes with the sheaf of tropical functions.

2.1.1 Cone complexes

All through this section, the letter N is used for a free $\mathbb{Z}$-module of finite rank and M denotes the dual of N, that is $M = N^\vee := \mathrm{Hom}(N, \mathbb{Z})$. We denote by $N_\mathbb{R}$ and $M_\mathbb{R}$ the corresponding real vector spaces which are dual to each other. Note that N and M form full rank lattices in $N_\mathbb{R}$ and $M_\mathbb{R}$, respectively. The duality pairing between M and N is denoted by $\langle\, , \rangle$. Recall that a *saturated sublattice* of N is a subgroup N' with the property that $N'_\mathbb{R} \cap N = N'$.

Definition 2.1.1 (Cones and cone complexes)**.**

1. A *rational polyhedral cone* in $N_\mathbb{R}$ is a set of the form

$$\sigma = \{x \in N_\mathbb{R} \mid \langle x, u_1\rangle \geq 0, \dots, \langle x, u_k\rangle \geq 0\}$$

   for some $u_1, \dots, u_k \in M = N^\vee$. We say that σ is *strictly convex* if it does not contain any line in $N_\mathbb{R}$.

2. A *rational polyhedral cone complex with (weak) integral structure* is a pair $(\Sigma, |\Sigma|)$ where $|\Sigma|$ is a topological space and Σ is a family of closed subsets of $|\Sigma|$ such that:

   (a) Each $\sigma \in \Sigma$ is enriched with a lattice N_σ and an identification of σ with a full dimensional rational strictly convex polyhedral cone in $N_{\sigma,\mathbb{R}}$.

   (b) These identifications are compatible in the sense that for each element $\sigma \in \Sigma$, faces of σ seen as a cone in $N_{\sigma,\mathbb{R}}$ correspond to elements τ of Σ. Under this identification, the lattice N_τ is identified with a saturated sublattice of N_σ.

   (c) As a set we have a disjoint union $|\Sigma| = \bigsqcup_{\sigma\in\Sigma} \mathring{\sigma}$ where $\mathring{\sigma}$ is the relative interior of σ (which make sense by (a)).

   (d) The intersection of two elements in Σ can be written as a union of elements in Σ.

We call $|\Sigma|$ the *support* of the cone complex, and the elements of Σ are called the cones or faces of the cone complex. By an abuse of the notation, we will only use Σ to refer to the pair $(\Sigma, |\Sigma|)$. For each cone σ, the lattice N_σ is called its *underlying integral structure* and we identify σ with its image in $N_{\sigma,\mathbb{R}}$.

(3) A cone of dimension one in Σ is called a *ray* and a cone of maximal dimension is called a *facet*. The set of all rays of Σ (resp. of a cone σ in Σ) is denoted by Σ_1 (resp.

σ_1). More generally, for any integer k, we denote by Σ_k (resp. σ_k) the set of all faces of Σ (resp. of σ) of dimension k.

Convention

In what follows, a rational strictly convex polyhedral cone will be simply called a cone as these are the only kind of cones we will deal with in this paper. Similarly, a rational polyhedral cone complex with a (weak) integral structure will be called simply a cone complex.

Remark 2.1.2. Definition 2.1.1 resembles the notion of a fan used in toric geometry but it differs from it in several ways. First, the lattices N_σ may not come simultaneously from a global ambient lattice N. Also, condition (b) allows an intersection of two cones to be a union of multiple faces instead of a single face as in the case of fans. In such a situation, the cone complex will have *parallel faces*, that is, two different faces τ and σ in Σ with the same set of rays $\tau_1 = \sigma_1$.

Definition 2.1.3 (Subdivision)**.** A *rational subdivision* of a cone complex Σ is a rational cone complex $\tilde{\Sigma}$ such that $|\Sigma| = |\tilde{\Sigma}|$ and for each cone $\tilde{\sigma} \in \tilde{\Sigma}$, there is a cone $\sigma \in \Sigma$ such that $\tilde{\sigma} \subseteq \sigma$ and $N_{\tilde{\sigma}}$ is a saturated sublattice of N_σ.

It follows from the definition that $\tilde{\sigma}$ is a rational cone in $N_{\sigma,\mathbb{R}}$. Again, rational subdivisions are the only ones appearing in this paper, so we drop sometimes the word rational and simply talk about subdivisions.

2.1.2 Dual complexes

We recall the concept of a simple normal crossing divisor (SNC) on a variety and its associated dual cone complex.

Definition 2.1.4 (SNC divisor and stratum)**.** Let X be a smooth variety and D a divisor on it.

1. The divisor D on X is called *simple normal crossing*, *SNC* in short, if

   - D is reduced, and
   - for each point $x \in X$, there is a Zariski neighborhood U_x of x and a regular system of parameters $z_1, \dots, z_r \in \mathcal{O}_{X,x}$ with $r = \operatorname{codim} \overline{\{x\}}$ such that the zero set of the product $z_1 \dots z_j$ over U_x coincides with $D \cap U_x$ for some non-negative integer $j = j_x \leq r$.

2. Given an SNC divisor D on X, we can write D as a sum $\sum_{i\in\mathcal{I}} D_i$ where D_i are the irreducible components of D. A connected component of an intersection of the form

$$D_I := \bigcap_{i\in I} D_i$$

for some $I \subseteq \mathcal{I}$ is called a *stratum* of D.

Remark 2.1.5. Notice that each SNC divisor is a Cartier divisor. Moreover, the SNC condition implies that each D_I appearing above is smooth, and in particular, has disjoint irreducible components, coinciding with its connected components.

Construction 2.1.6 (Dual complex)**.** Given a divisor $D = \sum_{i\in\mathcal{I}} D_i$ on a variety X we construct its *dual cone complex* $\Sigma(D)$ as follows. To each stratum S of D which is an irreducible component of D_I for a subset $I \subseteq \mathcal{I}$, one associates a cone σ_S which is a copy of $\mathbb{R}^I_+ \subseteq \mathbb{R}^{\mathcal{I}}$ with its natural integral structure given by the lattice $\mathbb{Z}^I \subset \mathbb{Z}^{\mathcal{I}}$. If a stratum S is included in another stratum T, then the subset $I \subseteq \mathcal{I}$ which corresponds to S should contain the subset $J \subseteq \mathcal{I}$ which corresponds to T. In particular, one can naturally identify the cone σ_T as a face of σ_S via the identification $\mathbb{R}^I_+ \subseteq \mathbb{R}^J_+$, as the set of all points with zero coordinates corresponding to elements of $J \setminus I$. The topological space $|\Sigma(D)|$ is defined as the gluing of all σ_S along these identifications and the set $\Sigma(D)$ is given as the image of the family $\{\sigma_S\}$ in the space $|\Sigma(D)|$. Sometimes we use the notation $\Sigma(X, D)$ to emphasize that D is a divisor in X. ◇

Proposition 2.1.7. *The pair* $(\Sigma(D), |\Sigma(D)|)$ *constructed above is a cone complex in the terminology of Definition 2.1.1.*

Proof. Part (a) in the definition follows from the definition of each σ_S and the fact they map injectively into the gluing $|\Sigma|$. For part (b) notice that given a cone $\sigma_S = \mathbb{R}^I_+$, a face of σ_S is then of the form $\mathbb{R}^J_+$ for some $J \subseteq I$. Therefor, the unique irreducible component T of D_J which contains S will give the unique $\sigma_T \in \Sigma(D)$ such that $\sigma_T = \mathbb{R}^J_+$. Part (c) follows from the corresponding fact on $\mathbb{R}^I$ by looking at the maximal elements in Σ. Finally, for part (d), notice that $\sigma_{S_1} \cap \sigma_{S_2}$ is equal the union of all σ_T where T is a minimal strata containing both S_1 and S_2. □

Notation 2.1.8. Notations as above, given a cone $\sigma \in \Sigma(D)$, we denote by S_σ the associated stratum. The generic point of S_σ is denoted by η_σ. If the divisor is given by $D = \sum_{i\in\mathcal{I}} D_i$, we denote by I_σ the subset $I \subseteq \mathcal{I}$ such that the stratum S_σ is open in a connected component of $D_I = \bigcap_{i\in I} D_i$.

2.1.3 Tropical functions

We endow a cone complex Σ with its *structure sheaf* $\mathcal{O}_\Sigma$ which is the sheaf of *tropical functions.*

Definition 2.1.9 (Tropical functions and the structure sheaf)**.** Let Σ be a cone complex with an integral structure and let U be an open subset of $\left|\Sigma\right|$. A function

$$F\colon U \to \mathbb{R}$$

is called *tropical* if there is a rational subdivision $\widetilde{\Sigma}$ of Σ such that for each $\sigma \in \widetilde{\Sigma}$, the restriction $F|_{\sigma\cap U}$ is integral linear, i.e., viewing σ in $N_{\sigma,\mathbb{R}}$, $F|_{\sigma\cap U}$ coincides with the restriction to $\sigma \cap U$ of an element in $M_\sigma \subseteq N^\vee_{\sigma,\mathbb{R}}$. The *structure sheaf* $\mathcal{O}_\Sigma$ is defined as the one whose sections on an open set U are given by the set of tropical functions on U. A tropical function on Σ is a global section of $\mathcal{O}_\Sigma$.

Remark 2.1.10. Let X be a smooth variety and D an SNC divisor on X. As we will see in the next section, when talking about tropicalization, the tropicalization of a rational function on X is a tropical function on $\Sigma(D)$. Later on we will prove that any tropical function is of this form.

2.1.4 Tangent cones

Now we see how to deal with tangent vectors in cone complexes. We are specially interested in those that *point inward* the cone complex. We first start by introducing a notion of tangent spaces adapted to our purposes.

Definition 2.1.11 (Tangent spaces)**.** Given a cone complex Σ and $x \in |\Sigma|$, the *tangent space at* x is the set

$$T_x\Sigma \coloneqq \bigsqcup_{\sigma \ni x} N_{\sigma,\mathbb{R}}$$

where the union goes over all faces of Σ containing x.

Remark 2.1.12. In general $T_x\Sigma$ is not a vector space, nonetheless given $w \in T_x\Sigma$ and $\lambda \in \mathbb{R}$ we can produce by addition the point $x + \lambda w \in \bigcup_{\sigma\in\Sigma} N_{\sigma,\mathbb{R}}$.

Definition 2.1.13 (Tangent cones)**.** Let Σ be a cone complex and $x \in |\Sigma|$.

1. The *tangent cone at* x denoted by $T\mathcal{C}_x\,\Sigma$ is the set of all $w \in T_x\Sigma$ for which $x + \varepsilon w \in |\Sigma|$ provided that $\varepsilon > 0$ is small enough.

2. More generally, for an integer $k \geq 1$, the *k-tangent cone at x* denoted by $\mathcal{TC}_x^k \Sigma$ consists of the set of all tuples $\underline{w} = (w_1, \dots, w_k)$ of vectors in $(T_x\Sigma)^k$ for which we have the following property:

   For any $r \in [k]$ and for $\varepsilon_i > 0$, $i \in [r]$, we have

$$x + \varepsilon_1 w_1 + \cdots + \varepsilon_r w_r \in |\Sigma|$$

   provided that ε_1 is sufficiently small and ε_j is sufficiently small with respect to ε_{j-1} for $1 < j \leq r$. Equivalently, if for any small enough $\varepsilon > 0$, we have

$$x + \varepsilon w_1 + \varepsilon^2 w_2 + \cdots + \varepsilon^r w_r \in |\Sigma|.$$

3. The *k-tangent cone bundle* is the set $\mathcal{TC}^k \Sigma := \bigsqcup_{x \in |\Sigma|} \mathcal{TC}_x^k \Sigma$. It comes with a natural projection map $\mathcal{TC}^k \Sigma \to \Sigma$ and its elements are denoted by $(x; w_1, \dots, w_k)$ or $(x; \underline{w})$, to make reference to the base point explicit.

We make two remarks. First, we note that the definition of the tangent cones given above guarantees, proceeding inductively on r, that for each $(x; \underline{w})$ and for $\{\varepsilon_i\}_{i=1}^k$ small enough, with $\epsilon_1 \gg \cdots \gg \epsilon_k > 0$, we have

$$x + \varepsilon_1 w_1 + \cdots + \varepsilon_r w_r \in |\Sigma|, \quad r \in [k].$$

In particular, the second point makes sense.

Moreover, it happens that we can actually ensure a stronger property, namely that for all $\{\varepsilon_i\}_{i=1}^k$ small enough and $\epsilon_1 \gg \cdots \gg \epsilon_k > 0$, all the above vectors fall in the same face of Σ. This is the content of the following proposition.

Proposition 2.1.14. *Given a cone complex Σ, we have*

$$\mathcal{TC}^k \Sigma = \bigcup_{\sigma \in \Sigma} \mathcal{TC}^k \sigma.$$

Proof. We will proceed by induction on k and show that for $(x; \underline{w}) \in \mathcal{TC}^k \Sigma$ there are $\varepsilon_1, \dots, \varepsilon_k \in \mathbb{R}_{>0}$ and a face σ of Σ containing x such that for each $1 \leq r \leq k$ we have

$$x + \varepsilon_1 w_1 + \cdots + \varepsilon_r w_r \in \sigma.$$

This will finish the proof as by convexity of σ, if $\delta_0, \delta_1, \dots, \delta_r$ are positive reals with $\sum_{i=0}^r \delta_i = 1$, we get

$$\sum_{i=0}^r \delta_i(x + \varepsilon_1 w_1 + \cdots + \varepsilon_r w_r) = x + \delta_1' w_1 + \cdots + \delta_r' w_r \in \sigma.$$

Making now δ_0 large enough and choosing $\delta_1 \gg \cdots \gg \delta_r > 0$ in an appropriate manner, we can ensure to get any $\delta_1' \gg \delta_2' \gg \cdots \gg \delta_r' > 0$ as long as they are small enough. Therefor, we get $(x; \underline{w}) \in T\mathcal{C}^k\,\sigma$.

For $k = 0$ there is nothing to prove as we can take any face of Σ containing x. Suppose now $k > 0$ and assume the result for $k-1$. For each natural number n, we use the assumption of the induction with $(x_n, \underline{w}')$ where $x_n = x + w_1/n$ and $w_i' = w_{i+1}$. In this way, we find positive numbers $\varepsilon_2^{(n)}, \dots, \varepsilon_k^{(n)}$ and a face σ_n of Σ such that

$$x_n, x_n + \epsilon_2(n) w_2, \dots, x_n + \varepsilon_2(n) w_2 + \cdots + \varepsilon_k(n) w_k \in \sigma_n.$$

The number of faces of Σ being finite, there is some $\sigma \in \Sigma$ such that $\sigma_n = \sigma$ for infinitely many n. Tending n to infinity through those n, we get $x_n = x + w_1/n \to x$ and hence $x \in \sigma$. So σ together with $1/n, \varepsilon_2(n), \dots, \varepsilon_k(n)$ for large enough n satisfy what we want. □

Remark 2.1.15. For a subdivision $\widetilde{\Sigma}$ of Σ, we have $T\mathcal{C}^k\,\Sigma = T\mathcal{C}^k\,\widetilde{\Sigma}$. Hence, by the proposition above, the subdivision $\widetilde{\Sigma}$ induces a subdivision $T\mathcal{C}^k\,\Sigma = \bigcup_{\sigma \in \tilde{\Sigma}} T\mathcal{C}^k\,\sigma$ of the tangent cone.

2.2 Tropicalization of rational functions

We now recall how to tropicalize rational functions on a variety into tropical functions on cone complexes. This is based on the idea that, given a point x in a variety X and a fix set of local parameters in $\mathcal{O}_{X,x}$ at the point, the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring at x becomes isomorphic to a power series ring in the local parameters. This isomorphism allows to see each rational function regular at x as a power series. We can then use the usual tropicalization procecdure with respect to the trivial valuation on the base field. Following this procedure, given an SNC divisor D, we can use the local equations of its components as local parameters to obtain for each rational function a tropical function over $\Sigma(D)$.

2.2.1 Admissible expansions

The following notion is useful to understand power series expansions directly in the ring $\widehat{\mathcal{O}}_{X,x}$. It is borrowed from [JM12].

Definition 2.2.1 (Admissible expansion)**.** Let R be a complete regular local κ-algebra and $z_1, \ldots, z_r$ with $r = \dim(R)$ a system of parameters for it. Given $f \in R$, an *admissible expansion for* f is an expression of the form

$$f = \sum_{\beta \in \mathbb{Z}_+^r} c_\beta z^\beta, \ c_\beta \in R, \tag{2.1}$$

in which the right hand side is a convergent series in which each coefficient c_β is either zero or a unit on R. The *support* of the admissible expansion is the set of all $\beta \in \mathbb{Z}_+^r$ with $c_\beta \neq 0$.

Here and in what follows, the notation z^β stands for the product $z_1^{\beta_1} \ldots z_r^{\beta_r}$ where $\beta_1, \ldots, \beta_r$ denote the coordinates of $\beta \in \mathbb{Z}^r$.

Remark 2.2.2. We will be essentially interested in the case in which R is equal to the completion $\widehat{\mathcal{O}}_{X,x}$ of the local ring of a point x in a smooth variety X. For technical reasons however we have defined it in this generality (see the proof of Proposition 2.7.5).

Remark 2.2.3. An element $f \in R$ has several admissible expansions and the support of these admissible expansions may vary. As an example, the identity $1 = (1 - z^\beta) + z^\beta$ shows two different admissible expansions with different supports for the constant function 1. Although admissible expansions are not unique, they always exist and as we will see next, the minimal terms of their supports form a uniquely determined set.

Proposition 2.2.4 (Existence of admissible expansions and uniqueness of the minimal elements of the support)**.** *Notations as in Definition 2.2.1, consider an element $f \in R$*

1. *There is an admissible expansion for f.*

2. *In the notation of (2.1), the set*

$$A_f := \min_{\leq_{\mathrm{cw}}} \{\beta \in \mathbb{Z}^r \mid c_\beta \neq 0\}$$

depends only on f and not in the choice of the admissible expansion.

3. *The set* A_f *does not change if we change the local parameters* $z_1, \dots, z_r$ *for some local parameters* $z'_1, \dots, z'_r$ *such that we have* $z'_i = z_i u_i$ *for some unit* $u_i \in R^\times$ *for each* $1 \le i \le r$.

Remark 2.2.5. A slightly weaker version of this proposition is stated in [JM12], where it is shown that the piecewise linear function defined by the admissible expansion is well-defined. Note that it might happen that two power series with different sets of minimal elements give the same piecewise linear function. The above proposition claims the uniqueness of minimal elements in different admissible expansions of a given rational function.

Proof of Proposition 2.2.4. (1) Denote by $\kappa(R)$ the residue field of R. By Cohen structure theorem [Coh46, Theorem 9], the ring $\widehat{\mathcal{O}}_{X,x}$ contains a coefficient field, that is, a field $\widetilde{\kappa} \subseteq \widehat{\mathcal{O}}_{X,x}$ such that the projection map $\widehat{\mathcal{O}}_{X,x} \to \kappa(R)$ restricts to an isomorphism $\widetilde{\kappa} \xrightarrow{\sim} \kappa(R)$. Moreover, this coefficient field induces a continuous isomorphism

$$\varphi\colon \kappa(R)[[\mathrm{x}_1, \dots, \mathrm{x}_r]] \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x} \tag{2.2}$$

which extends the isomorphism between $\kappa(R)$ and $\widetilde{\kappa}$ by sending x_i to z_i. Writing $\varphi^{-1}(f) = \sum_{\beta\in\mathbb{Z}_+^r} c_\beta \mathrm{x}^\beta$, we get an admissible expansion for f of the form

$$f = \sum_{\beta\in\mathbb{Z}_+^r} \varphi(c_\beta) z^\beta.$$

(2) Notations as in (1), let $f = \sum_{\beta\in\mathbb{Z}_+^r} a_\beta z^\beta$ be a second admissible expansion for f. Using the isomorphism (2.2) above, we can see each a_β for $\beta \in \mathbb{Z}_+^r$ as a power series with coefficients in $\kappa(R)$, that is as

$$\varphi^{-1}(a_\beta) = \sum_{\gamma\in\mathbb{Z}_+^r} a_{\beta,\gamma} \mathrm{x}^\gamma \in \kappa(R)[[\mathrm{x}_1, \dots, \mathrm{x}_r]].$$

We infer that

$$\begin{aligned}\sum_{\beta\in\mathbb{Z}_+^r} c_\beta \mathrm{x}^\beta = \varphi^{-1}(f) &= \sum_{\beta\in\mathbb{Z}_+^r} \varphi^{-1}(a_\beta)\mathrm{x}^\beta \\ &= \sum_{\beta\in\mathbb{Z}_+^r}\Bigg(\sum_{\gamma\in\mathbb{Z}_+^r} a_{\beta,\gamma}\mathrm{x}^\gamma\Bigg)\mathrm{x}^\beta = \sum_{\beta\in\mathbb{Z}_+^r}\Bigg(\sum_{0\le_{\mathrm{cw}}\gamma\le_{\mathrm{cw}}\beta} a_{\gamma,\beta-\gamma}\Bigg)\mathrm{x}^\beta\end{aligned}$$

which implies that $c_\beta = \sum_{0 \leq_{\text{cw}} \gamma \leq_{\text{cw}} \beta} a_{\gamma,\beta-\gamma}$. Now if β is a minimal element with $c_\beta \neq 0$, then $a_{\gamma,\beta-\gamma}$ is nonzero for some $\gamma \leq_{\text{cw}} \beta$, and therefor a_γ is nonzero. Conversely, if β is minimal among those β' such that $a_{\beta'} \neq 0$, then we have on one side $a_\beta = a_{\beta,0}$, and on the side, we have $a_{\beta,0} \neq 0$ because a_β is a unit. Combined together, we have shown that any minimal element in the support of one admissible expansion dominates a minimal element in the support of the second. This proves the statement in the proposition.

(3) The last point is straightforward. □

Remark 2.2.6.

1. Recall that a subset A of a partially ordered set is called an *antichain* if any pair of distinct elements in A are not comparable in the partial order. It is not hard to prove that an antichain in $(\mathbb{Z}^r_+, \leq_{\text{cw}})$ is necessarily finite. Since the sets A_f considered above are all antichains, we conclude that they must be finite.

