Higher Rank Tropical, Polyhedral and Analytic Geometry.

Hernan Iriarte

Supervisors: Omid Amini and Marco Maculan

École polytechnique
Sorbonne Université

February 10, 2022

This program has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 754362.

Overview

Introduction

The Tropical Approach

The tropical approach to algebraic geometry initiated in the 90s consist in the use of valuations to transform polynomial problems depending in the operations $+$ and $\times$ into piecewise linear problems depending on the operations $+$ and min.

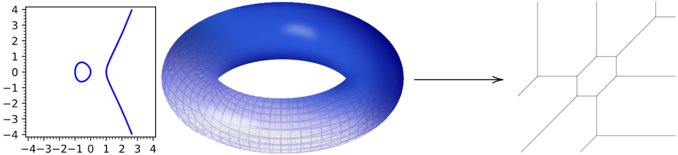


Figure: An elliptic curve and its tropical counterpart.

The Tropical Approach

The tropical approach to algebraic geometry initiated in the 90s consist in the use of valuations to transform polynomial problems depending in the operations $+$ and $\times$ into piecewise linear problems depending on the operations $+$ and min.

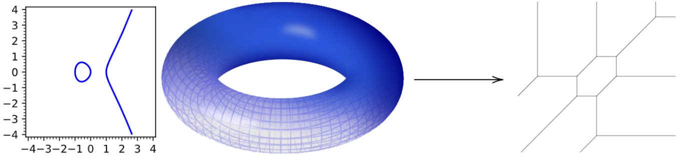


Figure: An elliptic curve and its tropical counterpart.

This thesis is inspired by two applications of the tropical approach:

1. The study of degenerations of algebraic varieties.
2. The asymptotic study of polarizations of algebraic varieties.

Valuations

Let K be a field and Γ an ordered abelian group. A valuation is a map

$$\nu : K^* \to \Gamma$$

satisfying

1. $\nu(fg) = \nu(f) + \nu(g)$
2. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$

Valuations

Let K be a field and Γ an ordered abelian group. A valuation is a map

$$\nu : K^* \to \Gamma$$

satisfying

1. $\nu(fg) = \nu(f) + \nu(g)$
2. $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$

Although the inequality in (2) can be strict, it is actually a equality in most instances.

Degenerations of Algebraic Varieties

Consider a family of algebraic curves parametrized by a variable $t \in \mathbb{C}^*$.

$$E_t = \{t^3x^3 + x^2y + xy^2 + t^3y + x^2 + t^{-1}xy + y^2 + x + y + t^3 = 0\}$$

What is the behaviour of the family when t approaches 0?

Degenerations of Algebraic Varieties

Consider a family of algebraic curves parametrized by a variable $t \in \mathbb{C}^*$.

$$E_t = \{t^3x^3 + x^2y + xy^2 + t^3y + x^2 + t^{-1}xy + y^2 + x + y + t^3 = 0\}$$

What is the behaviour of the family when t approaches 0?

We can think of the family E_t as a single curve defined over the field $\mathbb{C}(t)$. This is a valued field under the valuation

$$\nu : \sum_{n \in \mathbb{N}} a_n t^n \to \min\{n \in \mathbb{N} \mid a_n \neq 0\}.$$

Which we can use to tropicalize the polynomials above

$$\min\{3 + 3x, 2x + y, x + 2y, 3 + y, 2x, -1 + x + y, 2y, x, y, 3\}$$

The zeros of this polynomial encode the behaviour of the family around 0.

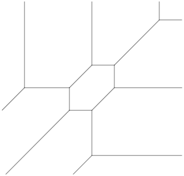
Degenerations of Algebraic Varieties

Consider a family of algebraic curves parametrized by a variable $t \in \mathbb{C}^*$.

$$E_t = \{t^3x^3 + x^2y + xy^2 + t^3y + x^2 + t^{-1}xy + y^2 + x + y + t^3 = 0\}$$

What is the behaviour of the family when t approaches 0?

We can think of the family E_t as a single curve defined over the field $\mathbb{C}(t)$. This is a valued field under the valuation



$$\nu : \sum_{n \in \mathbb{N}} a_n t^n \to \min\{n \in \mathbb{N} \mid a_n \neq 0\}.$$

Which we can use to tropicalize the polynomials above

$$\min\{3 + 3x, 2x + y, x + 2y, 3 + y, 2x, -1 + x + y, 2y, x, y, 3\}$$

The zeros of this polynomial encode the behaviour of the family around 0. What happens if we take families over more parameters?

Okounkov Bodies and Asymptotic of Polarizations

Let X be a projective variety over $\mathbb{C}$ and $E \subseteq X$ a divisor. Moreover, suppose that we have a valuation defined over its function field $\nu : K(X) \to \mathbb{R}^d$.