2. For $f, g \in R$, by manipulating admissible expansions, we can see that

$$\min_{\leq_{\text{cw}}}\Big(A_{f+g} \cup A_f \cup A_g\Big) = \min_{\leq_{\text{cw}}}\big(A_f \cup A_g\big)$$
$$\min_{\leq_{\text{cw}}}\Big(A_{f\cdot g} \cup \min_{\leq_{\text{cw}}}(A_f + A_g)\Big) = \min_{\leq_{\text{cw}}}\big(A_f + A_g\big).$$

Corollary 2.2.7. *Any function $f \in R$ admits an admissible expansion with finite support.*

Proof. Let $f \in R$ be an admissible expansion. By Proposition 2.2.4, f admits an admissible expansion $f = \sum_{\beta \in \mathbb{Z}^r_+} c_\beta z^\beta$. Rearranging terms, we can rewrite this in the form $f = \sum_{\beta \in A_f} \widetilde{c}_\beta z^\beta$ where each coefficient $\widetilde{c}_\beta$ can be written in form $\widetilde{c}_\beta = c_\beta + \sum_{\gamma >_{\text{cw}} \beta} c'_\gamma z^{\gamma-\beta}$, for c'_γ either 0 or equal to c_γ, and is still invertible. By Remark 2.2.6, the set A_f is finite. □

2.2.2 Conewise antichains associated to rational functions

Let D be an SNC divisor on X. For each cone $\sigma \in \Sigma(D)$ and for each $i \in I_\sigma$, consider a local equation z_i for D_i around η_σ. Then, the family $\{z_i\}_{i \in I_\sigma}$ provides a system of local parameters for the local ring $\widehat{\mathcal{O}}_{X,\eta_\sigma}$. For a function $f \in K(X)$ with $f \in \mathcal{O}_{X,\eta_\sigma}$, $\sigma \in \Sigma(D)$, we define the set

$$A^\sigma_f := \min_{\leq_{\text{cw}}}\{\beta \in \mathbb{Z}^{I_\sigma} \mid c_\beta \neq 0\}$$

for a given (and so for any)admissible expansion $f = \sum_{\beta \in \mathbb{Z}^r_+} c_\beta z^\beta$.

Definition 2.2.8 (Antichains attached to a rational function)**.** Notations as above, for a rational function f on X, we call the family $\mathcal{A}_f := \{A_f^\sigma \,|\, \sigma \in \Sigma(D) \text{ with } f \in \mathcal{O}_{X,\eta_\sigma}\}$ the *family of antichains attached to* f.

Remark 2.2.9. In practice, we reduce to rational functions f which belong to *all* local rings $\mathcal{O}_{X,\eta_\sigma}$ for $\sigma \in \Sigma(D)$. In this case, the family of antichains has an element A_f^σ for any $\sigma \in \Sigma(D)$. Any more general rational function h on X can be written as the ratio $h = f_1/f_2$ of two such rational functions, i.e., with f_1, f_2 belonging both to any local ring $\mathcal{O}_{X,\eta_\sigma}$ for $\sigma \in \Sigma(D)$.

Proposition 2.2.10 (Compatibility of the antichains)**.** *Let D be an SNC divisor on X. Fix a cone $\sigma \in \Sigma(D)$ and a face τ of σ. Consider the projection*

$$\begin{aligned} \mathrm{pr}_{\sigma\succ\tau} : \mathbb{R}^{I_\sigma} &\longrightarrow \mathbb{R}^{I_\tau} \\ (x_i)_{i\in I_\sigma} &\longmapsto (x_i)_{i\in I_\tau}. \end{aligned}$$

For each $f \in \mathcal{O}_{X,\eta_\sigma}$, we then have $f \in \mathcal{O}_{X,\eta_\tau}$ and an equality of the form

$$A_f^\tau = \min_{\leq_{\mathrm{cw}}}\big(\mathrm{pr}_{\sigma\succ\tau}(A_f^\sigma)\big).$$

Proof. Consider the diagram

$$\begin{array}{ccccc} & & \widehat{\mathcal{O}}_{X,\eta_\sigma} & & \\ & \overset{\iota_1}{\nearrow} & & \overset{\iota_3}{\searrow} & \\ \mathcal{O}_{X,\eta_\sigma} & & & & \widehat{\left(\widehat{\mathcal{O}}_{X,\eta_\sigma}\right)_{\mathfrak{p}_\tau}} \\ & \underset{\iota_2}{\searrow} & & \underset{\iota_4}{\nearrow} & \\ & & \widehat{\mathcal{O}}_{X,\eta_\tau} & & \end{array}$$

Here $\mathfrak{p}_\tau$ is the prime ideal in $\widehat{\mathcal{O}}_{X,\eta_\sigma}$ generated by $\{z_i \mid i \notin I_\tau\}$ and each completion is taken with respect to the maximal ideal. Moreover ι_1 is the inclusion in the completion, ι_2 and ι_3 are the composition of a localization with an inclusion into the corresponding completion, and ι_4 is obtained by functoriality by localizing ι_1 at $\mathfrak{p}_\tau$ and completing with respect to the maximal ideal. This is a commutative diagram of κ-algebras.

Given an element $f \in \mathcal{O}_{X,\eta_\sigma}$, by Corollary 2.2.7 we can find finite admissible expansions

$$\iota_1(f) = \sum_{\beta\in\mathbb{Z}^{I_\sigma}} a_\beta z^\beta \qquad\qquad \iota_2(f) = \sum_{\gamma\in\mathbb{Z}^{I_\tau}} b_\gamma z^\gamma$$

in $\mathcal{O}_{X,\eta_\sigma}$ and $\mathcal{O}_{X,\eta_\tau}$, respectively. We then get

$$\begin{aligned}\iota_3(\iota_1(f)) &= \sum_{\beta\in\mathbb{Z}^{I_\sigma}} \iota_3(a_\beta z^\beta) = \sum_{\gamma\in\mathbb{Z}^{I_\tau}} \left(\sum_{\mathrm{pr}_{\sigma\succ\tau}(\beta)=\gamma} \iota_3(a_\beta z^{\beta-\gamma}) \right) z^\gamma \\ \iota_4(\iota_2(f)) &= \sum_{\gamma\in\mathbb{Z}^{I_\tau}} \iota_4(b_\gamma) z^\gamma.\end{aligned}$$

As $\sum_{\mathrm{pr}_{\sigma\succ\tau}(\beta)=\gamma} a_\beta z^{\beta-\gamma} \notin \mathfrak{p}_\tau$, its image by ι_3 is invertible. Therefore, we obtain two admissible expansions for $\iota_3(\iota_1(f)) = \iota_4(\iota_2(f))$ inside $\widehat{\left(\widehat{\mathcal{O}}_{X,\eta_\sigma}\right)_{\mathfrak{p}_\tau}}$. By Proposition 2.2.4, we get

$$\min_{\leq_{\mathrm{cw}}}\Big\{\gamma\in\mathbb{Z}^{I_\tau} \,\Big|\, \iota_4(b_\gamma)\neq 0\Big\} = \min_{\leq_{\mathrm{cw}}}\Big\{\gamma\in\mathbb{Z}^{I_\tau} \,\Big|\, \iota_3(\textstyle\sum_{\mathrm{pr}_{\sigma\succ\tau}(\beta)=\gamma} a_\beta z^{\beta-\gamma})\neq 0\Big\}.$$

Since ι_3 and ι_4 are both injective, we infer $A_f^\tau = \min_{\leq_{\mathrm{cw}}}\left(\mathrm{pr}_{\sigma\succ\tau}(A_f^\sigma)\right)$, as required. □

2.2.3 Tropicalization

We now define the tropicalization of rational functions.

Construction 2.2.11 (Tropicalization)**.** Let X be a variety and let $D \subseteq X$ be an SNC divisor. Let $\sigma \in \Sigma(D)$ and let $x \in \sigma$.

- For $f \in \mathcal{O}_{X,\eta_\sigma}$, we define

$$\mathrm{trop}(f)(x) \coloneqq \min\big\{\langle x,\beta\rangle \mid \beta\in A_f^\sigma\big\}.$$

- For two elements $f_1, f_2 \in \mathcal{O}_{X,\eta_\sigma}$, we have

$$\mathrm{trop}(f_1f_2)(x) = \mathrm{trop}(f_1)(x) + \mathrm{trop}(f_2)(x).$$

This allows to extend the above definition to an arbitrary $g \in K(X)$. In this case, we write $g = f_1/f_2$ for $f_1, f_2 \in \mathcal{O}_{X,\eta_\sigma}$ and define for each $x \in \sigma$

$$\mathrm{trop}(g)(x) \coloneqq \mathrm{trop}(f_1)(x) - \mathrm{trop}(f_2)(x).$$

- Finally, by Proposition 2.2.10 above, this function is independent of the choice of the face of $\Sigma(D)$ which contains x. Hence, we obtain a well defined map

$$\mathrm{trop}(f) : |\Sigma(D)| \to \mathbb{R}$$

which we call the *tropicalization of f* with respect to D. ◇

Remark 2.2.12. In order to prove the second property, namely, that

$$\mathrm{trop}(f_1 f_2)(x) = \mathrm{trop}(f_1)(x) + \mathrm{trop}(f_2)(x)$$

for $f_1, f_2 \in \mathcal{O}_{X,\eta_\sigma}$, let $\mathrm{in}_x(A^\sigma_{f_i})$ be the subset of $A^\sigma_{f_i}$ consisting of all β with $\mathrm{trop}(f_i)(x) = \langle x, \beta\rangle$. Then, we get $\mathrm{in}_x(A^\sigma_{f_1 f_2}) \cap \Big(\mathrm{in}_x(A^\sigma_{f_1}) + \mathrm{in}_x(A^\sigma_{f_2})\Big) \neq \emptyset$. Combined with the second part of Remark 2.2.6, this gives the result.

Proposition 2.2.13. *The tropicalization of a rational function is a tropical function.*

Proof. For $\sigma \in \Sigma(D)$ and $f \in \mathcal{O}_{X,\eta_\sigma}$ the tropicalization $\mathrm{trop}(f)|_\sigma$ is the minimum of finitely many linear functions with integral coefficients. Therefore, this is an integral piecewise linear function on σ. More generally, for any element $f \in K(X)$, the tropicalization $\mathrm{trop}(f)$ can be written as the difference of two integral piecewise linear functions over each cone σ, and so it is itself integral piecewise linear on each cone. It follows that tropicalization of f is a tropical function. □

2.3 Quasi-monomial valuations of higher rank

In this section, we define quasi-monomial valuations as certain Krull valuations attached to a given SNC divisor. We study their basic properties and then relate their combinatorial structure with the one of the dual complex in the case the values are taken in $\mathbb{R}^k$ with its lexicographic order.

2.3.1 Definition

We start by giving the definition in the more general setting of totally ordered abelian group. The one important for us in this paper will be the additive group $\mathbb{R}^k$ endowed with the *lexicographic order* $\preceq_{\mathrm{lex}}$ that we sometime simply denote by $\preceq$. This is the order defined by $x \preceq_{\mathrm{lex}} y$ iff $x = y$ or there is an $1 \leq i \leq k$ such that $x_j = y_j$ for $j < i$ and $x_i < y_i$. This ordered group has specific properties, depicted in the presence of its two different natural topologies, which are exploited in this work.

Let $(\Gamma, \preceq)$ be a totally ordered abelian group and consider $\Gamma_{\succeq 0} = \{\alpha \in \Gamma \mid \alpha \succeq 0\}$. Let D be an SNC divisor in a smooth variety X. To a given cone $\sigma \in \Sigma(D)$ and a tuple

$\underline{\alpha} \in \Gamma_{\succeq 0}^{I_\sigma}$, we associate the valuation $\nu_{\sigma,\underline{\alpha}}$ by defining its value first at an element $f \in \mathcal{O}_{X,\eta_\sigma}$ by

$$\nu_{\sigma,\underline{\alpha}}(f) := \min_{\preceq} \left\{ \sum_{i \in I_\sigma} \beta_i \alpha_i \in \Gamma \mid \beta \in A_f^\sigma \right\}. \tag{2.3}$$

By Remark 2.2.6 and the argument used in 2.2.12, it is straightforward to see that

$$\nu_{\sigma,\underline{\alpha}}(fg) = \nu_{\sigma,\underline{\alpha}}(f) + \nu_{\sigma,\underline{\alpha}}(g), \quad \text{and}$$
$$\nu_{\sigma,\underline{\alpha}}(f+g) \succeq \min\{\nu_{\sigma,\underline{\alpha}}(f), \nu_{\sigma,\underline{\alpha}}(g)\}.$$

This shows that $\nu_{\sigma,\underline{\alpha}}$ verifies the properties of a valuation on $\mathcal{O}_{X,\eta_\sigma}$ and so uniquely extends to a valuation on $K(X)$, the fraction field of $\mathcal{O}_{X,\eta_\sigma}$.

Definition 2.3.1 (Quasi-monomial valuations)**.** Notations as above, the valuation $\nu_{\sigma,\underline{\alpha}}$ is called the Γ-*quasi-monomial valuation with respect to* σ *and* $\underline{\alpha}$. The set of all Γ-quasi-monomial valuations for a given cone $\sigma \in \Sigma(D)$ is denoted by $\mathscr{M}_\sigma^\Gamma(D)$. The set of all Γ-quasi-monomial valuations coming from any cone of $\Sigma(D)$ is denoted by $\mathscr{M}^\Gamma(D)$.

In the case the ordered group if the additive group $\mathbb{R}^k$ endowed with the lexicographic order, for a natural number k, we call the valuation $\nu_{\sigma,\underline{\alpha}}$ a *quasi-monomial valuation of rank bounded by* k. We denote simply by $\mathscr{M}_\sigma^k(D)$ and $\mathscr{M}^k(D)$ the corresponding sets of quasi-monomial valuations $\mathscr{M}_\sigma^{\mathbb{R}^k}(D)$ and $\mathscr{M}^{\mathbb{R}^k}(D)$, respectively. For $k = 1$, we further simplify $\mathscr{M}_\sigma^1(D)$ and $\mathscr{M}^1(D)$ to $\mathscr{M}_\sigma(D)$ and $\mathscr{M}(D)$, respectively.

In the rest of this paper, we will only consider quasi-monomial valuations of rank bounded by k for some positive integer k.

Remark 2.3.2. The integer k used in the definition of the quasi-monomial valuation makes reference to the rank of the codomain of the valuation. This should not be confused with the Krull dimension of the valuation ring of $\nu_{\sigma,\underline{\alpha}}$, neither with the rank of the value group of the valuation, as we allow the value group $\nu_{\sigma,\underline{\alpha}}(K(X))$ to be of rank strictly smaller than k. The idea of studying valuations of different ranks all together, simultaneously, is motivated from practical situations appearing in the study of multi-parameter degenerations of complex varieties, see for example [AN20; AN21].

2.3.2 The duality theorem

In this section, we provide a dual description of the set of quasi-monomial valuations of rank bounded by k.

Recall that for a variety X and a valuation $\nu : K(X) \to \Gamma$, the *center of* ν, if it exists, is the unique point of X denoted by c_ν such that ν is non-negative over $\mathcal{O}_{X,x}$ and strictly positive over its maximal ideal. The center of quasi-monomial valuations always exists.

Proposition 2.3.3. *Let D be an SNC divisor on a variety X and let Γ be a totally ordered abelian group. For $\sigma \in \Sigma(D)$ and $\underline{\alpha} \in \Gamma_+^{I_\sigma}$, consider the unique face τ of σ given by the rays $I_\tau = \{i \in I_\sigma \mid \alpha_i \succ 0\}$. Let $\underline{\alpha}_\tau = \mathrm{pr}_{\sigma\succ\tau}(\underline{\alpha})$ be the element $\Gamma_+^{I_\tau}$ whose coordinates are given by those of α.*

Then, we have $\nu_{\sigma,\underline{\alpha}} = \nu_{\tau,\underline{\alpha}_\tau}$. Moreover, the center of $\nu_{\sigma,\underline{\alpha}}$ exists and is equal to η_τ.

Proof. The first assertion follows directly from Proposition 2.2.10. To prove the second, notice that $\nu_{\tau,\underline{\alpha}_\tau}(f) \succeq 0$ for each $f \in \mathcal{O}_{X,\eta_\tau}$. Moreover, $\nu_{\tau,\underline{\alpha}_\tau}(f) = 0$ if and only if $0 \in A_f^\tau$, i.e., in the case f is invertible. This shows that the center of $\nu_{\tau,\underline{\alpha}_\tau}$ is η_τ. □

Consider now the case $\Gamma = \mathbb{R}$. In this case, the elements of $\mathbb{R}_+^{I_\sigma}$ can be naturally identified with the points of σ. From the compatibility in the above proposition, we get a natural bijection

$$\big|\Sigma(D)\big| \longrightarrow \mathscr{M}(D) \tag{2.4}$$

obtained by sending a point $x \in |\Sigma(D)|$ to the quasi-monomial valuation $\nu_{\sigma,\underline{\alpha}} \in \mathscr{M}(D)$ where σ is any cone of $\Sigma(D)$ which contains the point x and $\underline{\alpha}$ denotes the coordinates of x in that cone.

We now generalize this bijection to higher rank quasi-monomial valuations. First observe that there is a natural projection map $\pi\colon \mathscr{M}^k(D) \longrightarrow \mathscr{M}(D)$ defined as follows. Take a point $\underline{\alpha} = (\alpha_i)_{i\in I_\sigma} \in (\mathbb{R}^k)_+^{I_\sigma}$. Each α_i is an element of $(\mathbb{R}^k)_+ = (\mathbb{R}^k)_{\succeq 0}$ and we denote its coordinates by $\alpha_i = (\alpha_i^1, \dots, \alpha_i^k)$. Consider the projection to the first coordinate denoted by an abuse of the notation by π and given by

$$\pi\colon (\mathbb{R}^k)_+^{I_\sigma} \to \mathbb{R}_+^{I_\sigma}, \qquad \pi(\underline{\alpha}) = (\alpha_i^1)_{i\in I_\sigma}.$$

The projection map π is then defined by

$$\pi(\nu_{\sigma,\underline{\alpha}}) := \nu_{\sigma,\pi(\underline{\alpha})} \tag{2.5}$$

over each cone σ in $\Sigma(D)$. By Proposition 2.3.3, this map is well defined. It allows to view $\mathscr{M}^k(D)$ fibered over $\mathscr{M}(D)$.

Theorem 2.3.4 (Duality theorem)**.** *Notations as above, there is an isomorphism of bundles over* $\mathscr{M}(D) \simeq \big|\Sigma(D)\big|$

$$\begin{array}{ccc} \mathscr{M}^k(D) & \xrightarrow{\ \phi\ } & T\mathcal{C}^{k-1}\,\Sigma(D) \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ \mathscr{M}(D) & \longrightarrow & \big|\Sigma(D)\big| \end{array} \tag{2.6}$$

where:

- *the map* $\mathscr{M}(D) \to \big|\Sigma(D)\big|$ *on the base is the inverse of the isomorphism* (2.4)*, and*

- *the map* ϕ *is defined by a compatible family of maps*

$$\phi_\sigma : \mathscr{M}_\sigma(D) \longrightarrow T\mathcal{C}^{k-1}\,\sigma, \qquad \sigma \in \Sigma(D).$$

For $\sigma \in \Sigma(D)$*, the map* ϕ_σ *is defined as follows. Take a point* $\underline{\alpha} = (\alpha_i)_{i\in I_\sigma}$ *in* $(\mathbb{R}^k)_+^{I_\sigma}$*, let* $x = \pi(\underline{\alpha}) = (\alpha_i^1)_{i\in I_\sigma} \in \mathbb{R}_+^{I_\sigma}$*, and for each* $j = 2, \dots, k$*, define*

$$w_{j-1} \coloneqq (\alpha_i^j)_{i\in I_\sigma} \in \mathbb{R}^{I_\sigma}.$$

Then, the point $(x; w_1, \dots, w_{k-1})$ *belongs to* $T\mathcal{C}^{k-1}\,\sigma$*, and we set*

$$\phi_\sigma(\nu_{\sigma,\underline{\alpha}}) \coloneqq (x; w_1, \dots, w_{k-1}).$$

Remark 2.3.5. The proof of the duality theorem reduces to the following statement in coordinates. A real matrix $A \in \mathrm{Mat}_{k,r}(\mathbb{R})$ has columns in $(\mathbb{R}^k)_+$, with respect to the lexicographic order on $\mathbb{R}^k$, if and only if the family $(A^{\mathrm{t}}_{\bullet,1}; A^{\mathrm{t}}_{\bullet,2}, \dots, A^{\mathrm{t}}_{\bullet,k})$, given by the columns of the transpose A^{t} of A, belongs to the tangent cone $T\mathcal{C}^{k-1}\Big((\mathbb{R}_+)^r\Big)$. This justifies the name given to the theorem.

Proof of Theorem 2.3.4. We verify that each ϕ_σ is a bijection. Let σ be a cone in $\Sigma(D)$. By definition, an element $(\alpha_i)_{i\in I_\sigma} \in (\mathbb{R}^k)^{I_\sigma}$ gives a valuation $\nu_{\sigma,\underline{\alpha}}$ in the domain of ϕ_σ provided for each $i \in I_\sigma$, the vector α_i belongs to $(\mathbb{R}^k)_+$, that is, it is non-negative with respect to the lexicographic order. Denoting by $(\alpha_i^1, \dots, \alpha_i^k)$ the coordinates of α_i, this means that for each $i \in I_\sigma$, we must have

$$\begin{aligned}\text{either, } & \alpha_i^1 > 0 \\ & \text{or, } (\alpha_i^1 = 0 \text{ and } \alpha_i^2 > 0) \\ & \text{or, } (\alpha_i^1 = \alpha_i^2 = 0 \text{ and } \alpha_i^3 > 0) \\ & \quad \vdots \\ & \text{or, } (\alpha_i^1 = \cdots = \alpha_i^{k-1} = 0 \text{ and } \alpha_i^k \geq 0).\end{aligned} \tag{2.7}$$

On the other hand, for a collection of vectors x, $w_1, \dots, w_{k-1}$ in $\mathbb{R}^{I_\sigma}$, the family $(x; w_1, \dots, w_{k-1})$ by definition belongs to $T\mathcal{C}^{k-1}\sigma$ if and only if we have

$$\begin{aligned}& x \in \sigma \text{ and} \\ & x + \varepsilon_1 w_1 \in \sigma \text{ for } \varepsilon_1 > 0 \text{ small enough, and} \\ & x + \varepsilon_1 w_1 + \varepsilon_2 w_2 \in \sigma \text{ for } \varepsilon_1 \gg \varepsilon_2 > 0 \text{ small enough and} \\ & \quad \vdots \\ & x + \varepsilon_1 w_1 + \cdots + \varepsilon_{k-1} w_{k-1} \in \sigma \text{ for } \epsilon_{k-2} \gg \varepsilon_{k-1} > 0 \text{ small enough.}\end{aligned} \tag{2.8}$$

Specifying the collection of vectors $x, w_1, \dots, w_{k-1}$ to the ones given in the statement of the theorem, the conditions in 2.8 above can be rephrased as follows. For each $i \in I_\sigma$,

$$\begin{aligned}& \alpha_i^1 \geq 0 \text{ and} \\ & \alpha_i^1 + \varepsilon_1 \alpha_i^2 \geq 0 \text{ for } \varepsilon_1 > 0 \text{ small enough, and} \\ & \alpha_i^1 + \varepsilon_1 \alpha_i^2 + \varepsilon_2 \alpha_i^3 \geq 0 \text{ for } \varepsilon_1 \gg \varepsilon_2 > 0 \text{ small enough, and} \\ & \quad \vdots \\ & \alpha_i^1 + \varepsilon_1 \alpha_i^2 + \cdots + \varepsilon_{k-1} \alpha_i^k \geq 0 \text{ for } \varepsilon_{k-2} \gg \varepsilon_{k-1} > 0 \text{ small enough.}\end{aligned} \tag{2.9}$$

Clearly, Conditions (2.7) and (2.9) are equivalent, and we infer that φ_σ is a bijection.

Now to conclude, note that the family of maps $\{\phi_\sigma\}_\sigma$ is compatible with the descriptions of $\mathscr{M}^k(D)$ and $T\mathcal{C}^{k-1}(D)$ as the unions $\mathscr{M}^k(D) = \bigcup_\sigma \mathscr{M}^k_\sigma(D)$ and $T\mathcal{C}^{k-1}\Sigma(D) = \bigcup_\sigma T\mathcal{C}^{k-1}\sigma$, and so they can be glued together to define a map $\phi : \mathscr{M}^k(D) \to T\mathcal{C}^{k-1}\Sigma(D)$. Since each ϕ_σ is a bijection, so is ϕ. □

2.3.3 An analytic description of quasi-monomial valuations

We now explain how to understand higher quasi-monomial valuations from an analytic point of view, directly from tangent vectors, by taking directional derivatives. This leads to a description of the inverse ϕ^{-1} of the map ϕ appearing in the Duality Theorem.

We need to introduce the notion of *derivative of a function with respect to a tuple of inward tangent vectors* in the tangent cone.

Definition 2.3.6 (Directional derivatives)**.** Given a polyhedral complex Σ and a function $F : \left|\Sigma\right| \to \mathbb{R}$, the derivative of F at a point $x \in \left|\Sigma\right|$ along an inward vector $w \in T\mathcal{C}_x\,\Sigma$ is the limit

$$D_w F(x) \coloneqq \lim_{\varepsilon \to 0^+} \frac{F(x+\varepsilon w) - F(x)}{\varepsilon},$$

whenever this limit exists. More generally, we inductively define the derivative of F at a point $x \in |\Sigma|$ and with respect to the tuple $\underline{w} = (w_1 \dots, w_k) \in T\mathcal{C}^k_x\,\Sigma$ as the limit

$$D_{(w_1,\dots,w_k)} F(x) \coloneqq \lim_{\varepsilon \to 0^+} \frac{D_{(w_1,\dots,w_{k-1}+\varepsilon w_k)} F(x) - D_{(w_1,\dots,w_{k-1})} F(x)}{\varepsilon}, \tag{2.10}$$

whenever the directional derivatives $D_{(w_1,\dots,w_{k-1}+\varepsilon w_k)} F(x)$, for $\varepsilon \geq 0$ small enough, and the above limit exist.

In the case these limits exist for all points $x \in |\Sigma|$ and $\underline{w} \in T\mathcal{C}^k\,\Sigma$, we denote by $\mathscr{D}^k F$ the corresponding *derivative function* from $T\mathcal{C}^k\,\Sigma \to \mathbb{R}^{k+1}$. This is the function which to a point $x \in |\Sigma|$ and $\underline{w} = (w_1, \dots, w_k) \in T\mathcal{C}^k_x\,\Sigma$ associates the point

$$\mathscr{D}^k F(x; \underline{w}) \coloneqq \big(F(x), D_{w_1}F(x), D_{(w_1,w_2)}F(x), \dots, D_{(w_1,\dots,w_k)}F(x)\big) \in \mathbb{R}^{k+1}.$$

Remark 2.3.7. We make a few remarks.

1. When $(w_1, \dots, w_k) \in T\mathcal{C}^k\,\Sigma$, the points $(w_1, \dots, w_{k-2}, w_{k-2} + \epsilon w_{k-1})$, for $\varepsilon \geq 0$ small enough, all belong to $T\mathcal{C}^{k-1}\,\Sigma$. So the limit in 2.10 is well-posed.

2. At a smooth point $x \in \left|\Sigma\right|$, when x lies in the relative interior of a facet of Σ, and for a function $F : \left|\Sigma\right| \to \mathbb{R}$ which is smooth on a neighborhood of x, the definition of $D_{\underline{w}}F(x)$ for $\underline{w} = (w_1, \dots, w_k) \in T\mathcal{C}^k\,\Sigma$ coincides with the evaluation at the k-tuple of tangent vectors $\underline{w}$ of the k-th derivative of F at x. The definition is thus a natural extension to the case where F is not necessarily a smooth function and x is an arbitrary point of $\left|\Sigma\right|$.

The following proposition provides an alternative way of computing $D_{\underline{w}}F(x)$ when it exists.

Proposition 2.3.8. *Consider a point $x \in \Sigma$ and a tuple $\underline{w} \in T\mathcal{C}_x^k\,\Sigma$. Let $F : \Sigma \to \mathbb{R}$ be a function for which $D_{\underline{w}}F(x)$ exits. Then we have*

$$D_{\underline{w}}F(x) = \lim_{\varepsilon_k \to 0^+} \dots \lim_{\varepsilon_1 \to 0^+} \frac{1}{\varepsilon_1 \cdots \varepsilon_k}\Big(F(x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_k w_k) - F(x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_{k-1} w_{k-1})\Big).$$

Proof. For $k = 1$, this is the definition of $D_{\underline{w}}F(x)$. The general case can be obtained by induction. ☐

In this paper we are mainly interested in directional derivatives of tropical functions. In this case the derivatives always exist as the following proposition shows.

Proposition 2.3.9. *For any piecewise linear function $F : |\Sigma| \to \mathbb{R}$ and any $k \geq 0$ the derivative $\mathscr{D}^k F$ exists.*

Proof. Let $\widetilde{\Sigma}$ be a subdivision of Σ such that F is linear on each cone $\sigma \in \widetilde{\Sigma}$. By Remark 2.1.15, we have that $T\mathcal{C}^k\,\Sigma = \bigcup_{\sigma \in \widetilde{\Sigma}} T\mathcal{C}^k\,\sigma$, so given $(x; \underline{w}) \in T\mathcal{C}^k\,\Sigma(D)$, there is a cone $\sigma \in \widetilde{\Sigma}$ such that $(x; \underline{w}) \in T\mathcal{C}^k\,\sigma$. Denote by F_σ the linear function which is equal to the restriction of F to σ. A direct calculation shows that $\mathscr{D}^k F(x; \underline{w})$ exists and is given by

$$\mathscr{D}^k F(x; w_1, \dots, w_k) = (F_\sigma(x), F_\sigma(w_1), \dots, F_\sigma(w_k)).$$

☐

We now come back to the tropicalization of rational functions and its link to quasi-monomial valuations. From the very definition, it is clear that we can retrieve rank one quasi-monomial valuations by evaluating tropical functions at their corresponding point, that is, given $f \in K(X)^*$ and $x \in |\Sigma(D)|$, if ν_x denotes the valuation corresponding to x under the map in (2.4), then

$$\nu_x(f) = \mathrm{trop}(f)(x).$$

The following result extends this relation to higher rank quasi-monomial valuations.

Theorem 2.3.10 (Quasi-monomial valuations using derivatives). *Let $k \in \mathbb{N}$ be a natural number. Given $(x;\underline{w}) \in T\mathcal{C}^{k-1}\,\Sigma(D)$, consider the evaluation map*

$$\begin{aligned}\nu_{(x;\underline{w})} \colon K(X)^* &\longrightarrow \mathbb{R}^k\\ f &\longmapsto \mathscr{D}^k\,\mathrm{trop}(f)(x;\underline{w}).\end{aligned}$$

Then $\nu_{x;\underline{w}}$ is a well-defined function and it coincides with the valuation $\phi^{-1}(x;\underline{w})$ given by the Duality Theorem 2.3.4.

Proof. Fix a point $(x;\underline{w}) \in T\mathcal{C}^{k-1}\,\Sigma$ and let σ be a face of $\Sigma(D)$ containing x such that $(x;\underline{w}) \in T\mathcal{C}^{k-1}\,\sigma$. Then, by definition, for any $f \in \mathcal{O}_{X,\eta_\sigma}$, we have

$$\mathrm{trop}(f)(x) = \min\Big\{\langle x,\beta\rangle \,\Big|\, \beta \in A_f^\sigma\Big\}. \tag{2.11}$$

As $\mathrm{trop}(f)$ is piecewise linear, there is a subdivision Σ_f of $\Sigma(D)$ such that $\mathrm{trop}(f)$ is linear on each face of Σ_f. By Remark 2.1.15, there is a cone τ in Σ_f such that $(x;\underline{w}) \in T\mathcal{C}_x^{k-1}\,\tau$. Let $\beta_\tau \in A_f^\sigma$ be the exponent such that $\mathrm{trop}(f)(y) = \langle y,\beta_\tau\rangle$ for any $y \in \tau$. By Proposition 2.3.9, we get

$$\nu_{x;\underline{w}}(f) = \Big(\langle x,\beta_\tau\rangle, \langle w_1,\beta_\tau\rangle, \dots, \langle w_{k-1},\beta_\tau\rangle\Big). \tag{2.12}$$

We now show that $\nu_{x;\underline{w}} = \phi^{-1}(x;\underline{w})$, that is, $\nu_{x;\underline{w}} = \nu_{\sigma,\underline{\alpha}}$ where $\underline{\alpha} = (\alpha_i)_{i\in I_\sigma}$ and $\alpha_i = (x^i, w_1^i, \dots, w_{k-1}^i)$.

Note that here for $i \in I_\sigma$, x^i and w_j^i are the i-th coordinate of x and w_j, respectively. So with our previous notation, we have $\alpha_i^j = x_i$ for $j = 1$ and $\alpha_i^j = w_{j-1}^i$ for $j = 2, \dots, k$.

To show the above claim, note that for any $f \in \mathcal{O}_{\eta_\sigma}$, we have

$$\begin{aligned}v_{\sigma,\underline{\alpha}}(f) &= \min_{\preceq_{\mathrm{lex}}}\left\{\sum_{i\in I_\sigma}\alpha_i\beta_i \mid \beta \in A_f^\sigma\right\}\\ &= \sum_{i\in I_\sigma}\alpha_i\beta_{\underline{\alpha},i} = \sum_{i\in I_\sigma}\left(x^i, w_1^i, \dots, w_{k-1}^i\right)\beta_{\underline{\alpha},i} = \Big(\langle x,\beta_{\underline{\alpha}}\rangle, \langle w_1,\beta_{\underline{\alpha}}\rangle, \dots, \langle w_{k-1},\beta_{\underline{\alpha}}\rangle\Big)\end{aligned} \tag{2.13}$$

where $\beta_{\underline{\alpha}}$ is an exponent in A_f^σ which gives the minimum in the first equation above, and $\beta_{\underline{\alpha},i}$ is the i-th coordinate of $\beta_{\underline{\alpha}}$ for $i \in I_\sigma$. We thus need to prove that the two expressions in (2.12) and (2.13) are equal.

We will prove this by induction. The first entry in both expressions (2.12) and (2.13) coincide as they are both equal to $\mathrm{trop}(f)(x)$. Assuming the two expressions have the same j-entries for all $1 \leq j \leq \ell - 1$, we will prove that the ℓ-entries are also equal. The first $\ell - 1$ entries being equal,

$$\langle x, \beta_\tau \rangle = \langle x, \beta_{\underline{\alpha}} \rangle, \langle w_1, \beta_\tau \rangle = \langle w_1, \beta_{\underline{\alpha}} \rangle, \dots, \langle w_{\ell-1}, \beta_\tau \rangle = \langle w_{\ell-1}, \beta_{\underline{\alpha}} \rangle, \tag{2.14}$$

we infer that

$$\begin{aligned}
&\langle x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_\ell w_\ell, \beta_\tau \rangle \\
&\qquad = \mathrm{trop}(f)\big(x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_\ell w_\ell\big) \\
&\qquad = \min_{\preceq_{\mathrm{lex}}} \Big\{ \langle x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_\ell w_\ell, \beta \rangle \;\big|\; \beta \in A_f^\sigma \Big\} \\
&\qquad \overset{\star}{=} \min_{\preceq_{\mathrm{lex}}} \Big\{ \langle x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_\ell w_\ell, \beta \rangle \;\big|\; \beta \in A_f^\sigma \\
&\qquad\qquad\qquad \text{such that } \langle x, \beta \rangle = \langle x, \beta_\tau \rangle, \langle w_j, \beta \rangle = \langle w_j, \beta_\tau \rangle \text{ for } 1 \leq j \leq \ell - 1 \Big\} \\
&\quad = \langle x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_\ell w_\ell, \beta_{\underline{\alpha}} \rangle.
\end{aligned}$$

Here, in $\overset{\star}{=}$ we used the fact that to minimize $\langle x + \varepsilon_1 w_1 + \cdots + \varepsilon_1 \cdots \varepsilon_\ell w_\ell, \beta \rangle$ for $\beta \in A_f^\sigma$ and for $\varepsilon_1 \gg \varepsilon_2 \gg \dots \varepsilon_\ell > 0$ small enough, we need to first minimize $\langle x, \beta \rangle$, then minimize $\langle w_1, \beta \rangle$ and so on. By the hypothesis of our induction, β_τ does exactly this as it behaves like $\beta_{\underline{\alpha}}$ in those entries. From this equality, using Equation (2.14), we infer the equality $\langle w_\ell, \beta_\tau \rangle = \langle w_\ell, \beta_{\underline{\alpha}} \rangle$, as required.

This proves that $\nu_{x;\underline{w}}(f) = \nu_{\sigma,\underline{\alpha}}(f)$ for all $f \in \mathcal{O}_{X,\eta_\sigma}$. Using the relation $\mathrm{trop}(f/g) = \mathrm{trop}(f) - \mathrm{trop}(g)$ for two elements $f, g \in \mathcal{O}_{X,\eta_\sigma}$, we finally conclude that $\nu_{x;\underline{w}}(f) = \nu_{x;\underline{w}}(f)$ for all $f \in K(X)^*$ and the theorem follows. □

2.3.4 Flag valuations

In this section, we discuss an alternative way for getting valuations of higher rank on X based on flags of subvarieties, and explain the relation to our constructions above. More details on valuations associated to flags of subvarieties can be found in [LM09b; KK12], where they are used to define Newton-Okounkov bodies.

Consider a flag of subvarieties

$$\mathcal{F} : F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_k$$

where $F_0 = X$, and for each $1 \leq \ell \leq k$, F_i is a smooth irreducible subvariety of $F_{\ell-1}$ with $\mathrm{codim}_X(F_\ell) = \ell$.

Under these hypothesis, each F_ℓ defines a discrete valuation ord_{F_ℓ} over the function field of $F_{\ell-1}$. We choose a uniformizer t_ℓ for ord_{F_ℓ}. Using these orders of vanishing, we can construct a higher rank valuation on $K(X)$ as follows.

Proposition 2.3.11. *Notations as above, consider the map*

$$\begin{aligned} \nu_{\mathcal{F}} : K(X)^* &\to \mathbb{R}^k \\ f &\mapsto (\mathrm{ord}_{F_1}(f_1), \mathrm{ord}_{F_2}(f_2), \dots, \mathrm{ord}_{F_k}(f_k)) \end{aligned} \tag{2.15}$$

where $f_1 = f$ and $f_{\ell+1}$ is the restriction of $f_\ell \cdot t_\ell^{-\mathrm{ord}_{F_\ell}(f_\ell)}$ to $F_{\ell+1}$ viewed in the function field $K(F_{\ell+1})$. This is a rank k valuation which is independent of the choice of coordinates t_ℓ.

Given a nonempty SNC divisor $D = \sum_{i \in \mathcal{I}} D_i$, we can define a flag of subvarieties if we fix an ordered sequence of components $D_{i_1}, \dots, D_{i_k}$ of D, for $i_1, \dots, i_k \in \mathcal{I}$ with non-empty intersection, and an irreducible component S of the intersection $D_{i_1} \cap \cdots \cap D_{i_k}$. In this case, we set $F_0 = X$ and for each $1 \leq j \leq k$, we define F_j as the unique irreducible component of $D_{i_1} \cap \cdots \cap D_{i_j}$ which contains S. Then we have automatically $F_j \subsetneq F_{j-1}$.

Since D is SNC, each F_j is a smooth connected subvariety of codimension one inside F_{j-1}, and we get a flag of subvarieties

$$\mathcal{F} : X = F_0 \supsetneq F_1 \supsetneq \cdots \supsetneq F_k = S \tag{2.16}$$

which verify the hypothesis of Proposition 2.3.11.

We now prove that $\nu_{\mathcal{F}}$ corresponds to a quasi-monomial valuation defined in terms of D. For this let σ be the cone corresponding to the stratum F_k of D, this cone has rays indexed by $I_\sigma = \{i_1, \dots, i_k\} \subseteq \mathcal{I}$. Consider the standard basis $e_{i_1}, \dots, e_{i_r}$ of N_σ which is contained in σ where e_{i_j} is the primitive vector of the ray corresponding to i_j.

Theorem 2.3.12. *Let $\mathcal{F}$ the flag on (2.16) and $\nu_{\mathcal{F}}$ the valuation defined by Proposition 2.3.11. Then $\nu_{\mathcal{F}} = \nu_{x,\underline{w}}$ for $(x; \underline{w}) = (e_{i_1}; e_{i_2}, \dots, e_{i_k}) \in T\mathcal{C}^{k-1}\,\Sigma(D)$.*

Proof. Without loss of generality, we can assume that $D_{i_1} \cap \cdots \cap D_{i_k} = \{p\}$ is a closed point of X. Indeed, if this is not the case, we can extend the flag in 2.16 to a complete flag, by adding, if needed, more components to the divisor D, and then work with this complete flag. The result then follows by taking the projection to the first k components of the valuation.

Now take $z_1, \dots, z_k$ equations for $D_{i_1}, \dots, D_{i_k}$ around p. Using these elements, for each $1 \leq r \leq k$, we get a restriction map (called as well *reduction map* in the literature)

$$\begin{aligned} \operatorname{res}_j : K(F_{j-1}) &\longrightarrow K(F_j), \\ f &\longmapsto f z_j^{-\operatorname{ord}_{F_j}(f)}\big|_{F_j}, \end{aligned}$$

which satisfies $\operatorname{res}_j(\mathcal{O}_{F_{j-1},x}) \subseteq \mathcal{O}_{F_j,x}$.

The elements $z_j, \dots, z_k$ give us a local system of parameters for the local ring $\mathcal{O}_{F_{j-1},x}$ and hence they induce an isomorphism $\widehat{\mathcal{O}}_{F_{j-1},x} \simeq k[[z_j, \dots, z_k]]$.

In this way, we obtain an extension to the power series ring for res_j as follows

$$\begin{array}{ccc} \mathcal{O}_{F_{j-1},x} & \xrightarrow{\operatorname{res}_j} & \mathcal{O}_{F_j,x} \\ \big\downarrow & & \big\downarrow \\ \widehat{\mathcal{O}}_{F_{j-1},x} & & \widehat{\mathcal{O}}_{F_j,x} \\ \wr\| & & \wr\| \\ k[[z_j, \dots, z_k]] & \xrightarrow{\operatorname{res}_j} & k[[z_{j+1}, \dots, z_k]] \\ f = \sum_\beta a_\beta z^\beta & \longmapsto & \big(z_j^{-\operatorname{ord}_{z_j}(f)} \sum_\beta a_\beta z^\beta\big)\big|_{z_j=0} \end{array}$$

Now, given $f \in \mathcal{O}_{X,x}$, if we write $f = \sum_\beta a_\beta z^\beta \in \widehat{\mathcal{O}}_{X,x}$, then by Theorem 2.3.10, we have

$$\nu_{x,\underline{w}}(f) = \big(\langle e_{i_1}, \beta_\alpha\rangle, \langle e_{i_2}, \beta_\alpha\rangle, \dots, \langle e_{i_k}, \beta_\alpha\rangle\big)$$

where $\beta_{\underline{\alpha}} \in A_f^\sigma$ is the exponent which minimizes the right hand side with respect to the lexicographic order in $\mathbb{R}^k$ given in the proof of that theorem. On the other hand, we have by definition

$$\nu_{\mathcal{F}}(f) = \big(\operatorname{ord}_{z_1}(f_1), \operatorname{ord}_{z_2}(f_2), \dots, \operatorname{ord}_{z_k}(f_k)\big)$$

where $f_1 = f$ and $f_r = \operatorname{res}_r(f_{r-1})$.

Now notice that

$$\begin{aligned}\operatorname{ord}_{z_1}(f) &= \min\{\langle e_1, \beta\rangle \mid \beta \in \operatorname{supp}(f)\}\\ &= \min\{\langle e_1, \beta\rangle \mid \beta \in A_f^\sigma\}\\ &= \langle e_1, \beta_{\underline{\alpha}}\rangle.\end{aligned}$$

This shows that the first coordinates of $\nu_{\mathcal{F}}(f)$ and $\nu_{x,\underline{w}}(f)$ are equal. Proceeding by induction, suppose the first j coordinates of $\nu_{\mathcal{F}}(f)$ and $\nu_{x,\underline{w}}(f)$ are equal. We get

$$\begin{aligned}\operatorname{ord}_{z_{j+1}}(f_{j+1}) &= \operatorname{ord}_{z_{j+1}}(\operatorname{res}_j(f_j))\\ &= \operatorname{ord}_{z_{j+1}}\left(z_j^{-\operatorname{ord}_{z_j}(f)} \sum_{\beta\in\operatorname{supp}(f_j)} a_\beta z^\beta\right)\Bigg|_{z_j=0}\\ &= \min\big\{\langle e_{j+1}, \beta\rangle \mid \beta \in \operatorname{supp}(f_j) \text{ and } \langle e_j, \beta\rangle = \langle e_j, \beta_{\underline{\alpha}}\rangle\}\\ &= \langle e_{j+1}, \beta_{\underline{\alpha}}\rangle \qquad \text{(by the definition of } \beta_{\underline{\alpha}})\end{aligned}$$

and so the $j+1$-coordinates of $\nu_{\mathcal{F}}(f)$ and $\nu_{x,\underline{w}}(f)$ coincide as well. This proves that the valuations are equal on $\mathcal{O}_{X,x}$ and so they coincide on $K(X)$. The theorem follows. □

2.4 Tropical weak approximation theorem

The aim of this section is to prove the weak approximation theorem in the tropical setting stated in the introduction.

2.4.1 Statement of the theorem

Recall that a subset $A \subseteq \mathbb{Z}_+^I$ is called an *antichain* for the partial order $\leq = \leq_{\text{cw}}$ if any pair of distinct elements $\beta, \gamma \in A$ are not comparable, i.e., $\beta \not\leq \gamma$ and $\gamma \not\leq \beta$. (This implies that A is necessarily finite.)

Definition 2.4.1 (Coherent family of antichains associated to cones)**.** Suppose for any cone σ, we have an antichain $A^\sigma \subseteq \mathbb{Z}_+^{I_\sigma}$. We call the collection $\mathcal{A} = \{A^\sigma \mid \sigma \in \Sigma(X,D)\}$ *coherent* if for any inclusion of faces $\tau \subseteq \sigma$, we have the relation

$$A^\tau = \min_{\leq} \operatorname{pr}_{\sigma \succ \tau}(A^\sigma).$$

Here $\mathrm{pr}_{\sigma\succ\tau}$ is the projection map $\mathbb{R}^{I_\sigma} \to \mathbb{R}^{I_\tau}$.

Theorem 2.4.2 (Tropical weak approximation theorem). *Let X be a smooth quasi-projective variety over a field k and let D be an SNC divisor on X. Let $\mathcal{A} = \{A^\sigma \mid \sigma \in \Sigma(X, D)\}$ be a coherent family of antichains. There exists then a rational function $f \in K(X)$ such that for each cone σ of $\Sigma(X, D)$, we have $f \in \mathcal{O}_{X,\eta_\sigma}$ and $A^\sigma = A^\sigma_f$.*

Remark 2.4.3. The theorem should be regarded as a tropical analogue of the weak approximation theorem in number theory. Stronger versions of this theorem might be true. Namely, given admissible expansions $f_\sigma \in \mathcal{O}_{X,\eta_\sigma}$ for each $\sigma \in \Sigma(X, D)$ such that each f_σ has only finitely many non-zero terms, and such that for inclusion of faces $\tau \subseteq \sigma$, we have $\iota_{\sigma\supset\tau}(f_\sigma) = f_\tau$, one might wonder whether there exists a rational function $f \in K(X)$ such that $f - f_\sigma$ has an admissible expansion in $\widehat{\mathcal{O}}_{X,\eta_\sigma}$ in which every monomial is divisible by a monomial in f_σ.

A corollary of the theorem is the following.

Corollary 2.4.4 (Approximation theorem for tropical functions). *Let X be a smooth quasi-projective variety over a field k and let D be an SNC divisor on X. For any tropical function $F : \Sigma(X, D) \to \mathbb{R}$, there is a rational function $f \in K(X)$ such that* $\mathrm{trop}(f) = F$.

The rest of this section is devoted to the proof of the above theorems. We first prove Theorem 2.4.2 and then later explain how to deduce the above corollary from this result.

2.4.2 Proof of Theorem 2.4.2 in the toric case

It would be more instructive to first treat the case of a toric variety with the arrangement of the corresponding toric divisors. In this situation, we can drop the quasi-projectivity condition.

Let Σ be a unimodular fan of dimension d in the real vector space $N_\mathbb{R}$ of the same dimension, and let $\mathbb{P}_\Sigma$ be the corresponding toric variety. Each ray ϱ in Σ gives the corresponding divisor D_ϱ in $\mathbb{P}_\Sigma$. By unimodularity assumption on Σ, the divisor $D = \cup_{\varrho\in\Sigma_1} D_\varrho$ is SNC.

Let σ be a cone in Σ, and denote by $\varrho_1, \dots, \varrho_d$ the rays of σ. Denote the rays of the dual cone $\sigma^\vee$ by $\zeta_1, \dots, \zeta_d$. Let $n_1, \dots, n_d$ be the primitive vectors of $\varrho_1, \dots, \varrho_d$ and denote by $m_1, \dots, m_d$ the primitive vectors of the rays $\zeta_1, \dots, \zeta_d$, respectively. Note that $\langle m_j, n_i\rangle = \delta_{i,j}$, where $\langle . , . \rangle$ denotes the duality pairing between N and M.

For each point $\underline{a} = (a_1, \dots, a_d) \in A^\sigma$, consider the rational function

$$f_{\sigma,\underline{a}} := \frac{(\chi^{m_1})^{a_1} \cdots (\chi^{m_d})^{a_d}}{(\chi^{m_1} + \cdots + \chi^{m_d} + 1)^\ell}$$

for a large enough integer ℓ to be determined later.

Let ϱ be a ray of Σ with primitive vector $n \in N$. The order of vanishing of $f_{\sigma,\underline{a}}$ along the component D_ϱ of D can be obtained as follows. First note that the order of vanishing of χ^{m_j} along D_ϱ is equal to $\langle m_j, n\rangle$. Moreover, the order of vanishing of $\chi^{m_1} + \cdots + \chi^{m_d} + 1$ along D_ϱ is equal to $\min\{0, \langle m_1, n\rangle, \langle m_2, n\rangle, \dots, \langle m_d, n\rangle\}$. Therefor, we get

$$\mathrm{ord}_{D_\varrho}(f_{\sigma,\underline{a}}) = \langle a_1 m_1 + \cdots + a_d m_d, n\rangle - \ell \cdot \min\{0, \langle m_1, n\rangle, \dots, \langle m_d, n\rangle\}.$$

In particular, for each $j = 1, \dots, d$, we get $\mathrm{ord}_{D_{\varrho_j}}(f_{\sigma,\underline{a}}) = a_j$. Moreover, if ϱ is different from $\varrho_1, \dots, \varrho_d$, then there exists an integer j among $1, \dots, d$ such that $\langle m_j, n\rangle < 0$. This implies that if ℓ is chosen to be large enough, the rational function $f_{\sigma,\underline{a}}$ will have a huge order of vanishing along D_ϱ.

Consider now the rational function f_σ in $K(\mathbb{P}_\Sigma)$ defined as

$$f_\sigma := \sum_{\underline{a} \in A^\sigma} f_{\sigma,\underline{a}} = \sum_{\underline{a}=(a_1,\dots,a_d)\in A^\sigma} \frac{(\chi^{m_1})^{a_1} \cdots (\chi^{m_d})^{a_d}}{(\chi^{m_1} + \cdots + \chi^{m_d} + 1)^\ell}.$$

In the completed local ring $\widehat{\mathcal{O}}_{\mathbb{P}_\Sigma, x_\sigma}$ we have the equality

$$\frac{(\chi^{m_1})^{a_1} \cdots (\chi^{m_d})^{a_d}}{(\chi^{m_1} + \cdots + \chi^{m_d} + 1)^\ell} = (\chi^{m_1})^{a_1} \cdots (\chi^{m_d})^{a_d} \cdot \left(1 + \sum_{k\geq 1} (-1)^k (\chi^{m_1} + \cdots + \chi^{m_d})^k\right)^\ell,$$

which gives an admissible expansion of f_σ with respect to the local parameters $\chi^{m_1}, \dots, \chi^{m_d}$ around η_σ, the point of intersection of $D_{\varrho_1}, \dots, D_{\varrho_d}$.

From this, we see that $A^\sigma_{f_{\sigma,\underline{a}}} = \{\underline{a}\}$, and since A^σ is an antichain, it follows $A^\sigma_{f_\sigma} = A^\sigma$.

Now let τ be another facet and denote by $\{\rho_j\}_{j=1}^d$ its rays. They correspond to the components $D_{\rho_1}, \dots, D_{\rho_d}$ of D with the torus-invariant point η_τ as the point of intersection. From the preceding discussion, we infer that if ρ_j is not a ray of σ, then, choosing ℓ large enough, we can ensure that f_σ has a large order of vanishing along D_{ρ_j}.

Since the order of vanishing of f_σ along such a component D_{ρ_j} is equal to the minimum j-th coordinate of any element of $A^\tau_{f_\sigma}$, we see that all the elements of $A^\tau_{f_\sigma}$ have large j-th coordinates. On the other hand, on the intersection face $\delta = \tau \cap \sigma$, we have

$$\mathrm{pr}_{\tau \succ \delta}(A^{\tau}_{f_\sigma}) = \mathrm{pr}_{\sigma \succ \delta}(A^{\sigma}_{f_\sigma})$$

where $\mathrm{pr}_{\cdot \succ \cdot}$ denote the corresponding projection maps, as in the previous sections.

In particular, by the coherence of the collection A^σ, this shows that if ℓ is chosen to be large enough, then any element in $A^{\tau}_{f_\sigma}$ dominates an element of A^τ, that is,

$$A^\tau = \min_{\leq}\Big(A^\tau \cup A^{\tau}_{f_\sigma}\Big).$$

Now we choose ℓ large enough, and for each facet σ of Σ, we define the rational function f_σ as above. Let

$$f := \sum_{\sigma \in \Sigma_d} \lambda_\sigma f_\sigma, \tag{2.17}$$

for generic choices of λ_σ in the base field.

Observe that for any facet τ, for any pair of rational functions h, g, and generic choice of scalars λ, μ in the base field, we have

$$A^{\tau}_{\lambda h + \mu g} = \min_{\leq_{\mathrm{cw}}}\big(A^{\tau}_{h} \cup A^{\tau}_{g}\big).$$

We thus infer that for any facet τ of Σ and for the function f defined in (2.17), we have

$$A^{\tau}_{f} = \min_{\leq}\left(\bigcup_{\sigma \in \Sigma_d} A^{\tau}_{f_\sigma}\right) = A^{\tau}_{f_\tau} = A^\tau.$$

To conclude, we note that the coherence condition implies that more generally, for each face δ of Σ, we have $A^{\delta}_{f} = A^{\delta}$ and the result follows.

2.4.3 Proof of Theorem 2.4.2

We now treat the theorem in its full generality. In the following, we will use the following terminology borrowed from lattice theory concerning the combinatorial structure of faces in a cone complex.

Definition 2.4.5 (The (multivalued) meet and join operations $\wedge$ and $\vee$)**.** Given two faces τ and σ in a cone complex Σ, we denote by $\tau \wedge \sigma$ the set of all maximal common faces between τ and σ. If τ and σ are faces of a cone ζ, we denote by $\tau \vee_\zeta \sigma$ the unique minimal face of ζ that contains both τ and σ. Notice that if Σ does not have parallel faces, then $\tau \wedge \sigma$ is a single cone and $\tau \vee_\zeta \sigma$ is independent of ζ, so in this case, we denote this cone by $\tau \vee \sigma$.

In the rest of this section we assume given an SNC divisor $D = \sum_{i \in \mathcal{I}} D_i$ in X, and we consider the dual cone complex $\Sigma(X, D)$.

Adapted family of rational functions

Proceeding somehow similarly as in the proof of the toric case, we will prove the existence of a family of rational functions with nice properties depicted in the following theorem.

Theorem 2.4.6. *Let σ be a face of $\Sigma(X, D)$. There exists a rational function $u_\sigma \in K(X)$ with the following properties:*

(P1) u_σ belongs to all local rings $\mathcal{O}_{x,\eta_\delta}$, for $\delta \in \Sigma(X, D)$, and is invertible in $\mathcal{O}_{X,\eta_\sigma}$.

(P2) It has a zero along the divisor D_j for each $j \notin I_\sigma$.

(P3) For any face τ of $\Sigma(X, D)$ the following holds. If $\zeta \in \tau \wedge \sigma$ is a maximal common face of τ and σ, then the restriction $u_{\sigma|S_\zeta}$ of u_σ on the stratum S_ζ has a zero along all the strata $S_{\zeta \vee_\tau \varrho_j} \subseteq S_\tau$ for any $j \in I_\tau \setminus I_\zeta$.

Definition 2.4.7 (Adapted family of rational functions)**.** Given a dual cone complex $\Sigma(X, D)$, the collection of rational functions u_σ, $\sigma \in \Sigma(X, D)$, verifying the properties (P1), (P2), and (P3) in the above theorem is called an *adapted family of rational functions* for the dual cone complex.

In order to prepare for the proof of the above theorem, we start by stating two lemmas concerning the existence of rational functions with prescribed regularity on a given finite set of points.

Lemma 2.4.8. *Let $Y \subseteq X$ be a closed irreducible set and let x be a (non-necessarily closed) point in $X \setminus Y$. Then there exists an irreducible divisor $E \subset X$ which contains Y but not x.*

Proof. Denote by η_Y the generic point of Y. As X is separated we have $\mathcal{O}_{X,x} \setminus \mathcal{O}_{X,\eta_Y} \neq \varnothing$. Take $f \in \mathcal{O}_{X,x} \setminus \mathcal{O}_{X,\eta}$, then η_Y is contained in the indeterminacy set of f. As X is smooth the indeterminacy set is the support of the negative part of $\mathrm{div}(f)$, hence there is a component E of this negative part containing Y but not x. $\square$

Lemma 2.4.9. *Suppose X is quasi-projective. Given a hypersurface $E \subset X$, points $x_1, \dots, x_n \in E$ and a point $x \notin E$, there is a rational function u which vanishes on each component of E with order of vanishing one, belongs to each local ring $\mathcal{O}_{X,x_i}$ for any $i = 1, \dots, n$, and which is invertible at x.*

Proof. Taking a projective compactification, we can assume without loss of generality that X is projective. Consider an ample divisor H not containing any of the points $x, x_1, \dots, x_n$ and not sharing any component with E. Then, for some large integer number n, the divisor $nH - E$ is very ample, and so base point free. Therefor, there is a section u of $\mathcal{O}(nH - E)$ which does not vanish on x. The corresponding rational function satisfies all the required properties. □

We are now ready to proof the existence of adapted families of rational functions.

Proof of Theorem 2.4.6. In order to show this, we first apply Lemma 2.4.8 to each stratum S_τ not contained in S_σ to get an irreducible divisor $E_\tau \subset X$ which contains S_τ but not S_σ. Let

$$E := \sum_{\tau\colon S_\tau \not\supset S_\sigma} E_\tau.$$

We now apply Lemma 2.4.9 to E, the points η_δ for $\delta \in \Sigma(X, D)$, and the point η_σ, which clearly does not belong to E. We infer the existence of a rational function u_σ in $K(X)$ that vanishes on each component E_τ of E, which belongs to $\mathcal{O}_{X,\eta_\delta}$ for any point $\delta \in \Sigma(X, \Sigma)$, and which is invertible in $\mathcal{O}_{X,\eta_\sigma}$. We claim u_σ verifies all the claimed properties (P1), (P2), and (P3).

The first claim (P1) is clearly satisfied by the construction of u_σ.

Also, notice that if $j \notin I_\sigma$, then $S_{\varrho_j} = D_j$. Since both D_j and E_{ϱ_j} are irreducible, we must have $E_{\varrho_j} = D_j$. Since $j \notin I_\sigma$, we have $\eta_\sigma \notin D_j$, and so by the choice of u_σ, it should vanish on E_{ϱ_j}. This shows that u_σ verifies Property (P2).

Finally, let ζ be a maximal common face of τ and σ, and let $j \in I_\tau \setminus I_\zeta$. Note that by the maximality of ζ, the cone $\zeta \vee_\tau \varrho_j$ is not a face of σ. This means that η_σ does not belong to $S_{\zeta \vee_\tau \varrho_j}$, and so u_σ vanishes on $E_{\zeta \vee_\tau \varrho_j}$. Notice as well that $u_\sigma \in \mathcal{O}_{X,\eta_\zeta}$ and $S_{\zeta \vee_\tau \varrho_j} \subseteq E_{\zeta \vee_\tau \varrho_j} \cap S_\zeta$. It follows that the restriction $u_{\sigma|S_\zeta}$ of u_σ to S_ζ vanishes on $S_{\zeta \vee_\tau \varrho_j}$. This proves that u_σ verifies also Property (P3). □

Proof of the weak approximation theorem

Now, we come back to the proof of the approximation theorem. Let σ be a face of $\Sigma(X,D)$. Applying again Lemma 2.4.9, we find a local equation z_i for D_i around η_σ for each $i \in I_\sigma$ with the additional property that $z_i \in \mathcal{O}_{X,\tau}$ for each cone τ that is not a face of σ. With this choice of local parameters, we define

$$f_\sigma := u_\sigma^\ell \sum_{\underline{a} \in A^\sigma} \prod_{i \in I_\sigma} z_i^{a_i}, \tag{2.18}$$

for a large enough number ℓ which will be precised in a moment. Notice that f_σ is defined in each local ring $\mathcal{O}_{X,\tau}$ for any cone $\tau \in \Sigma(X,D)$. We prove the following.

Proposition 2.4.10. *Provided ℓ is large enough, f_σ verifies the following two properties.*

1. *The set $A^\sigma_{f_\sigma}$ is equal to A^σ, and*

2. *For each face τ of $\Sigma(X,D)$ different from σ, we have*

$$\min_{\leq}(A^\tau_{f_\sigma} \cup A^\tau) = A^\tau.$$

Using this proposition, we can finish the proof of our approximation theorem.

Proof of Theorem 2.4.2. Let

$$f := \sum_\sigma \lambda_\sigma f_\sigma$$

where λ_σ is a generic choice of coefficients for each face of $\Sigma(X,D)$. Then, applying the above proposition, we get for each face τ of $\Sigma(X,D)$,

$$A^\tau_f = \min_{\leq} \bigcup_\tau A^\tau_{f_\sigma} = A^\tau.$$

In other words, f is the rational function we have been looking for. □

At this point, we are only left with the proof of Proposition 2.4.10.

Proof of Proposition 2.4.10. We use the notations preceding the proposition. By invertibility of u_σ in $\mathcal{O}_{X,\eta_\sigma}$, the expression (2.18) gives an admissible expansion of f_σ, and so we clearly have $A^\sigma_{f_\sigma} = A^\sigma$. This shows the assertion (1) in the proposition.

Now, in order to prove Claim (2), let $\tau \neq \sigma$ be a face of $\Sigma(X, D)$, and take local parameters w_i for each D_i around η_τ, for $i \in I_\tau$. The element u_σ lives in $\mathcal{O}_{X,\eta_\tau}$ and so in $\widehat{\mathcal{O}}_{X,\eta_\tau}$. Consider an admissible expansion in $\widehat{\mathcal{O}}_{X,\eta_\tau}$ for u_σ

$$u_\sigma = \sum_\beta c_\beta w^\beta. \tag{2.19}$$

Property (P2) above implies that for each $j \in I_\tau \setminus I_\sigma$ and for each $\alpha = (\alpha_i)_{i \in I_\tau}$ in the support of (2.19), we should have

$$\alpha_j \geq \mathrm{ord}_{D_j}(u_\sigma) \geq 1.$$

More generally, we claim the following.

Claim 2.4.11. *For each α in the support of the admissible expansion (2.19), there is a maximal common face ζ of τ and σ such that for each $j \in I_\tau \setminus I_\zeta$, we have $\alpha_j \geq 1$.*

Proof. Let α be an element in the support of the admissible expansion (2.19), and consider $J = \{i \in I_\tau \mid \alpha_i = 0\}$. Let τ_J be the face of τ corresponding to $J \subseteq I_\tau$. It will be enough to show that $\tau_J \subseteq \sigma$. Indeed, in this case, τ_J will be a common face of τ and σ, and so there exists a face $\zeta \in \tau \wedge \sigma$ which contains τ_J. For any $j \in I_\tau \setminus I_\zeta$, we have $\alpha_j \geq 1$, and so the claim follows.

For the sake of a contradiction, suppose τ_J is not a face of σ, and let ζ be a maximal common face of σ and τ_J. In particular, $\zeta \subsetneq \tau_J$, which implies that $I_\zeta \subsetneq J$.

We have a projection

$$\pi : \mathcal{O}_{X,\eta_\tau} \to \mathcal{O}_{S_\zeta,\eta_\tau}$$
$$h \mapsto h_{|S_\zeta}$$

that extends by continuity to a projection $\pi : \widehat{\mathcal{O}}_{X,\eta_\tau} \to \widehat{\mathcal{O}}_{S_\zeta,\eta_\tau}$. Using this, we obtain an admissible expansion for $u_{\sigma|S_\zeta}$ in $\widehat{\mathcal{O}}_{S_\zeta,\eta_\tau}$ in terms of local parameters $w_{i|S_\zeta}$ for $S_{\zeta \vee_\tau \varrho_i}$, for $i \in I_\tau \setminus I_\zeta$. This is obtained by applying the projection π to both sides of (2.19). Indeed, for each β, the restriction $c_{\beta|S_\zeta}$ is still a unit in $\widehat{\mathcal{O}}_{S_\zeta,\eta_\tau}$, and so we get

$$u_{\sigma|S_\zeta} = \sum_\beta c_\beta w^\beta_{|S_\zeta},$$

which is an admissible expansion in $\widehat{\mathcal{O}}_{S_\zeta,\eta_\tau}$. Note in particular that since $I_\zeta \subseteq J$, and since

$\alpha_i = 0$ for all $i \in J$, we get that $\pi_{I_\tau \setminus I_\zeta}(\delta)$ is in the support of the admissible expansion for $u_{\sigma|S_\zeta}$.

Now take $j \in J \setminus I_\zeta$. As $u_{\sigma|S_\zeta}$ vanishes along the divisor $S_{\zeta \vee_\tau \varrho_j}$ and a local equation for this is given by $w_{j|S_\zeta}$, we should have that $w_{j|S_\zeta}$ divides $u_{\sigma|S_\zeta}$ inside $\mathcal{O}_{S_\zeta, \eta_\tau}$. In particular, this implies that for any β in the admissible expansion $u_{\sigma|S_\zeta}$, we must have $\beta_j > 0$. In particular, this gives $\alpha_j > 0$ which contradicts the definition of J, so the claim follows. □

Let $h_\sigma = \sum_{\underline{a} \in A^\sigma} \prod_{j \in I_\sigma} z_j^{a_j}$. We have $f_\sigma = u_\sigma^\ell h_\sigma$, from which we get the inclusion

$$A^\tau_{f_\sigma} \subseteq A^\tau_{u_\sigma^\ell} + A^\tau_{h_\sigma}.$$

Moreover, we have

$$A^\tau_{u_\sigma^\ell} \subseteq \underbrace{A^\tau_{u_\sigma} + \cdots + A^\tau_{u_\sigma}}_{\ell \text{ times}}.$$

Let now β be an element of $A^\tau_{f_\sigma}$. It follows that we can write β as the sum of ℓ elements in $A^\tau_{u_\sigma}$ and an element $\gamma \in A^\tau_{h_\sigma}$.

By what preceded, we have for each $j \in I_\tau \setminus I_\sigma$ and each element α in $A^\tau_{u_\sigma}$ that $\alpha_j \geq 1$. It follows that we have $\beta_j \geq \ell$ for all $j \in I_\tau \setminus I_\sigma$. For ℓ large enough, this is certainly larger than the j-coordinate of any element in A^τ.

We now show how to control the j-coordinates of β for $j \in I_\tau \cap I_\sigma$. For this we write

$$\beta = \alpha^1 + \cdots + \alpha^\ell + \gamma$$

for $\alpha^1, \ldots, \alpha^\ell \in A^\tau_{u_\sigma}$ and $\gamma \in A^\tau_{h_\sigma}$.

Applying Claim 2.4.11, for each α^i, which is in the support of the admissible expansion (2.19) of u_σ, we infer the existence of a maximal common face ζ_i of τ and σ such that for each $j \in I_\tau \setminus I_{\zeta_i}$, we have $\alpha^i_j \geq 1$. Here α^i_j is the j-coordinate of α^i.

Let r be the number of elements of $\tau \wedge \sigma$. By the pigeonhole principle, there is a maximal common face ζ of both σ and τ such that we have $\zeta_i = \zeta$ for at least ℓ/r indices $i \in [\ell]$. We thus get for each $j \in I_\tau \setminus I_\zeta$, the inequality

$$\beta_j = \alpha^1_j + \cdots + \alpha^\ell_j + \gamma_j \geq \ell/r + \gamma_j.$$

We infer again that if ℓ is large enough, the j-coordinate of β is larger than the j-coordinate of any element in A^τ provided that j is in $I_\tau \setminus I_\zeta$.

Now to finish the proof of Property (2), note that since ζ is a common face for τ and σ, we have by the coherence property that

$$\min_{\leq} \mathrm{pr}_{\tau\succ\zeta}(A^{\tau}_{f_\sigma}) = \min_{\leq} \mathrm{pr}_{\sigma\succ\zeta}(A^{\sigma}_{f_\sigma}) \qquad \text{and} \qquad \min_{\leq} \mathrm{pr}_{\tau\succ\zeta}(A^{\tau}) = \min_{\leq} \mathrm{pr}_{\sigma\succ\zeta}(A^{\sigma}).$$

Since $A^{\sigma}_{f_\sigma} = A^{\sigma}$, this shows that

$$\min_{\leq} \mathrm{pr}_{\tau\succ\zeta}(A^{\tau}_{f_\sigma}) = \min_{\leq} \mathrm{pr}_{\tau\succ\zeta}(A^{\tau}).$$

Hence, since $\beta \in A^{\tau}_{f_\sigma}$, there is an element $\beta' \in A^{\tau}$ such that $\mathrm{pr}_{\tau\succ\zeta}(\beta) \geq \mathrm{pr}_{\tau\succ\zeta}(\beta')$. Moreover, by what we discussed above, all the j-coordinates of β for j outside I_ζ are also larger than the corresponding j-coordinates of β' (if we choose ℓ large enough). This shows that we actually have $\beta \geq \beta'$ and the claim in (2) follows, namely that

$$\min_{\leq}(A^{\tau}_{f_\sigma} \cup A^{\tau}) = A^{\tau}.$$

□

2.4.4 Proof of Corollary 2.4.4

The proof of this result is based on the following proposition

Proposition 2.4.12. *Any tropical function F on the cone complex $\Sigma(X, D)$ can be written as the difference of two tropical functions F_1 and F_2 such that the restriction of F_i to each cone $\sigma \in \Sigma(X, D)$ is convex and non-negative.*

Proof. In order to prove the existence of F_1, F_2, it will be enough to prove

(i) there exists a non-negative tropical function G that is convex on each cone of the original dual complex $\Sigma(X, D)$, and

(ii) if G is such a function, then for any large enough integer ℓ, the function $F + \ell G$ is a non-negative convex tropical function.

In fact, given this, we can write $F = (F + \ell G) - \ell G$ and take $F_1 = F + \ell G$ and $F_2 = \ell G$ which verify the convexity condition.

Assuming (i), we now prove (ii). Let G be a convex tropical function on $\Sigma(X, D)$, and let $\Sigma(G)$ be the corresponding subdivision of $\Sigma(X, D)$. For each cone σ in $\Sigma(X, D)$, we get a quasi-projective subdivision Δ_σ of σ, which can be completed to a projective rational fan in $N_{\sigma,\mathbb{R}}$. Let X_σ be the projective toric variety associated to this complete fan, and let E_σ be the corresponding toric divisor, which is thus an ample divisor. The restriction $F_{|\sigma}$ gives a divisor L_σ in X. Since E_σ is very ample, the divisor $\ell E_\sigma + L_\sigma$ remains very ample for a large enough integer number ℓ.

Given the bijection between order functions on Δ_σ and rational ideal sheafs on X, this bijection makes a correspondence between ample divisors and order functions that are convex on Δ_σ, hence the fact that $F_{|\sigma} + \ell G_{|G}$ remains convex for some large enough number n follows from the fact that $\ell E_\sigma + L_\sigma$ is very ample in X. □

Proof of Corollary 2.4.4. By the proposition above, there are tropical functions F_1, F_2 such that each F_i is non-negative and convex on each facet of $\Sigma(X, D)$ and $F_1 - F_2 = F$. Each F_i, for $i = 1, 2$, is given by a coherent family of antichains A_i^σ in the sense that for each cone σ we have

$$F_i(x) = \min\{\langle x, \beta\rangle \mid \beta \in A_i^\sigma\}.$$

By approximation theorem, there are rational functions f_1, f_2 such that $A_{f_i}^\sigma = A_i^\sigma$ for $i = 1, 2$ and fo each cone σ on $\Sigma(X, D)$. We therefor get $F = \mathrm{trop}(f_1/f_2)$ and the theorem follows. □

2.5 Tropical topology on tangent cone bundles

In this section we study the *tropical topology* on the spaces of quasi-monomial valuations. By our duality and approximation theorems proved in the previous sections, this coincides with the coarsest topology on the tangent cone which makes the directional derivatives of tropical functions all continuous. Therefore, we define the tropical topology in the general framework of cone complexes and their tangent cones.

2.5.1 Definition of the topology

In order to motivate what follows, we first observe that tangent cone bundles inherit a natural Euclidean topology defined as follows.

Definition 2.5.1 (Euclidean Topology)**.** Let Σ be a cone complex and k a non-negative integer number. The Euclidean topology on the tangent cone $T\mathcal{C}^k\,\Sigma$ is the topology defined by the inclusion of

$$TC^k\,\Sigma \hookrightarrow \bigcup_{\sigma\in\Sigma} N^k_{\sigma,\mathbb{R}}$$

where the space on the right hand side is obtained by gluing the vector spaces $N^k_{\sigma,\mathbb{R}}$ with the quotient topology, and the topology on each $N^k_{\sigma,\mathbb{R}}$ is the topology of a finite dimensional real vector space.

This topology however turns out to be not properly adapted to the study of valuation theory and tropical geometry in the higher rank context. This is suggested by the following example of a tropical function whose derivative is not continuous with respect to the Euclidean topology.

Example 2.5.2. Let $\sigma = \mathbb{R}_+ \times \mathbb{R}_+$ and consider the tropical function

$$\begin{aligned} F : \sigma &\longrightarrow \mathbb{R} \\ (x_1, x_2) &\longmapsto \min\{x_1, x_2\}. \end{aligned}$$

For the first directional derivative of F, we have

$$\begin{aligned} \mathscr{D}F : TC\,\sigma &\longrightarrow \mathbb{R}^2 \\ ((x_1,x_2);(y_1,y_2)) &\longmapsto \begin{cases} (x_1,y_1) & \text{if } x_1 < x_2 \text{ or } (x_1 = x_2 \text{ and } y_2 < y_1) \\ (x_2,y_2) & \text{if } x_1 > x_2 \text{ or } (x_1 = x_2 \text{ and } y_2 < y_1). \end{cases} \end{aligned}$$

This map is not continuous with respect to the Euclidean topology. To see this, consider the map

$$t \mapsto \mathscr{D}F((t, 1-t); (y_1, y_2))$$

for $y_1 \neq y_2$. This function has a discontinuity at $t = \frac{1}{2}$.

In order to use topological tools in the context of higher rank valuation theory, and in view of the analytic description of higher rank quasi-monomial valuations, we are naturally led to introduce the following topology.

Definition 2.5.3 (Tropical topology)**.** Let Σ be a cone complex and $TC^k\,\Sigma$ its tangent cone. We consider $\mathbb{R}^{k+1}$ with its Euclidean topology and define the *tropical topology* on $TC^k\,\Sigma$ as the coarsest topology which makes all the maps

$$\mathscr{D}^k F : TC^k\,\Sigma \longrightarrow \mathbb{R}^{k+1}$$

continuous for any tropical function $F : \left|\Sigma\right| \to \mathbb{R}$.

Remark 2.5.4. In the case $\Sigma = \Sigma(X, D)$ for a smooth quasi-projective variety X and an SNC divisor D on X, the tropical topology on $T\mathcal{C}^k\,\Sigma$ is the coarsest topology such that for any rational function $f \in K(X)^\times$, the directional derivative $\mathscr{D}^k \operatorname{trop}(f)$ is a continuous function from $T\mathcal{C}^k\,\Sigma \to \mathbb{R}^{k+1}$. This is a direct consequence of the approximation theorem.

2.5.2 Description of the topology

The aim of this section is to give a description of this topology by introducing a basis of open sets. This will be based on the following definition.

Definition 2.5.5 ($\widetilde{\Sigma}$-open sets)**.** Let Σ be a cone complex and $\widetilde{\Sigma}$ be a rational subdivision of it. A set $U \subset T\mathcal{C}^k\,\Sigma$ is called a $\widetilde{\Sigma}$-*open set* if $U \cap T\mathcal{C}^k\,\sigma$ is open in $T\mathcal{C}^k\,\sigma$ with respect to the Euclidean topology for every cone $\sigma \in \widetilde{\Sigma}$.

Here is the main theorem of this section.

Theorem 2.5.6. *Let Σ be a cone complex and consider its tangent cone $T\mathcal{C}^k\,\Sigma$ for k a non-negative integer number. Then*

1. *For each subdivision $\widetilde{\Sigma}$ of Σ, the $\widetilde{\Sigma}$-open sets of $T\mathcal{C}^k\,\Sigma$ are open with respect to the tropical topology.*
2. *The union of all $\widetilde{\Sigma}$-open sets for $\widetilde{\Sigma}$ a rational subdivision of Σ form a basis of opens sets for the tropical topology.*

The proof of this theorem is given in the next section. We state the following corollary.

Corollary 2.5.7. *Let Σ be a cone complex and $T\mathcal{C}^k\,\Sigma$ its tangent cone endowed with the tropical topology. Then*

1. *The tropical topology is finer than the Euclidean topology, in particular it is both Hausdorff and normal.*
2. *A set is dense in $T\mathcal{C}^k\,\Sigma$ with respect to the tropical topology if and only if it is dense with respect to the Euclidean topology.*
3. *$T\mathcal{C}^k\,\Sigma$ is not locally compact in general (in fact, as soon as $k > 0$ and the dimension of Σ is at least two).*

Proof. As any Euclidean open set is $\widetilde{\Sigma}$-open for any subdivision $\widetilde{\Sigma}$ of Σ we get that the Euclidean topology is finer than the tropical topology. Hence it is Hausdorff and normal.

For point (2), it is enough to notice that for any subdivision $\widetilde{\Sigma}$ of Σ and any $\widetilde{\Sigma}$-open set U, there is an Euclidean open set V contained in U. For example, we can take a non-empty set consisting of of all the points whose first coordinate is in the relative interior of some given cone in $\widetilde{\Sigma}$.

For point (3), notice that if U is open and $\overline{U}$ is compact, then as we saw in the proof of (2), we could find an open set $V \subseteq U$ such that V is open in the Euclidean topology and moreover, the closure $\overline{V}$ with respect to the tropical topology is compact. Denoting by $\overline{V}_{\text{Trop}}$ and $\overline{V}_{\text{Euc}}$ the set $\overline{V}$ endowed with its tropical and Euclidean topology, respectively, we see first that the identity map

$$\mathrm{id} : \overline{V}_{\text{Trop}} \to \overline{V}_{\text{Euc}}$$

is continuous. Moreover, as $\overline{V}_{\text{Trop}}$ is compact and $\overline{V}_{\text{Euc}}$ is Hausdorff, the identify would be a homeomorphism. This implies that the tropical and the Euclidean topologies agree on V. However this is not possible if we have both $k > 0$ and $\dim \Sigma > 1$ as otherwise, we could choose a subdivision $\widetilde{\Sigma}$ of Σ subdividing V. Then for some $W \subseteq V$ and some $\sigma \in \widetilde{\Sigma}$ we would have that $W \cap T\mathcal{C}\,\sigma$ is $\widetilde{\Sigma}$-open but not an Euclidean open set. □

2.5.3 Proof of Theorem 2.5.6

We adapt the following terminology in the sequel. For a cone σ of a cone complex Σ, and a point $(x; \underline{w}) \in T\mathcal{C}^k\,\Sigma$, by saying σ *supports the point* $(x; \underline{w})$ we mean the point $(x; \underline{w})$ belongs to $T\mathcal{C}^k\,\sigma$.

We first prove the second point assuming the first.

Proof of (2). Let $F\colon \left|\Sigma\right| \to \mathbb{R}$ be a tropical function and consider a rational subdivision $\widetilde{\Sigma}$ of Σ such that F is linear on each cone of $\widetilde{\Sigma}$. Let $V \subset \mathbb{R}^{k+1}$ be an open set for the Euclidean topology. We show that $\left(\mathscr{D}^k F\right)^{-1}(V)$ is $\widetilde{\Sigma}$-open. This proves the result.

Let δ be a cone of $\widetilde{\Sigma}$. By the choice of $\widetilde{\Sigma}$, there is a linear function F_δ on $N_{\delta,\mathbb{R}}$ such that for any point $(x; \underline{w}) \in T\mathcal{C}^k\,\delta$, with $\underline{w} = (w_1, \dots, w_k)$, we have

$$\mathscr{D}^k F(x; \underline{w}) = (F_\delta(x), F_\delta(w_1), \dots, F_\delta(w_k)).$$

The intersection

$$\left(\mathscr{D}^k F\right)^{-1}(V) \cap T\mathcal{C}^k\,\delta = \big(\underbrace{F_\delta \times \cdots \times F_\delta}_{(k+1)\text{ times}}\big)^{-1}(V) \cap T\mathcal{C}^k\,\delta$$

is clearly an open set in $T\mathcal{C}^k\,\delta$ for the Euclidean topology, and the claim follows.

□

Proof of (1). Let $\widetilde{\Sigma}$ be a rational subdivision of Σ and let U be a $\widetilde{\Sigma}$-open set. We have to show that U is open for the tropical topology of $T\mathcal{C}^k\,\Sigma$.

We first observe that if $\widetilde{\Sigma}'$ is a subdivision of $\widetilde{\Sigma}$, any $\widetilde{\Sigma}$-open set is also $\widetilde{\Sigma}'$-open. Therefore, in order to prove the above claim, we can assume that $\widetilde{\Sigma}$ is simplicial.

Take $(x;\underline{w}) \in U$. We will prove that $(x;\underline{w})$ is an interior point of U for the tropical topology, which clearly implies the result. For this, we will explicitly construct a neighborhood of $(x;\underline{w})$ for the tropical topology included in U.

Let ζ be the minimal face of $\widetilde{\Sigma}$ which supports $(x;\underline{w})$. For each facet δ of $\widetilde{\Sigma}$ we find a rational subdivision $\widetilde{\Sigma}_\delta$ of $\widetilde{\Sigma}$ with the following properties.

(a) $\widetilde{\Sigma}_\delta$ is simplicial.

(b) There is a unique facet in $\widetilde{\Sigma}$ denoted by $\gamma = \gamma_{\zeta,\delta}$ which contains ζ and which is contained in δ.

(c) For each pair (δ, ϱ) consisting of δ and a ray ϱ of δ, we can find a tropical function $F^{\delta,\varrho}$ on $\left|\Sigma\right|$ such that the following properties hold:

   (1) $F^{(\delta,\varrho)}$ is linear on each cone of $\widetilde{\Sigma}_\delta$.

   (2) over the facet γ of $\widetilde{\Sigma}_\delta$, we have $F^{\delta,\varrho}|_\gamma = \chi^m|_\gamma$ where m is the primitive element in the ray dual to ϱ in $\delta^\vee$ and χ^m is the linear function induced by this vector. In other words, $F^{\delta,\varrho}$ takes value one on the primitive vector of ϱ and value zero on all the rays of δ.

For such a fixed facet δ of $\widetilde{\Sigma}$ which contains ζ, consider the function Φ_δ defined by the collection of functions $F^{\delta,\varrho}$ for ϱ a ray of δ, so

$$\Phi_\delta := \left(F^{\delta,\varrho}\right)_{\varrho \text{ ray of } \delta} : \Sigma \to \mathbb{R}^{\dim(\delta)}.$$

Let $\gamma = \gamma_{\zeta,\delta}$ be the facet of $\widetilde{\Sigma}(\zeta)$ which contains ζ and which is contained in δ. By Property (c), the linear functions $F^{\delta,\varrho}$ for ϱ a ray of δ are linearly independent on γ, and so Φ_δ restricted to γ is a homeomorphism with its image in $\mathbb{R}^{\dim(\delta)}$.

We remark now that the directional derivative map

$$\Psi_\delta = (\mathscr{D}^k F^{\delta,\varrho})_{\varrho \text{ ray of } \delta} : T\mathcal{C}^k\,\Sigma \to (\mathbb{R}^{\dim(\delta)})^{k+1} \tag{2.20}$$

is a homeomorphism with its image when restricted to $T\mathcal{C}^k\,\gamma$, when we put on $T\mathcal{C}^k\,\gamma$ its Euclidean topology. Indeed, restricted to $T\mathcal{C}^k\,\gamma$, Ψ_δ can be identified with the restriction to $T\mathcal{C}^k\,\gamma$ of the invertible linear map

$$\big(\underbrace{\Phi_\delta \times \Phi_\delta \times \cdots \times \Phi_\delta}_{(k+1)\text{ times}}\big) : \mathbb{R}^{\dim(\delta)\times(k+1)} \to \mathbb{R}^{\dim(\delta)\times(k+1)}.$$

Hence, there is an open set $U_\delta \subseteq (\mathbb{R}^{k+1})^{\dim\delta}$ such that its preimage under the map (2.20) satisfies

$$\Psi_\delta^{-1}(U_\delta) \cap T\mathcal{C}^k\,\gamma = U \cap T\mathcal{C}^k\,\gamma.$$

Note that

$$\Psi_\delta^{-1}(U_\delta) = \bigcap_{\delta,\varrho \text{ ray in } \delta} \big(\mathscr{D}^k F^{(\delta,\varrho)}\big)^{-1}(U_\delta),$$

and so $\Psi_\delta^{-1}(U_\delta)$ is an open set in the tropical topology and therefore a neighbourhood of $(x;\underline{w}) \in U$. This proves that $(x;\underline{w})$ is an interior point of the intersection $\bigcap_\delta \Psi_\delta^{-1}(U_\delta)$ for δ running over all facets which contain ζ, which is an open set for the tropical topology. Denote by W this intersection. Note that we have $W \cap T\mathcal{C}^k\,\gamma_{\zeta,\delta} \subset U \cap T\mathcal{C}^k\,\gamma_{\zeta,\delta}$ for each facet δ which contains δ.

Let now $\widetilde{\Sigma}'$ be a rational subdivision of Σ with the following properties:

- $\widetilde{\Sigma}'$ is finer than $\widetilde{\Sigma}_\delta$ for all facets δ which contain ζ.
- there exists a tropical function G which is linear on each cone of $\widetilde{\Sigma}'$, and which is strictly positive on the relative interior of each cone τ of $\widetilde{\Sigma}'$ which supports $(x;\underline{w})$ and which is non-positive everywhere else.

Then, $(\mathscr{D}^k G)^{-1}(\mathbb{R}_{>0} \times \mathbb{R}^k)$ is an open set in the tropical topology, and moreover, it is contained in the union $\bigcup_\tau T\mathcal{C}^k\,\tau$ where the union goes over all cones τ of $\widetilde{\Sigma}'$ which support $(x;\underline{w})$.

It follows finally with these constructions that

$$(x;\underline{w}) \in (\mathscr{D}^k G)^{-1}(\mathbb{R}_{>0} \times \mathbb{R}^k) \cap \bigcap_{\delta} \Psi_{\delta}^{-1}(U_{\delta}) \subseteq \bigcup_{\delta} U \cap T\mathcal{C}^k\,\delta$$

where δ runs over facets of $\widetilde{\Sigma}$ which contain $(x;\underline{w})$.

We finally infer that U is a neighborhood of $(x;\underline{w})$ in the tropical topology, and the theorem follows. □

2.6 Spaces of valuations and the retraction map

For a given variety X, we introduce some spaces of valuations and show how for an SNC divisor D, the tangent cone $T\mathcal{C}^k\,\Sigma(D)$ endowed with its tropical topology naturally fits inside them.

2.6.1 Higher rank analytification and its centroidal filtration

Definition 2.6.1. Given a variety X, we define the *birational analytification of X of rank bounded by k* as the set

$$X^{\mathrm{bir},k} := \big\{\nu : K(X)^* \to \mathbb{R}^k \mid \nu \text{ is a valuation}\big\}$$

endowed with the coarsest topology which makes continuous all the evaluation maps

$$\begin{aligned} \mathrm{ev}_f : X^{\mathrm{bir},k} &\longrightarrow \mathbb{R}^k \\ \nu &\longmapsto \nu(f), \end{aligned}$$

for any $f \in K(X)^*$, where $\mathbb{R}^k$ is given its Euclidean topology. We define moreover the following subspaces of $X^{\mathrm{bir},k}$

$$\begin{aligned} X^{\exists,k} &:= \big\{\nu \in X^{\mathrm{bir},k} \mid \nu \text{ has center in } X\big\} \\ X^{\neg,k} &:= \big\{\nu \in X^{\mathrm{bir},k} \mid \nu \text{ does not have center in} X\big\} \end{aligned}$$

and endow them with the topology induced by that of $X^{\mathrm{bir},k}$.

Remark 2.6.2. Notice that $X^{\mathrm{bir},k} = X^{\exists,k} \sqcup X^{\neg,k}$ and $X^{\mathrm{bir},k} = X^{\exists,k}$ if X is proper. In the terminology of [FR16b], the space $X^{\mathrm{bir},k}$ coincides with the subspace of all valuations defined over the generic point in the Hahn analytification of X endowed with the extended Euclidean topology. Moreover, the notation $X^{\dot{\beth},k}$ is used in analogy with the analytic space $X^{\beth}$ of Berkovich [Ber96] and Thuillier [Thu07], where we have used a dot as a remainder that we are considering only the birational parts.

We now introduce a flag of subspaces on $X^{\mathrm{bir},k}$ which interpolate between $X^{\dot{\beth},k}$ and $X^{\mathrm{bir},k}$.

Definition 2.6.3 (The centroidal filtration)**.** For $0 \leq r \leq k$ we consider the set

$$\mathscr{F}^r X^{\mathrm{bir},k} = \{\nu \in X^{\mathrm{bir},k} \mid \mathrm{proj}_r(\nu) \text{ has center in } X\}$$

where $\mathrm{proj}_r(\upsilon)$ is the composition of υ with the projection $\mathbb{R}^k \to \mathbb{R}^r$ to the first r coordinates. In other words,

$$\mathscr{F}^r X^{\mathrm{bir},k} = \mathrm{proj}_r^{-1} X^{\dot{\beth},r}.$$

This give a decreasing filtration

$$X^{\mathrm{bir},k} = \mathscr{F}^0 X^{\mathrm{bir},k} \supseteq \mathscr{F}^1 X^{\mathrm{bir},k} \supseteq \cdots \supseteq \mathscr{F}^k X^{\mathrm{bir},k} = X^{\dot{\beth},k}.$$

Many of the constructions we will do in the following will be compatible or can be extended to this centroidal filtration, and we will do so.

2.6.2 Inclusion of tangent cones in the analytification

Proposition 2.6.4 (The inclusion map)**.** *Given an SNC divisor D on the variety X, by Theorem 2.3.10 we get an inclusion map*

$$\begin{aligned} \iota : T\mathcal{C}^{k-1}\,\Sigma(D) &\longrightarrow X^{\dot{\beth},k} \\ (x; \underline{w}) &\longmapsto \nu_{x;\underline{w}}. \end{aligned}$$

Then, the map ι induces a homeomorphism between $T\mathcal{C}^k\,\Sigma(D)$ endowed with the tropical topology and its image with the topology induced by $X^{\dot{\beth},k}$. In the case in which X is proper we can restrict the codomain to an inclusion

$$\iota : T\mathcal{C}^{k-1}\,\Sigma(D) \longrightarrow (X \setminus D)^{\dot{\daleth},k}.$$

Proof. By Proposition 2.3.3 the center of a quasi-monomial valuation defined by D is always on D, so we can restrict to the codomain above in each case. Moreover, the fact that the map induces a homeomorphism with its image is a direct consequence of the approximation theorem. In fact, any tropical function is of the form $\operatorname{trop}(f)$ for a rational function $f \in K(X)^*$. Since $\operatorname{ev}_f \circ \iota = \operatorname{trop}(f)$, the tropical topology coincides with the induced topology by ι, and so ι will be a homeomorphism to its image. $\square$

Regarding the center, the following result will be useful later. For a valuation ν in $X^{\beth,k}$, we denote by c_ν the center of ν in X.

Proposition 2.6.5. *Let X be a variety, then the map*

$$c_X : X^{\beth,k} \longrightarrow X$$
$$\nu \longmapsto \mathrm{c}_\nu$$

that assigns to each valuation its center in X is anticontinuous.

Proof. Let $U = \operatorname{Spec}(A) \subseteq X$ be an affine open set. Then for a valuation $\nu \in X^{\beth,k}$, we have

$$\mathrm{c}_\nu \in U \iff \nu\big|_A \geq 0 \iff \nu \in \bigcap_{f \in A} \operatorname{ev}_f^{-1}[0,\infty).$$

Hence, $c_X^{-1}(U) = \bigcap_{f\in A} \operatorname{ev}_f^{-1}[0,\infty)$ is closed. Now if V is an arbitrary open set, then, as X is Noetherian, we have a finite cover $V = \bigcup_i U_i$ by open affine subsets and so $c_X^{-1}(V) = \bigcup_i c_X^{-1}(U_i)$ is closed. $\square$

2.6.3 The retraction map

Let X be a smooth variety and D an SNC divisor on X. Endowing the tangent cone with the tropical topology, Proposition 2.6.4 gives an inclusion of $T\mathcal{C}^{k-1}\Sigma(D)$ as a topological subspace of $X^{\beth,k}$. In this section we will construct a retraction of $X^{\beth,k}$ onto $T\mathcal{C}^{k-1}\Sigma(D)$ for this inclusion and study its basic properties. This generalizes the picture from rank one to higher rank.

Definition of the retraction map

We start by recalling how to apply a valuation to a divisor when the valuation has a center in the variety.

Definition 2.6.6. Let E be a cartier divisor in a variety X. Given a valuation $\nu \in X^{\beth,k}$ with center c_ν in X, we define $\nu(E) := \nu(z)$ where $z \in \mathcal{O}_{X,\mathrm{c}_\nu}$ is a local equation for E around the point c_ν.

As two local equations differ by a unit, this is independent of the choice of local equation. Using this we can introduce the retraction map. We identify $\mathscr{M}^k(D)$ with $T\mathcal{C}^{k-1}\,\Sigma(D)$ using the duality Theorem.

Definition 2.6.7 (Retraction)**.** Let D be an SNC divisor on a variety X. *The retraction to* $T\mathcal{C}^{k-1}\,\Sigma(D)$ is the map

$$r_{_D} : X^{\beth,k} \to T\mathcal{C}^{k-1}\,\Sigma(D) \tag{2.21}$$

given by sending any valuation $\nu \in X^{\beth,k}$ to the unique pair $(x; \underline{w}) \in T\mathcal{C}^{k-1}\,\Sigma(D)$, with corresponding quasi-monomial valuation $\nu_{x,\underline{w}}$, which verifies for any component D_i of D, the equality

$$\nu_{x,w}(D_i) = \nu(D_i). \tag{2.22}$$

Proposition 2.6.8. *The map $r_{_D}$ verifies the following properties:*

1. *It is well-defined.*

2. *It is continuous.*

3. *It is a retraction for the inclusion ι from Proposition 2.6.4, that is, $r_{_D} \circ \iota = \mathrm{Id}$.*

Proof. (1) We need to prove that for any valuation ν there is a quasi-monomial valuation which satisfies 2.22. For this, let S_σ be the smallest stratum in D which contains c_ν, the center of ν. We note that a component D_i contains c_ν if and only if $i \in I_\sigma$. Moreover, if a component D_j does not contain c_ν, then its local equation around c_ν is invertible so $\nu(D_j) = 0$. Hence, we can associate to ν the pair $(x; \underline{w})$ such that $\nu_{x,\underline{w}}$ corresponds to the quasi-monomial valuation in $\mathscr{M}_\sigma(D)$ which takes the value $\nu(D_i)$ for any $i \in I_\sigma$.

(2) The proof will be based on the Topology-Mixing Lemma 2.6.9 below, and is given in Section 2.6.5.

(3) To show that $r_{_D}$ is a retraction, we note that if $\nu_{x,\underline{w}} \in \mathscr{M}^k(D)$ is a quasi-monomial valuation defined by vectors $\alpha_i \in \mathbb{R}^k_{\sqsupseteq_{\mathrm{lex}} 0}$, then for each component D_i of D, we have by definition $\nu_{x\underline{w}}(D_i) = \alpha_i$ and so $r_{_D}(\nu) = r_{_D}(\iota(x; \underline{w})) = (x; \underline{w})$. This shows that the map $r_{_D}$ is a retraction.

□

2.6.4 Topology-Mixing Lemma

In this section we prove the following lemma.

Lemma 2.6.9 (Topology-Mixing Lemma)**.** *Let X be an algebraic variety and fix an element $f \in K(X)^*$. Then the set*

$$\mathrm{ev}_f^{-1}\big((-\infty, 0]\big) = \big\{\nu \in X^{\beth,k} \mid \nu(f) \preceq_{\mathrm{lex}} 0\big\}$$

is a closed set inside $X^{\beth,k}$.

Remark 2.6.10. Notice that the interval $(-\infty, 0]$ in $\mathbb{R}^k$ constructed with the lexicographic order is not closed with respect to the Euclidean topology. Therefore the lemma does not follow directly from the definition of the tropical topology on $X^{\beth,k}$, and it might appear to be somehow unexpected as it happens to mix the Euclidean and ordered topologies (where the name given to the result). The statement might be not true when the interval $(-\infty, 0]$ is replaced by other half intervals.

Proof. The statement is equivalent to showing that

$$\mathrm{ev}_f^{-1}\big((0, \infty)\big) = \{\nu \in X^{\beth,k} \mid \nu(f) \succ_{\mathrm{lex}} 0\}$$

is an open set inside $X^{\beth,k}$. For this we will show that any element $\nu \in \mathrm{ev}_f^{-1}(0, \infty)$ is an interior point of $\mathrm{ev}_f^{-1}(0, \infty)$.

So let ν be such an element. First, notice that since $\nu(f) \succ 0$, we must have $\nu(f+1) = 0$. This shows $\nu(f) \neq \nu(f+1)$. Take now two disjoint open neighborhoods U and V of $\nu(f)$ and $\nu(f+1)$ in $\mathbb{R}^k$, respectively, so that we have $\nu \in \mathrm{ev}_f^{-1}(U) \cap \mathrm{ev}_f^{-1}(V)$. For any valuation $\tilde{\nu} \in \mathrm{ev}_f^{-1}(U) \cap \mathrm{ev}_f^{-1}(V)$, we have $\tilde{\nu}(f) \neq \tilde{\nu}(f+1)$, which implies that

$$0 = \tilde{\nu}(1) = \min_{\preceq_{\mathrm{lex}}}\big\{\tilde{\nu}(f+1), \tilde{\nu}(f)\big\}.$$

Since $\tilde{\nu}(f) \in U$, and $0 \notin U$, we get $\tilde{\nu}(f) \neq 0$. This implies that $\tilde{\nu}(f) \succ_{\mathrm{lex}} 0$ and so $\tilde{\nu} \in \mathrm{ev}_f^{-1}\big((0, \infty)\big)$. We infer that the open neighborhood $\mathrm{ev}_f^{-1}(U) \cap \mathrm{ev}_f^{-1}(V)$ of ν in $X^{\beth,k}$ is contained in $\mathrm{ev}_f^{-1}\big((0, \infty)\big)$, from which the result follows. □

2.6.5 Continuity of the retraction map

In this section, we prove part (2) of Proposition 2.6.8 by using the Topology-Mixing Lemma.

By the definition of the tropical topology on $T\mathcal{C}^{k-1}\Sigma(D)$, in order to prove r_D is continuous, it will be enough to prove that for each tropical function $F : \Sigma(D) \to \mathbb{R}$ the composition

$$\mathcal{F} := \mathscr{D}^{k-1}F \circ r_D : X^{\beth,k} \to \mathbb{R}^k$$

is continuous. Moreover, by our approximation theorem, we can find a rational function f such that $F = \mathrm{trop}(f) : \Sigma(D) \to \mathbb{R}$ and then $\mathcal{F} = \mathscr{D}^{k-1}\,\mathrm{trop}(f) \circ r_D$. We will fix such a function.

In order to prove the continuity of $\mathcal{F}$, we will construct a sequence of covers $X^{\beth,k} = \bigcup_i G_i$ by finitely many closed sets in which $\mathcal{F}$ behaves better in G_i than in $X^{\beth,k}$. Then to show the continuity of the function $\mathcal{F}$, it will be enough to prove the continuity of $\mathcal{F}$ restricted to each of these sets G_i. Indeed, for any Euclidean closed set $C \subseteq \mathbb{R}^k$ we have that

$$\mathcal{F}^{-1}(C) = \bigcup_{\sigma\in\Sigma(D)} \mathcal{F}\big|_{G_i}^{-1}(C)$$

and so $\mathcal{F}^{-1}(C)$ is closed provided that $\mathcal{F}\big|_{G_i}^{-1}(C)$ are all closed for each G_σ.

Let us start by taking a finite affine open cover $X = \bigcup_j U_j$ with the property that each component D_i of D is a principal divisor over each U_j. Then we have $X^{\beth,k} = \bigcup_i U_i^{\beth,k}$ where

$$U_i^{\beth,k} = \{\nu \in X^{\beth,k} \mid c_\nu \in U_i\}$$

is closed by Proposition 2.6.5, hence it is a finite closed cover. By the observation above it will be enough to prove that $\mathcal{F}$ is continuous restricted to each $U_i^{\beth,k}$, that is, we can assume that X is affine and each divisor D_i is principal, that is, $\div(z_i) = D_i$ for some regular function z_i. Moreover, we can assume as well that the rational function f defining $\mathcal{F}$ is a regular function on X.

Now, for each cone $\sigma \in \Sigma(D)$, consider the set

$$G_\sigma := \big\{\nu \in X^{\beth,k} \mid \mathrm{c}_\nu \in X \setminus \bigcup_{j\notin I_\sigma} D_j\big\}.$$

That is, G_σ is the set of all valuations $\nu \in X^{\beth,k}$ whose center c_ν does not belong to any component D_i with $i \notin I_\sigma$. As we have $c_X^{-1}(X \setminus \bigcup_{j\notin I_\sigma} D_j)$, by Proposition 2.6.5 this set is a closed subset of $X^{\mathrm{bir},k}$.

Proposition 2.6.11 (The closed cover G_σ)**.** *The above family of closed sets G_σ, σ a face* *of* $\Sigma(D)$*, forms a closed cover of* $X^{\beth,k}$*, namely,* $X^{\beth,k} = \bigcup_\sigma G_\sigma$ *where the union goes over all faces* σ *of* $\Sigma(D)$*.*

Proof. Given $\nu \in X^{\beth,k}$, two cases can occur. Either, $\mathrm{c}_\nu \notin D$, in which case we get $\nu \in G_\sigma$ for all facets σ of $\Sigma(D)$. Or, $\mathrm{c}_\nu \in D$, in which case, there exists a component D_i such that $\mathrm{c}_\nu \in D_i$. Let σ be the face of $\Sigma(D)$ whose associated stratum S_σ contains c_ν. We have $\nu \in G_\sigma$, and the proposition follows. □

Hence, it is enough to prove the continuity over each G_σ. Without loss of generality, assume that $D_1, \dots, D_r$ are all the components of D which contain η_σ.

As f is regular, we have $f \in \mathcal{O}_{\eta_\sigma}$. Consider now an admissible expansion

$$f = \sum_\beta c_\beta z^\beta \in \mathcal{O}_{X,\eta_\sigma}$$

of f around the point η_σ where $\div(z_i) = D_i$. Then $\mathcal{F}(\nu) = \min\Big\{\nu(z^\beta) \mid \beta \in A_f^\sigma\Big\}$ for each $\nu \in G_\sigma$. In order to prove the continuity of $\mathcal{F}$ on G_σ, we further decompose G_σ as a finite union of closed sets as follows. For each $\beta \in A_f^\sigma$, consider the set

$$G_{\sigma,\beta} := \big\{\nu \in G_\sigma \mid \mathcal{F}(\nu) = \nu(z^\beta)\big\}.$$

Proposition 2.6.12 (The closed cover $G_{\sigma,\beta}$)**.** *Notations as above, the family* $G_{\sigma,\beta}$*,* $\beta \in A_f^\sigma$*, is a closed cover of* G_σ*.*

Proof. We need to show that $G_{\sigma,\beta}$ is closed for any element $\beta \in A_f^\sigma$. We have

$$\begin{aligned} G_{\sigma,\beta} &= \big\{\nu \in G_\sigma \mid \nu(z^\beta) \preceq_{\mathrm{lex}} \nu(z^{\beta'})\ \forall \beta' \in A_f^\sigma\big\} \\ &= G_\sigma \cap \bigcap_{\beta' \in A_f^\sigma} \big\{\nu \in G_\sigma \mid \nu(z^{\beta-\beta'}) \preceq_{\mathrm{lex}} 0\big\} \end{aligned}$$

which is a closed set by the Topology-Mixing Lemma 2.6.9 applied to rational functions $z^{\beta-\beta'}$. □

We are now ready to finish the proof of the continuity of the retraction map.

Proof of part (2) of Proposition 2.6.8. By the above discussion, we are reduced to show that $\mathcal{F}$ is continuous over each G_σ for $\sigma \in \Sigma(D)$. Moreover, we have $G_\sigma = \bigcup_{\beta \in A_f^\sigma} G_{\sigma,\beta}$, and $G_{\sigma,\beta}$ are all closed. Hence, again it is enough to prove that the restriction of $\mathcal{F}$ on each $G_{\sigma,\beta}$ is continuous. But this is clear because the restriction of $\mathcal{F}$ to $G_{\sigma,\beta}$ equals ev_{z^β} which is continuous by definition. □

2.7 Log-smooth pairs

In the previous section, given a variety X we constructed a retraction from $X^{\beth,k}$ to the tangent cone $T\mathcal{C}^{k-1}\,\Sigma(D)$ associated to SNC divisor D on X. In this section, we will introduce other instances for which we can construct dual complexes, tangent cones, and corresponding retractions. The results will be of use in the subsequent section in order to prove the limit formulae. We start by the following definition.

Definition 2.7.1. Let X be a smooth variety

1. A *log-smooth pair over* X is the data of a pair $\mathbf{Y} = (Y, D)$ consisting of a smooth variety Y and an SNC divisor D on Y together with a proper morphism $\varphi : Y \to X$ such that the restriction

   $$\varphi|_{Y\setminus D} : Y \setminus D \longrightarrow X \setminus f(D)$$

   is an isomorphism. The morphism φ is called the *structure morphism of the log-smooth pair.*

   Given log-smooth pairs $\mathbf{Y}' = (Y', D')$ and $\mathbf{Y} = (Y, D)$, a morphism $\mathbf{Y}' \to \mathbf{Y}$ between them is a proper morphism

   $$f : Y' \longrightarrow Y$$

   that commutes with the structure map of Y and Y' and such that $\mathrm{Supp}\big(f^*(D)\big) \subseteq \mathrm{Supp}\big(D'\big)$.

   We denote by $\mathbf{LSP}_X$ the category of log-smooth pairs over X.

2. A *log-smooth compactification of* X is a proper variety Y containing X as an open subvariety such that $Y \setminus X$ is an SNC divisor on Y.

   A morphism between log-smooth compactifications Y' and Y is a morphism $f : Y' \to Y$ between the underlying varieties such that $f^{-1}(X) = X$ and $f|_X$ is an isomorphism. The category of log-smooth compactifications of X will be denoted by $\mathbf{LSC}_X$.

   Notice that for a morphism as above we have $f^{-1}(Y \setminus X) = Y' \setminus X$.

3. A *compactified log-smooth pair* is the data of a pair $\overline{\mathbf{Y}} = (Y, D)$ consisting of a proper variety Y and an SNC divisor $D \subset Y$ together with a birational map $\varphi : Y \dashrightarrow X$ such that the divisor D can be decomposed as $D = D^\circ + D^\infty$ where D° and D^∞ does not have any component in common, and such that

   (i) the domain of definition of φ is $Y \setminus D_\infty$, that is,
   $$\varphi : Y \setminus D^\infty \longrightarrow X$$
   is well-defined and $Y \setminus D^\infty$ is the maximum open set with this property.

   (ii) the pair $(Y \setminus D^\infty, D^\circ|_{Y\setminus D^\infty})$ is a log-smooth pair for X, i.e., $\varphi|_{Y\setminus D^\infty}$ is a proper morphism from $Y \setminus D^\infty$ to X and the restriction
   $$Y \setminus (D^\circ \cup D^\infty) \longrightarrow X \setminus \varphi(D^\circ)$$
   is an isomorphism.

   A morphism $\overline{\mathbf{Y}}' \to \overline{\mathbf{Y}}$ between compactified log-smooth pairs $\overline{\mathbf{Y}}' = (Y', D')$ and $\overline{\mathbf{Y}} = (Y, D)$ is a proper morphism $f : Y' \to Y$ which commutes with the structure map φ and such that $f^*(D) \subseteq D'$. Notice that in this case we have $f^*(Y \setminus X) = Y' \setminus X$. The category of compactified log smooth pairs will be denoted by $\mathbf{CLSP}_X$.

4. Given compactified log-smooth pairs $\overline{\mathbf{Y}}'$ and $\overline{\mathbf{Y}}$. We will say that $\overline{\mathbf{Y}}'$ dominates $\overline{\mathbf{Y}}$ if there is a morphism $\overline{\mathbf{Y}}' \to \overline{\mathbf{Y}}$, and we will denote this by $\overline{\mathbf{Y}}' \geq \overline{\mathbf{Y}}$. Similar notations are given for log-smooth pairs and log-smooth compactifications.

Proposition 2.7.2. *The categories $\mathbf{CLSP}_X$, $\mathbf{LSP}_X$ and $\mathbf{LSC}_X$ are filtered. That is, for any two objects Y_1, Y_2 there is a third object Y_3 together with morphisms $Y_3 \to Y_1$ and $Y_3 \to Y_2$.*

Proof. We will give a proof for $\mathbf{CLSP}_X$, for the other categories similar constructions work. So consider two compactified log-smooth pairs $\overline{\mathbf{Y}}_1 = (Y_1, D_1)$ and $\overline{\mathbf{Y}}_2 = (Y_2, D_2)$ and the diagonal birational map
$$X \to Y_1 \times Y_2$$
induced by the inverse of the structure maps. If Z denotes the closure of the image of this map, then we have projection maps $\mathrm{pr}_i : Z \to Y_i$ for $i = 1, 2$. By taking an embedded resolution of singularities with respect to $\mathrm{pr}_1^*(D_1) \cup \mathrm{pr}_2^*(D_2)$, we get a variety Y_3 together

with a simple normal crossing divisor E on it. Let $D_3 = E_{\mathrm{red}}$, and define D_3^∞ as the divisor generated by the components of D_3 lying over D_1^∞ or D_2^∞, and D_3° as the divisor defined by the other components of D_3. Then $Y_3 \setminus \mathbb{D}_3^\infty$ is the domain of definition of the rational map $\varphi_3 : Y_3 \dashrightarrow X$ and $Y_3 \setminus D \dashrightarrow X \setminus \varphi(D^\circ)$ is an isomorphism, hence the pair $\overline{\mathbf{Y}}_3 = (Y_3, D_3)$ is a compactified log-smooth pair which dominates both $\overline{\mathbf{Y}}_1$ and $\overline{\mathbf{Y}}_2$. □

Definition 2.7.3. Given a compactified log-smooth pair $\overline{\mathbf{Y}} = (Y, D)$, we denote by $\mathscr{M}^k(\mathbf{Y}) = \mathscr{M}^k(Y, D)$ the set consisting of all rank k quasi-monomial valuations on Y relative to the divisor D, denote by $\Sigma(\overline{\mathbf{Y}}) = \Sigma(Y, D)$ the dual cone complex to the divisor D on Y and by $T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$ its tangent cone. Similar notations will be used for log-smooth pairs and log-smooth compactifications.

2.7.1 The retraction map revisited

Proposition 2.7.4. *Let X be a smooth variety.*

1. *For each log-smooth pair $\mathbf{Y} = (Y, D)$ over X, there is a continuous retraction*
$$r_{\mathbf{Y}} : X^{\beth,k} \longrightarrow T\mathcal{C}^{k-1}\,\Sigma(\mathbf{Y}).$$
2. *For each log-smooth compactification Y of X, there is a continuous retraction*
$$r_Y : X^{\daleth,k} \longrightarrow T\mathcal{C}^{k-1}\,\Sigma(Y) \setminus \{0\}.$$
3. *For each compactified log-smooth pair $\overline{\mathbf{Y}} = (Y, D)$ over X there is a continuous retraction*
$$r_{\overline{\mathbf{Y}}} : X^{\mathrm{bir},k} \longrightarrow T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}).$$

Proof. (1) As the structure map $\varphi : Y \to X$ is birational, it induce by pullback an isomorphism $K(Y) \cong K(X)$ from which we get $Y^{\mathrm{bir},k} \cong X^{\mathrm{bir},k}$. Moreover, as $\varphi : Y \to X$ is a proper map, by the valuative criterion of properness, a valuation ν has a center in Y if and only if it has a center in X. Therefore $Y^{\beth,k} \cong X^{\beth,k}$.

Now the retraction $r_{\mathbf{Y}}$ is given by the composition
$$X^{\beth,k} \xrightarrow{\varphi^*} Y^{\beth,k} \xrightarrow{r_D} T\mathcal{C}^{k-1}\,\Sigma(\mathbf{Y}),$$
where r_D is given by Definition 2.6.7 applied to the pair (Y, D).

(2) If Y is a log smooth compactification of X, then $D = Y \setminus X$ is an SNC divisor on Y. Applying Definition 2.6.7 to this SNC divisor and using that $X^{\mathrm{bir},k} = Y^{\mathrm{bir},k}$, we get a map

$$X^{\mathrm{bir},k} \longrightarrow T\mathcal{C}^{k-1}\,\Sigma(Y).$$

Moreover, a valuation ν goes to $0 \in T\mathcal{C}^{k-1}\,\Sigma(Y)$ iff ν is centered outside D, hence by restriction we get a map

$$X^{\urcorner,k} \longrightarrow T\mathcal{C}^{k-1}\,\Sigma(Y) \setminus \{0\}.$$

(3) Suppose the compactified log-smooth pair is $\overline{\mathbf{Y}} = (Y, D)$. As in (1) we obtain the retraction as the composition

$$X^{\beth,k} \xrightarrow{\varphi^*} Y^{\beth,k} \xrightarrow{r_D} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$$

where r_D is Definition 2.6.7 applied to the divisor D inside Y and φ^* is the pullback along the rational map $\varphi : Y \to X$. $\square$

Proposition 2.7.5. *The retractions from Proposition 2.7.4 are compatible in the sense that if we have a morphism $\overline{\mathbf{Y}'} \to \overline{\mathbf{Y}}$ of compactified log-smooth pairs, then we have*

$$r_{\overline{\mathbf{Y}}} \circ r_{\overline{\mathbf{Y}'}} = r_{\overline{\mathbf{Y}}}.$$

Similar statements hold for log-smooth pairs and log-smooth compactifications.

Proof. Let $\nu \in Y^{\mathrm{bir},k}$ be a valuation. Consider a compactified log-smooth pair $\overline{\mathbf{Y}}' \geq \overline{\mathbf{Y}}$ above $\overline{\mathbf{Y}}$ and denote by $D'_1, \ldots, D'_l$ all the components of D' in Y'.

Let D_i be a component of D in Y. There exists a subset $J_i \subseteq [l]$ such that $\pi^*(D_i) = \sum_{j \in J_i} n_j D'_j$.

Let h_j be local parameters for D'_j around the center c'_ν of ν in Y'. The product $\prod_{j\in J_i} h_j$ is a local equation for D_i around the center c_ν of ν in Y. We have

$$\nu(D_i) = \nu(\prod_{j\in J_i} h_j^{n_j}) = \sum_{j\in J_i} n_j \nu(h_j) = \sum_{j\in J_i} n_j r_{\overline{\mathbf{Y}'}}(\nu)(h_j) = r_{\overline{\mathbf{Y}'}}(\nu)(\prod h_j^{n_j}) = r_{\overline{\mathbf{Y}'}}(v)(D_i).$$

This implies that the two valuations ν and $r_{\overline{\mathbf{Y}'}}(\nu)$ are mapped to the same point by the retraction map $r_{\overline{\mathbf{Y}}}$, i.e., $r_{\overline{\mathbf{Y}}}(\nu) = r_{\overline{\mathbf{Y}}}(r_{\overline{\mathbf{Y}'}}(\nu))$. Since this holds for all valuations ν, the compatibility of the retraction maps $r_{\overline{\mathbf{Y}}}$ and $r_{\overline{\mathbf{Y}'}}$ follows. $\square$

2.7.2 The retraction inequality

We finish this section by recalling the following useful statement from [JM12] that we call *the retraction inequality* which will be used in the next section.

Proposition 2.7.6 (Retraction inequality). *Let $\mathbf{Y} = (Y, D) \in \mathbf{CLSP}_X$ be a compactified log-smooth pair and let $\nu \in X^{\mathrm{bir},k}$ be a valuation with center the point x of Y. Then for each $f \in \mathcal{O}_{Y,x}$ we have the inequality*

$$\nu(f) \succeq (r_{\mathbf{Y}}(\nu))(f) \tag{2.23}$$

with equality when the zero set $V(f)$ of f is included in D locally around x.

Proof. Denote by $D_1, \ldots, D_m$ the components of D which pass through x and take local equations z_i for each component D_i around x. The family $\{z_i\}_{i=1}^m$ can be extended to a set of local parameters $\{z_i\}_{i=1}^r$ for Y at x. By Corollary 2.2.7 there is a finite admissible expansion of the form

$$f = \sum_{\beta \in A_f} a_\beta u_\beta z^\beta.$$

Now we have

$$\nu(f) \succeq_{\mathrm{lex}} \min_{\beta \in A_f} \left\{ \nu(a_\beta u_\beta z^\beta) \right\} \succeq_{\mathrm{lex}} \min_{\beta \in A_f} \{\nu(z^\beta)\} = (r_{\mathbf{Y}}(\nu))(f),$$

which is the stated inequality.

Suppose now that $V(f) \subseteq D$ locally around x. Then we can write f in $\mathcal{O}_{Y,x}$ as $f = u \prod_{i=1}^m z_i^{n_i}$ for a unit $u \in \mathcal{O}_{Y,x}$ and non-negative integers n_i. We conclude by observing that

$$\nu(f) = \sum_{i=1}^m n_i \nu(z_i) = \sum_i n_i r_{\mathbf{Y}}(\nu)(z_i) = (r_{\mathbf{Y}}(\nu))(f),$$

as required. □

2.8 Limit formulae

Let X be a smooth variety over an algebraically closed field k. In this section we will see how it is possible to reconstruct the space $X^{\beth,k}$ of valuations with center inside X in terms of the spaces $T\mathcal{C}^{k-1}\Sigma(\mathbf{Y})$ for log-smooth pairs $\mathbf{Y}$ studied in the previous section (Theorem

2.8.1 below). After that we will give a similar result for the set $X^{\daleth,k}$ of valuations whose center is outside X in terms of a limit $T\mathcal{C}^{k-1}\,\Sigma(Y)$ over the compactification of X (Theorem 2.8.7 below) and similarily we can recontruct the centroidal filtration $\mathscr{F}^r X^{\mathrm{bir},k}$ in terms of a centroidal filtration for $T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$ over compactified log-smooth pairs $\overline{\mathbf{Y}}$.

2.8.1 Limit formula for $X^{\beth,k}$

The compatibility shown in 2.7.5 for the retraction maps presented in Proposition 2.7.4 implies that there exist natural continuous maps

$$r : X^{\beth,k} \longrightarrow \varprojlim_{\mathbf{Y}\in\mathbf{LSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\mathbf{Y}) \tag{2.24}$$

$$r : X^{\daleth,k} \longrightarrow \varprojlim_{Y\in\mathbf{LSC}_X} T\mathcal{C}^{k-1}\,\Sigma(Y)\setminus\{0\} \tag{2.25}$$

$$r : X^{\mathrm{bir},k} \longrightarrow \varprojlim_{\overline{\mathbf{Y}}\in\mathbf{CLSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}). \tag{2.26}$$

The objective of this section is to prove the following theorem.

Theorem 2.8.1 (Limit formula)**.** *The maps 2.24, 2.25 and 2.26 above are all homeomorphism.*

In order to show this, it will be enough to construct an inverse for each of these maps. We will do this for the case of 2.26 as the proof in the other cases are essentially the same.

The inverse for this map is the function

$$\begin{aligned} q : \varprojlim_{\overline{\mathbf{Y}}\in\mathbf{CLSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) &\longrightarrow X^{\mathrm{bir},k} \\ s = [(x;\underline{w})]_{\overline{\mathbf{Y}}} &\longmapsto \nu_s \end{aligned} \tag{2.27}$$

where ν_s is the valuation defined by

$$\nu_s(f) = \sup_{\overline{\mathbf{Y}}} \nu_{s,\overline{\mathbf{Y}}}(f), \text{ for every } f \in \bigcap_{\overline{\mathbf{Y}}} \mathcal{O}_{X,\eta_{s,\overline{\mathbf{Y}}}}$$

and $\nu_{s,\overline{\mathbf{Y}}} := \nu_{x,\underline{w}}$ for the element $(x;\underline{w})$ in the position indexed by $\overline{\mathbf{Y}}$ on the sequence s and $\eta_{s,\overline{\mathbf{Y}}}$ is the center of $\nu_{s,\overline{\mathbf{Y}}}$.

Proposition 2.8.2. *The map 2.27 is well defined and it is an inverse for the map 2.26.*

Proof. Let us start by noticing that if $\overline{\mathbf{Y}}' \geq \overline{\mathbf{Y}}$ then $r_D(\nu_{s,\overline{\mathbf{Y}}'}) = \nu_{s,\overline{\mathbf{Y}}}$ and therefore $\eta_{s,\overline{\mathbf{Y}}'} \in \overline{\{\eta_{s,\overline{\mathbf{Y}}'}\}}$. Hence, the sequence of points $\eta_{s,\overline{\mathbf{Y}}}$ is decreasing for the order given by specialization, and therefore it becomes eventually constant equal to some η. We get the equality $\bigcap_{\overline{\mathbf{Y}}} \mathcal{O}_{X,\eta_{s,\mathbf{Y}}} = \mathcal{O}_{X,\eta}$.

Now, notice that by Proposition 2.7.6 the sequence

$$[\nu_{s,\overline{\mathbf{Y}}}(f)]_{\overline{\mathbf{Y}}}$$

is increasing. Moreover, if we fix $\overline{\mathbf{Y}} = (Y, D)$, a compactified log smooth pair, and we take $\overline{\mathbf{Y}}' = (Y', D'_{\text{red}})$ where (Y', D') is an embedded resolution of singularities for $V(f) \cup D^{\infty} \subseteq Y$, we get that

$$V(f) \cup \operatorname{Supp}(D^{\infty}) \subseteq \operatorname{Supp}(D_{\text{red}}).$$

Now, for any $\overline{\mathbf{Y}}'' \geq \overline{\mathbf{Y}}'$ we will get $D'' \supseteq V(f)$. By the equality part of Proposition 2.7.6, we infer that

$$\nu_{s,\overline{\mathbf{Y}}''}(f) = \nu_{s,\overline{\mathbf{Y}}'}(f).$$

Hence the sequence is eventually constant. This shows that for each $f \in \mathcal{O}_{X,\eta}$ the supremum is attained.

Moreover, Proposition 2.7.2 shows that given $f, g \in \mathcal{O}_{X,\eta}$, there is a compactified log-smooth pair $\overline{\mathbf{Y}}$ in which the sequences $[\nu_{s,\overline{\mathbf{Y}}}(f)]_{\overline{\mathbf{Y}}}, [\nu_{s,\overline{\mathbf{Y}}}(g)]_{\overline{\mathbf{Y}}}, [\nu_{s,\overline{\mathbf{Y}}}(f+g)]_{\overline{\mathbf{Y}}}, [\nu_{s,\overline{\mathbf{Y}}}(fg)]_{\overline{\mathbf{Y}}}$ are all constant at the same time from $\overline{\mathbf{Y}}$ onwards. Hence for this $\overline{\mathbf{Y}}$ we have

$$\nu_s(fg) = \nu_{s,\overline{\mathbf{Y}}}(fg) = \nu_{s,\overline{\mathbf{Y}}}(f) + \nu_{s,\overline{\mathbf{Y}}}(g) = \nu_s(f) + \nu_s(g)$$

and similarly $\nu_s(f+g) \geq \min\{\nu_s(f), \nu_s(g)\}$, so ν_s is a valuation. This shows that q is well defined.

Now, let us see that that q is the inverse of r. For this we should check that the composition over each side gives the identity. This translate in the equalities

1. $[r_{\overline{\mathbf{Y}}}(\nu_s)]_{\overline{\mathbf{Y}}} = s$ for any $s = [(x;w)]_{\overline{\mathbf{Y}}} \in \varprojlim_{\overline{\mathbf{Y}} \in \mathbf{CLSP}_X} T\mathcal{C}^{k-1}\Sigma(\overline{\mathbf{Y}})$, and

2. $\nu_s = \nu$ for any $\nu \in X^{\text{bir},k}$ where $s = [r_{\overline{\mathbf{Y}}}(\nu)]_{\overline{\mathbf{Y}}}$.

For the first equality, we need to prove that for any $\overline{\mathbf{Y}} \in \mathbf{CLSP}_X$ we have $r_{\overline{\mathbf{Y}}}(\nu_s) = (x;w)$ for $(x;w)$ in the $\overline{\mathbf{Y}}$ instance of the sequence s. In order to do this consider $\{z_i\}_i$ local equations for the components D_i of the divisor D defining $\overline{\mathbf{Y}}$ around the center of ν_s

in Y. There is a compactified log-smooth pair $\overline{\mathbf{Y}}' = (Y', D')$ in which we simultaneously have the equalities $\nu_s(z_i) = \nu_{s,\overline{\mathbf{Y}}'}(z_i)$ for each i. In this case, we get $r_{\overline{\mathbf{Y}}}(\nu_s) = r_{\overline{\mathbf{Y}}}(\nu_{s,\overline{\mathbf{Y}}'})$ and by the compatibility of the retraction maps, we infer that $r_{\overline{\mathbf{Y}}}(\nu_{s,\overline{\mathbf{Y}}'}) = (x; w)$.

For the second equality, it is enough to prove that for each $f \in \mathcal{O}_{X,\eta}$ we have $\nu_s(f) = \nu(f)$, but this follows directly by the equality part in Proposition 2.7.6. □

Proposition 2.8.3. *The map 2.27 is continuous.*

Proof. For this, recall that the topology of $X^{\mathrm{bir},k}$ is generated by open sets of the form

$$U = \{\nu \in X^{\mathrm{bir},k} \mid \nu(f) \in A\}$$

for $A \subseteq \mathbb{R}^k$ an euclidean open set and $f \in K(X)^*$ a rational function. Hence, given a fixed sequence $s = [(x; w)]_{\overline{\mathbf{Y}}}$ such that $q(s) \in U$, it is enough to find a neighborhood V of s such that $q(V) \subseteq U$. For this, take a compactified log-smooth pair $\overline{\mathbf{Y}}$ and consider $f = \frac{g}{h}$ where $g, h \in \mathcal{O}_{Y,c_Y(q(s))}$. Now take a compactified log-smooth pair $\overline{\mathbf{Y}}'$ by choosing an embedded resolution of $V_Y(g) \cup V_Y(h) \cup D$. Consider then

$$V_g = \left\{ t \in \varinjlim_{\overline{\mathbf{Y}} \in \mathbf{CLSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \mid \nu_{t,\overline{\mathbf{Y}}} \text{ has center inside } V_{Y'}(g) \right\}, \text{ and}$$

$$V_h = \left\{ t \in \varinjlim_{\overline{\mathbf{Y}} \in \mathbf{CLSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \mid \nu_{t,\overline{\mathbf{Y}}} \text{ has center inside } V_{Y'}(h) \right\}.$$

By Proposition 2.6.5, for the center map $T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \to Y'$, we see that both V_g and V_h are open neighborhoods of s in the direct limit. By Proposition 2.7.6, for each $t \in V_g \cap V_h$ we have $\nu_t(g) = \nu_{t,\overline{\mathbf{Y}}}(g)$ and $\nu_t(h) = \nu_{t,\overline{\mathbf{Y}}}(h)$. We thus get $\nu_t(f) = \nu_{t,\overline{\mathbf{Y}}}(f)$. Now consider

$$V' = \left\{ t \in \varinjlim_{\overline{\mathbf{Y}} \in \mathbf{CLSP}_X} T\mathcal{C}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \mid \nu_{t,\overline{\mathbf{Y}}}(f) \in A \right\}.$$

This is another open neighborhood of s in the direct limit. To conclude, note that for $V = V_g \cap V_h \cap V'$ and $t \in V$, we have

$$\nu_{t,\mathbf{Y}}(f) = \nu_t(f) \in A$$

which shows that $q(V) \subseteq U$. This proves the continuity. □

Corollary 2.8.4. *The family of all quasi-monomial valuations $\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$ for some compactified log-smooth pair $\overline{\mathbf{Y}}$ is dense in $X^{\mathrm{bir},k}$.*

2.8.2 Refined limit formula

Definition 2.8.5 (Centroidal filtration)**.** Given a compactified log-smooth pair $\overline{\mathbf{Y}} = (Y, D)$ over X we have a decomposition of D as $D = D^\circ \cup D^\infty$. This give us the subcomplex $\Sigma(D^\circ)$ inside $\Sigma(\overline{\mathbf{Y}})$ which we denote by $\Sigma(\overline{\mathbf{Y}}^\circ)$. We define the *centroidal filtration* of $\mathcal{TC}^{k-1}\,\overline{\mathbf{Y}}$ to be the filtration

$$\mathscr{F}^0\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \supseteq \mathscr{F}^0\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \supseteq \cdots \supseteq \mathscr{F}^k\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$$

given for $0 \le r \le k$ by

$$\mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \coloneqq \{(x;\underline{w}_{k-1}) \in \mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \mid (x;\underline{w}_{r-1}) \in \mathcal{TC}^{r-1}\,\Sigma(\overline{\mathbf{Y}}^\circ)\}.$$

Remark 2.8.6. For $i < j$, we have that $(x;\underline{w}_j) \in \mathcal{TC}^j\,\Sigma(\overline{\mathbf{Y}}^\circ)$ implies $(x;\underline{w}_i) \in \mathcal{TC}^i\,\Sigma(\overline{\mathbf{Y}}^\circ)$. Therefore, the sequence $\big(\mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}})\big)_r$ is indeed decreasing. Moreover, we have

$$\mathscr{F}^0\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) = \mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \text{ and } \mathscr{F}^k\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) = \mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}^\circ).$$

This is similar to the centroidal filtration on $X^{\mathrm{bir},k}$.

The limit formula for compactified log-smooth pairs in the previous subsection preserves the centroidal filtrations. We thus obtain a limit description of each term of the filtration.

Theorem 2.8.7. *For each $0 \le r \le k$, the isomorphism of Theorem 2.8.1 restricts to a homeomorphism*

$$\mathscr{F}^r X^{\mathrm{bir},k} \longrightarrow \varprojlim_{\overline{\mathbf{Y}} \in \mathbf{CLSP}_X} \mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}).$$

Proof. We first note that given $(x;\underline{w}) \in \mathscr{F}^r\,\mathcal{TC}^k\,\Sigma(\overline{\mathbf{Y}})$, the center of the valuation $\mathrm{proj}_r(\nu_{x,\underline{w}})$ is the point η_σ where σ is the smallest face such that $(x,\underline{w}_r) \in \mathcal{TC}^r\,\sigma$. Hence, if $(x;\underline{w}) \in \mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}^\circ)$, then the center of $\nu_{x,\underline{w}}$ is inside $D^\circ \subseteq X$. This proves that the inclusion $\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \hookrightarrow X^{\mathrm{bir},k}$ restricts to an inclusion $\mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}) \hookrightarrow \mathscr{F}^r X^{\mathrm{bir},k}$

Moreover, since for any compactified log-smooth pair, the center of ν is a specialization of the center of $r_{\overline{\mathbf{Y}}}(\nu)$, we can see that for each pair $\overline{\mathbf{Y}}' \ge \overline{\mathbf{Y}}$, the retraction map in 2.7.4 induces a map

$$\mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}}') \longrightarrow \mathscr{F}^r\,\mathcal{TC}^{k-1}\,\Sigma(\overline{\mathbf{Y}})$$

and these maps are still compatible. This implies that once we take the inverse limit, we obtain a natural map

$$r : \mathscr{F}^r X^{\mathrm{bir},k} \longrightarrow \varprojlim_{\mathbf{Y} \in \mathbf{CLSP}_X} \mathscr{F}^r T\mathcal{C}^{k-1}\Sigma(\overline{\mathbf{Y}}).$$

On the other hand, the inverse map q defined in 2.27 also restricts to a map

$$q : \mathscr{F}^r T\mathcal{C}^{k-1}\Sigma(\overline{\mathbf{Y}}) \longrightarrow \mathscr{F}^r X^{\mathrm{bir},k}.$$

Indeed, if $s = [(x;w)]_{\mathbf{Y}}$ is a sequence of elements in $\mathscr{F}^r T\mathcal{C}^{k-1}\Sigma(\overline{\mathbf{Y}})$, then the center of the elements $\mathrm{proj}_r(\nu_{s,\overline{\mathbf{Y}}})$ in X are points $\eta_{s,\overline{\mathbf{Y}}}$ with the property that $\eta_{s,\overline{\mathbf{Y}}'}$ specialize $\eta_{\overline{\mathbf{Y}}}$ if $\overline{\mathbf{Y}}'$ dominates $\overline{\mathbf{Y}}$. As X is Noetherian, the sequence $[(\eta_{s,\overline{\mathbf{Y}}})]_{\overline{\mathbf{Y}}}$ is eventually constant, and hence $\eta_{s,\overline{\mathbf{Y}}}$ is in the projection of a strata of D° onto X, that is, the center of $\mathrm{proj}_r(q(s))$ is on X. Hence $q(s) \in \mathscr{F}^r X^{\mathrm{bir},k}$. The two maps r and q are still inverses to each other and this finishes the proof. □

Corollary 2.8.8. *The limit above can be restricted to each stratum in the centroidal filtration, that is, for each r, we have a homeomorphism*

$$\mathscr{F}^r X^{\mathrm{bir},k} \setminus \mathscr{F}^{r+1} X^{\mathrm{bir},k} \longrightarrow \varprojlim_{\overline{\mathbf{Y}} \in \mathbf{CLSP}_X} \mathscr{F}^r T\mathcal{C}^{k-1}\Sigma(\overline{\mathbf{Y}}) \setminus \mathscr{F}^{r+1} T\mathcal{C}^{k-1}\Sigma(\overline{\mathbf{Y}}).$$

2.9 Variations of Okounkov bodies

Let X be a smooth algebraic variety of dimension d over the algebraically closed base field κ and with function field $K(X)$. Consider a big line bundle $L = \mathcal{O}(E)$ on X with the corresponding graded algebra

$$H_\bullet = \bigoplus_{n \geq 0} H_n$$

where each $H_n = H^0(X, \mathcal{O}(nE))$ is a κ-vector subspace of $K(X)$ of finite dimension.

Consider the space $\mathrm{BC}(\mathbb{R}^k)$ of compact subsets of $\mathbb{R}^k$ endowed with the Hausdorff distance. We consider the map

$$\begin{aligned} \Delta : X^{\mathrm{bir},k} &\longrightarrow \mathrm{BC}(\mathbb{R}^n) \\ v &\longmapsto \Delta_v = \overline{\bigcup_{n\geq 0} \left\{ \frac{v(f)}{n} \mid f \in H_n \right\}} \end{aligned} \tag{2.28}$$

which attaches to each valuation the corresponding Newton-Okounkov body.

Conjecture 2.9.1. The restriction of the map Δ in 2.28 on each higher rank skeleton is continuous.

In the following we will give a heuristic argument for the validity of this conjecture. We start by the following result.

Theorem 2.9.2. *Let $(C_n)_n \in \mathrm{BC}(\mathbb{R}^k)$ be a sequence of full dimensional compact convex subsets of $\mathbb{R}^k$. Then we have that*

$$C_n \overset{n\to\infty}{\longrightarrow} C \in \mathrm{BC}(\mathbb{R}^k)$$

with the Hausdorff distance if and only if for each continuous function $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ of compact support we have

$$\int_{C_n} f(x)dx \overset{n\to\infty}{\longrightarrow} \int_C f(x)dx$$

Proof. ($\Rightarrow$) If we denote by $B(C;\varepsilon)$ the ε-neighbourhood of C, we have the inequality

$$\mathrm{Vol}(C_n\Delta C) \leq \mathrm{Vol}(B(C_n; d_H(C_n, C)) \setminus C_n) + \mathrm{Vol}(B(C; d_H(C_n, C)) \setminus C).$$

The left hand size goes to 0 as n goes to infinity. We thus get $\mathrm{Vol}(C_n\Delta C) \longrightarrow 0$. Hence we have the almost everwhere convergence $f \cdot \mathbb{1}_{C_n} \to f \cdot \mathbb{1}_C$, and so by the dominated convergence theorem the integrals converge.

($\Leftarrow$) Recall that

$$d_H(C_n, C) = \left\{ \sup_{x\in C_n} \{d(x, C)\}, \sup_{y\in C}\{d(C_n, y)\} \right\}$$

If C_n does not converge to C, passing to a subsequence if necessary, we get the existence of $\varepsilon > 0$ such that $d_H(C_n, C) \geq \varepsilon$ for all n. We thus have either

$$\sup_{x\in C_n} \{d(x, C)\} \geq \varepsilon \qquad \text{or} \qquad \sup_{y\in C}\{d(C_n, y)\} \geq \varepsilon$$

happen infinitely many times.

In the first case, as C_n is compact for each n, there is an $x_n \in C_n$ for which the supremum is attained. Let x' be an accumulation point of (x_n).

Consider an open ball of the form $B(y; \varepsilon)$ contained in C. If for each continuous function f the integrals above converge, by taking f as a bump function supported exactly on $B(y;\varepsilon)$, we get that $\mathrm{Vol}(B(y;\varepsilon)\setminus C_n)\to 0$. Therefore, since C_n and $B(y,\varepsilon)$ are convex, for n big enough, we would get that C_n contains the ball $B(y;\varepsilon/2)$. This implies that for each such n we have

$$C_n \setminus C \supseteq \mathrm{conv}(B(y;\varepsilon/2)\cup\{x\})\setminus C.$$

The set appearing in the right hand side is independent of n, and has nonempty interior. Taking again a bump function supported in this set we see that the integrals could not converge which would give a contradiction.

In the second case, for each n, there is a $y_n \in C$ for which the supremum is attained. By compactness, and passing to a subsequence if necessary, we can assume that y_n converge to a point in C that we denote by y'. For n big enough, we have $d(C_n, y') \geq \varepsilon/2$, as $d(C_n, y') \geq |d(C_n, y_n) - d(y_n, y')|$. It follows that $B(y';\varepsilon/2)\cap C$ is disjoint from C_n for infinitely many n. By taking f as a bump function supported on $B(y';\varepsilon/2)\cap C$ we see that C_n would not converge to C in a weak sense, which would be a contradiction. □

Heuristic argument for the validity of the conjecture. We will use the fact that the family of sets

$$\Delta_v^n = \left\{\frac{v(f)}{n} \mid f \in H_n\right\}$$

is equidistributed on the set Δ_v. That is, for each continuous function h of compact support we have

$$\lim_{n\to\infty}\frac{1}{N_n}\sum_{x\in\Delta_v^n} h(x) = \int_{\Delta_v} h(x)dx$$

where N_n is the dimension of H_n over κ, which is equal to the size of Δ_v^n.

If v_j is a sequence of valuations converging to v, we would like to interchange the limits in the following

$$\begin{aligned}\lim_{j\to\infty}\int_{\Delta_{v_k}} h(x)dx &= \lim_{j\to\infty}\lim_{n\to\infty}\frac{1}{N_n}\sum_{x\in\Delta_{v_j}^n} h(x) \overset{?}{=} \lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{N_n}\sum_{x\in\Delta_{v_j}^n} h(x)\\ &= \lim_{n\to\infty}\frac{1}{N_n}\sum_{x\in\Delta_v^n} h(x) = \int_{\Delta_v} h(x)dx\end{aligned}$$

and then use Theorem 2.9.2 to get $\Delta_{v_k}\to\Delta_v$ with the Hausdorff distance. Hence, the main issue in proving the conjecture would be to justify the possibility in changing the orders in the limit. □

Bibliography

[AB15] O. Amini and M. Baker. "Linear series on metrized complexes of algebraic curves". In: *Math. Ann.* 362.1-2 (2015), pp. 55–106.

[ACP15] D. Abramovich, L. Caporaso, and S. Payne. "The tropicalization of the moduli space of curves". In: *Ann. Sci. Éc. Norm. Supér. (4)* 48.4 (2015), pp. 765–809.

[AGT16] F. Aroca, C. Garay, and Z. Toghani. "The fundamental theorem of tropical differential algebraic geometry". In: *Pacific Journal of Mathematics* 283.2 (2016), pp. 257–270.

[AHK18] K. Adiprasito, J. Huh, and E. Katz. "Hodge theory for combinatorial geometries". In: *Ann. of Math. (2)* 188.2 (2018), pp. 381–452.

[AI21] O. Amini and H. Iriarte. "Geometry of higher rank valuations". In: *preprint* (2021).

[All+14] X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig. "Combinatorial simplex algorithms can solve mean payoff games". In: *SIAM Journal on Optimization* 24.4 (2014), pp. 2096–2117.

[All+15] X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig. "Tropicalizing the simplex algorithm". In: *SIAM Journal on Discrete Mathematics* 29.2 (2015), pp. 751–795.

[All+18] X. Allamigeon, P. Benchimol, S. Gaubert, and M. Joswig. "Log-barrier interior point methods are not strongly polynomial". In: *SIAM Journal on Applied Algebra and Geometry* 2.1 (2018), pp. 140–178.

[Ami14] O. Amini. "Equidistribution of Weierstrass points on curves over non-Archimedean fields". In: *arXiv preprint arXiv:1412.0926* (2014), revised version 2021.

[AN20] O. Amini and N. Nicolussi. "Moduli of hybrid curves I: Variations of canonical measures". In: *arXiv preprint arXiv:2007.07130* (2020).

[AN21] O. Amini and N. Nicolussi. "Moduli of hybrid curves II: Tropical and hybrid Laplacians". In: *preprint* (2021).

[Aro10a] F. Aroca. "Krull-tropical hypersurfaces". In: *Annales de la Faculté des sciences de Toulouse: Mathématiques* 19.3–4 (2010), pp. 525–538.

[Aro10b] F. Aroca. "Tropical geometry for fields with a Krull valuation: first definitions and a small result". In: *Boletin de la Sociedad Matemática Mexicana* 16.1 (2010), pp. 9–14.

[Ban15] S. D. Banerjee. "Tropical geometry over higher dimensional local fields". In: *Journal für die reine und angewandte Mathematik* 2015.698 (2015), pp. 71–87.

[Bar21] D. Bartl. "A discrete variant of Farkas' lemma and related results". In: *Optimization* 70.9 (2021), pp. 2031–2060.

[BC11] S. Boucksom and H. Chen. "Okounkov bodies of filtered linear series". In: *Compositio Mathematica* 147.4 (2011), pp. 1205–1229.

[Ben90] C. D. Bennett. *Affine lambda-buildings*. Thesis (Ph.D.)–The University of Chicago. ProQuest LLC, Ann Arbor, MI, 1990, (no paging).

[Ber12] V. G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*. 33. American Mathematical Soc., 2012.

[Ber96] V. Berkovich. "Vanishing cycles for formal schemes. II". In: *Inventiones mathematicae* 125.2 (1996), pp. 367–390.

[BFJ16] S. Boucksom, C. Favre, and M. Jonsson. "Singular semipositive metrics in non-Archimedean geometry". In: *J. Algebraic Geom* 25.1 (2016), pp. 77–139.

[BJ18] S. Boucksom and M. Jonsson. "Singular semipositive metrics on line bundles on varieties over trivially valued fields". In: *arXiv preprint arXiv:1801.08229* (2018).

[BL85] S. Bosch and W. Lütkebohmert. "Stable reduction and uniformization of abelian varieties I". In: *Mathematische Annalen* 270.3 (1985), pp. 349–379.

[Bos+17] L. Bossinger, S. Lamboglia, K. Mincheva, and F. Mohammadi. "Computing toric degenerations of flag varieties". In: *Combinatorial algebraic geometry*. Springer, 2017, pp. 247–281.

[Bos+18] L. Bossinger, X. Fang, G. Fourier, M. Hering, and M. Lanini. "Toric degenerations of Gr (2, n) and Gr (3, 6) via plabic graphs". In: *Annals of Combinatorics* 22.3 (2018), pp. 491–512.

[Bos21] L. Bossinger. "Full-rank valuations and toric initial ideals". In: *International Mathematics Research Notices* 2021.10 (2021), pp. 7715–7763.

[Bou12] S. Boucksom. "Corps d'Okounkov". In: *Séminaire Bourbaki* 65 (2012), pp. 1–38.

[Bou14] S. Boucksom. "Corps d'Okounkov (d'après Okounkov, Lazarsfeld-Mustaţă et Kaveh-Khovanskii)". In: *Astérisque* 361 (2014), Exp. No. 1059, vii, 1–41.

[BPR13] M. Baker, S. Payne, and J. Rabinoff. "On the structure of non-Archimedean analytic curves". In: *Tropical and non-Archimedean geometry* 605 (2013), pp. 93–121.

[BPR16] M. Baker, S. Payne, and J. Rabinoff. "Nonarchimedean geometry, tropicalization, and metrics on curves". In: *Algebr. Geom* 3.1 (2016), pp. 63–105.

[BR15] M. Baker and J. Rabinoff. "The skeleton of the Jacobian, the Jacobian of the skeleton, and lifting meromorphic functions from tropical to algebraic curves". In: *International Mathematics Research Notices* 2015.16 (2015), pp. 7436–7472.

[Bru+15] E. Brugallé, I. Itenberg, G. Mikhalkin, and K. Shaw. "Brief introduction to tropical geometry". In: *Proceedings of the Gökova Geometry-Topology conference*. 2015, pp. 1–75.

[Cam+18] A. Camara et al. "Newton–Okounkov bodies and reified valuations of higher rank". In: Springer, 2018, pp. 113–118.

[CGP18] M. Chan, S. Galatius, and S. Payne. "Tropical curves, graph complexes, and top weight cohomology of M_g". In: (2018). arXiv: 1805.10186 [math.AG].

[Cil+17] C. Ciliberto, M. Farnik, A. Küronya, V. Lozovanu, J. Roé, and C. Shramov. "Newton–Okounkov bodies sprouting on the valuative tree". In: *Rendiconti del Circolo Matematico di Palermo Series 2* 66.2 (2017), pp. 161–194.

[CMM21] Y. Cid-Ruiz, F. Mohammadi, and L. Monin. "Multigraded algebras and multigraded linear series". In: *arXiv preprint arXiv:2104.05397* (2021).

[Coh46] I. S. Cohen. "On the structure and ideal theory of complete local rings". In: *Transactions of the American Mathematical Society* 59 (1946), pp. 54–106.

[Con08] B. Conrad. "Several approaches to non-Archimedean geometry". In: *p-adic geometry*. Vol. 45. Univ. Lecture Ser. Amer. Math. Soc., Providence, RI, 2008, pp. 9–63.

[DJ14] N. Dinh and V. Jeyakumar. "Farkas' lemma: three decades of generalizations for mathematical optimization". In: *TOP* 22.1 (2014), pp. 1–22.

[Duc12] A. Ducros. "Les espaces de Berkovich sont modérés". In: *Séminaire Bourbaki* (2012), p. 64.

[DY07] M. Develin and J. Yu. "Tropical polytopes and cellular resolutions". In: *Experimental Mathematics* 16.3 (2007), pp. 277–291.

[EH19] L. Escobar and M. Harada. "Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian". In: *arXiv preprint arXiv:1912.04809* (2019).

[EKL06] M. Einsiedler, M. Kapranov, and D. Lind. "Non-Archimedean amoebas and tropical varieties". In: *J. Reine Angew. Math.* 601 (2006).

[FGP14] T. Foster, P. Gross, and S. Payne. "Limits of tropicalizations". In: *Israel Journal of Mathematics* 201.2 (2014), pp. 835–846.

[FJ04] C. Favre and M. Jonsson. *The valuative tree*. Springer, 2004.

[FK13] K. Fujiwara and F. Kato. "Foundations of rigid geometry I". In: *arXiv preprint arXiv:1308.4734* (2013).

[Fos16] T. Foster. "Introduction to adic tropicalization". In: *Nonarchimedean and tropical geometry*. Simons Symp. Springer, [Cham], 2016, pp. 337–364.

[FR16a] T. Foster and D. Ranganathan. "Degenerations of toric varieties over valuation rings". In: *Bull. Lond. Math. Soc.* 48.5 (2016), pp. 835–847.

[FR16b] T. Foster and D. Ranganathan. "Hahn analytification and connectivity of higher rank tropical varieties". In: *Manuscripta Math.* 151.3-4 (2016), pp. 353–374.

[GG14] J. Giansiracusa and N. Giansiracusa. "The universal tropicalization and the Berkovich analytification". In: *arXiv preprint arXiv:1410.4348* (2014).

[GG16] J. Giansiracusa and N. Giansiracusa. "Equations of tropical varieties". In: *Duke Mathematical Journal* 165.18 (2016), pp. 3379–3433.

[GKZ08] I. M. Gelfand, M. Kapranov, and A. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*. Springer Science & Business Media, 2008.

[Gro18] A. Gross. "Intersection theory on tropicalizations of toroidal embeddings". In: *Proceedings of the London Mathematical Society* 116.6 (2018), pp. 1365–1405.

[GRW16] W. Gubler, J. Rabinoff, and A. Werner. "Skeletons and tropicalizations". In: *Advances in Mathematics* 294 (2016), pp. 150–215.

[GRW17] W. Gubler, J. Rabinoff, and A. Werner. "Tropical skeletons". In: 67.5 (2017), pp. 1905–1961.

[Gub13] W. Gubler. "A guide to tropicalizations". In: *Algebraic and combinatorial aspects of tropical geometry* 589 (2013), pp. 125–189.

[Hah95] H. Hahn. "Über die nichtarchimedischen Größensysteme". In: *Hans Hahn Gesammelte Abhandlungen Band 1/Hans Hahn Collected Works Volume 1*. Springer, 1995, pp. 445–499.

[HHM06] D. Haskell, E. Hrushovski, and D. Macpherson. "Definable sets in algebraically closed valued fields: elimination of imaginaries". In: *J. Reine Angew. Math.* 597 (2006), pp. 175–236.

[HIL20] A. Hébert, D. Izquierdo, and B. Loisel. "Λ-buildings associated to quasi-split groups over Λ-valued fields". In: *arXiv preprint arXiv:2001.01542* (2020).

[HKT09] P. Hacking, S. Keel, and J. Tevelev. "Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces". In: *Inventiones mathematicae* 178.1 (2009), pp. 173–227.

[HKW20] C. Haase, A. Küronya, and L. Walter. "Toric Newton-Okounkov functions with an application to the rationality of certain Seshadri constants on surfaces". In: *arXiv preprint arXiv:2008.04018* (2020).

[HL16] E. Hrushovski and F. Loeser. *Non-Archimedean Tame Topology and Stably Dominated Types*. Vol. 192. Annals of Mathematical Studies. Princeton University Press, 2016.

[Hub94] R. Huber. "A generalization of formal schemes and rigid analytic varieties". In: *Mathematische Zeitschrift* 217.1 (1994), pp. 513–551.

[IMS09] I. Itenberg, G. Mikhalkin, and E. Shustin. *Tropical algebraic geometry*. Vol. 35. Oberwolfach seminars. Springer Science & Business Media, 2009.

[Iri21] H. Iriarte. "Higher rank polyhedral and tropical geometry". In: *preprint* (2021).

[IW20] N. Ilten and M. Wrobel. "Khovanskii-finite valuations, rational curves, and torus actions". In: *Journal of Combinatorial Algebra* 4.2 (2020), pp. 141–166.

[JM12] M. Jonsson and M. Mustaţă. "Valuations and asymptotic invariants for sequences of ideals". In: *Ann. Inst. Fourier (Grenoble)* 62.6 (2012), 2145–2209 (2013).

[Joh18] I. T. Johnson. *Chain Posets*. 2018. arXiv: 1802.05813 [math.CO].

[JS18] M. Joswig and B. Smith. "Convergent Puiseux Series and Tropical Geometry of Higher Rank". In: *arXiv preprint arXiv:1809.01457* (2018).

[JSY18] P. Jell, C. Scheiderer, and J. Yu. "Real tropicalization and analytification of semialgebraic sets". In: *arXiv preprint arXiv:1810.05132* (2018).

[Jun17] J. Jun. "Geometry of hyperfields". In: *arXiv preprint arXiv:1707.09348* (2017).

[Kap00] M. M. Kapranov. "Amoebas over non-archimedean fields". In: *preprint* (2000).

[Ked15] K. Kedlaya. "Reified valuations and adic spectra". In: *Research in Number Theory* 1.1 (2015), p. 20.

[KK12] K. Kaveh and A. G. Khovanskii. "Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory". In: *Annals of Mathematics* (2012), pp. 925–978.

[KM19a] K. Kaveh and C. Manon. "Khovanskii bases, higher rank valuations, and tropical geometry". In: *SIAM Journal on Applied Algebra and Geometry* 3.2 (2019), pp. 292–336.

[KM19b] K. Kaveh and C. Manon. "Toric bundles, valuations, and tropical geometry over semifield of piecewise linear functions". In: *arXiv preprint arXiv:1907.00543* (2019).

[KM19c] K. Kaveh and C. Manon. "Toric flat families, valuations, and tropical geometry over the semifield of piecewise linear functions". In: *arXiv preprint arXiv:1907.00543* (2019).

[KS06] M. Kontsevich and Y. Soibelman. "Affine structures and non-Archimedean analytic spaces". In: *The unity of mathematics*. Springer, 2006, pp. 321–385.

[LM09a] R. Lazarsfeld and M. Mustaţă. "Convex bodies associated to linear series". In: *Ann. Sci. Éc. Norm. Supér. (4)* 42.5 (2009), pp. 783–835.

[LM09b] R. Lazarsfeld and M. Mustaţă. "Convex bodies associated to linear series". In: *Annales Scientifiques de l'École Normale Supérieure* 42 (2009), pp. 783–835.

[Lor15] O. Lorscheid. "Scheme theoretic tropicalization". In: *arXiv preprint arXiv:1508.07949* (2015).

[Mac20] A. Macpherson. "Skeleta in non-Archimedean and tropical geometry". In: *Annales de la Faculté des sciences de Toulouse: Mathématiques* 29.2 (2020), pp. 431–506.

[Mik05] G. Mikhalkin. "Enumerative tropical algebraic geometry in $\mathbb{R}^2$". In: *J. Amer. Math. Soc.* 18.2 (2005), pp. 313–377.

[Mik06] G. Mikhalkin. "Tropical geometry and its applications". In: *International Congress of Mathematicians. Vol. II.* Eur. Math. Soc., Zürich, 2006, pp. 827–852.

[Mik07] G. Mikhalkin. "Tropical geometry and its applications". In: *Proceedings of the International Congress of Mathematicians Madrid, August 22–30, 2006*. 2007, pp. 827–852.

[MR09] G. Mikhalkin and J. Rau. *Tropical geometry.* Vol. 8. MPI for Mathematics, 2009.

[MR20] D. Maclagan and F. Rincón. "Tropical schemes, tropical cycles, and valuated matroids". In: *Journal of the European Mathematical Society* 22.3 (2020), pp. 777–796.

[MS15] D. Maclagan and B. Sturmfels. *Introduction to tropical geometry.* Vol. 161. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2015, pp. xii+363.

[Oko96] A. Okounkov. "Brunn–Minkowski inequality for multiplicities". In: *Inventiones mathematicae* 125.3 (1996), pp. 405–411.

[Pay09] S. Payne. "Analytification is the limit of all tropicalizations". In: *Mathematical Research Letters* 16.3 (2009), pp. 543–556.

[RW19] K. Rietsch and L. Williams. "Newton–Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians". In: *Duke Mathematical Journal* 168.18 (2019), pp. 3437–3527.

[SS04] D. Speyer and B. Sturmfels. "The tropical Grassmannian". In: *Adv. Geom.* 4.3 (2004).

[Ste06] D. A. Stepanov. "A remark on the dual complex of a resolution of singularities". In: *Uspekhi Mat. Nauk* 61.1(367) (2006), pp. 185–186.

[Tat71] J. Tate. "Rigid analytic spaces". In: *Invent. Math.* 12 (1971), pp. 257–289.

[Tem10] M. Temkin. "Stable modification of relative curves". In: *Journal of Algebraic Geometry* 19.4 (2010), pp. 603–677.

[Tem11] M. Temkin. "Relative Riemann-Zariski spaces". In: *Israel Journal of Mathematics* 185.1 (2011), p. 1.

[Thu05] A. Thuillier. "Théorie du potentiel sur les courbes en géométrie analytique non archimédienne. applications à la théorie d'Arakelov". PhD thesis. 2005.

[Thu07] A. Thuillier. “Géométrie toroidale et géométrie analytique non archimédienne. Application au type d’homotopie de certains schémas formels”. In: *Manuscripta Math.* 123.4 (2007), pp. 381–451.

[Uli17] M. Ulirsch. “Functorial tropicalization of logarithmic schemes: the case of constant coefficients”. In: *Proceedings of the London Mathematical Society* 114.6 (2017), pp. 1081–1113.

[Vir10] O. Viro. “Hyperfields for tropical geometry I. Hyperfields and dequantization”. In: *arXiv preprint arXiv:1006.3034* (2010).

[Zar39] O. Zariski. “The reduction of the singularities of an algebraic surface”. In: *Annals of Mathematics* (1939), pp. 639–689.

[Zar44] O. Zariski. “Reduction of the singularities of algebraic three dimensional varieties”. In: *Annals of Mathematics* (1944), pp. 472–542.