With the aim of study the asymptotic behaviour of the spaces

$$H^0(nE) = \{f \in K(X) \mid \mathrm{div}(f) + nE \geq 0\}$$

we consider the following set

$$\Delta_\nu(E) = \overline{\bigcup_{n\geq 1} \left\{ \frac{\nu(f)}{n} \middle| f \in H^0(nE) \setminus \{0\} \right\}}.$$

Okounkov Bodies and Asymptotic of Polarizations

Let X be a projective variety over $\mathbb{C}$ and $E \subseteq X$ a divisor. Moreover, suppose that we have a valuation defined over its function field $\nu : K(X) \to \mathbb{R}^d$.

With the aim of study the asymptotic behaviour of the spaces

$$H^0(nE) = \{f \in K(X) \mid \operatorname{div}(f) + nE \geq 0\}$$

we consider the following set

$$\Delta_\nu(E) = \overline{\bigcup_{n \geq 1} \left\{ \frac{\nu(f)}{n} \middle| f \in H^0(nE) \setminus \{0\} \right\}}.$$

- This is a closed subset of $\mathbb{R}^d$.
- As $\frac{1}{2}\left(\frac{\nu(f)}{n} + \frac{\nu(g)}{n}\right) = \frac{\nu(fg)}{2n}$ it is a convex body.

We call $\Delta_\nu(E)$ the Okounkov body of E with respect to ν.

Okounkov Bodies and Asymptotic of Polarizations

We know that, for any large n, the dimension $\dim_{\mathbb{C}} H^0(nE)$ is a polynomial function on the variable n called the Hilbert polynomial H_E of E.

Surprisingly, the geometry of the Okounkov body is linked to the Hilbert polynomial.

Theorem (Lazarfeld-Mustata 08', Kaveh-Khovanskii 09')

If the valuation ν comes from a flag of subvarieties, then:

1. $\deg H_E = \dim \Delta_\nu(E)$
2. If E is big. Leading coefficient of $H_E = \mathrm{Vol}(\Delta_\nu(E))$

This statement give us plenty of liberty to choose ν.

Okounkov Bodies and Asymptotic of Polarizations

We know that, for any large n, the dimension $\dim_{\mathbb{C}} H^0(nE)$ is a polynomial function on the variable n called the Hilbert polynomial H_E of E.

Surprisingly, the geometry of the Okounkov body is linked to the Hilbert polynomial.

Theorem (Lazarfeld-Mustata 08', Kaveh-Khovanskii 09')

If the valuation ν comes from a flag of subvarieties, then:

1. $\deg H_E = \dim \Delta_\nu(E)$
2. If E is big. Leading coefficient of $H_E = \mathrm{Vol}(\Delta_\nu(E))$

This statement give us plenty of liberty to choose ν.

→ What can we say about the function $\nu \mapsto \Delta_\nu(E)$?

→ Is it continuous?

Variation of Okounkov Bodies

Understanding $\Delta_\nu(E)$ as ν changes is an open project with many challenges.

1. Find the right setting for the variation: Spaces of Valuations

   Khovanskii bases, higher rank valuations and tropical geometry

   Kiumars Kaveh, Christopher Manon

2. Understand the continuity: Mutations

   Newton-Okounkov bodies sprouting on the valuative tree

   C. Ciliberto, M. Farnik, A. Küronya, V. Lozovanu, J. Roé, C. Shramov

   Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian

   Laura Escobar, Megumi Harada

3. Find the hidden information: Canonical Measures

   Equidistribution of Weierstrass points on curves over non-Archimedean fields

   Omid Amini

Variation of Okounkov Bodies

Understanding $\Delta_\nu(E)$ as ν changes is an open project with many challenges.

1. Find the right setting for the variation: Spaces of Valuations

   Khovanskii bases, higher rank valuations and tropical geometry
   Kiumars Kaveh, Christopher Manon

2. Understand the continuity: Mutations

   Newton-Okounkov bodies sprouting on the valuative tree
   C. Ciliberto, M. Farnik, A. Küronya, V. Lozovanu, J. Roé, C. Shramov

   Wall-crossing for Newton-Okounkov bodies and the tropical Grassmannian
   Laura Escobar, Megumi Harada

3. Find the hidden information: Canonical Measures

   Equidistribution of Weierstrass points on curves over non-Archimedean fields
   Omid Amini

In the following we will describe some developments in the direction of the first question.

Spaces of Valuations

What can be the domain of the map $\nu \mapsto \Delta_\nu(E)$?

- Riemann-Zariski space
- Huber Analytification (Adic Spaces)
- Berkovich Analytification
- The valuative tree

Analytic Geometry of Higher Rank

Monomial Valuations

Consider $X = \mathbb{A}^2 = \mathrm{Spec}\ \mathbb{C}[x, y]$ and fix $\alpha, \beta \in \mathbb{R}_{\geq 0}$. Then, the map

$$\begin{aligned} \nu_{\alpha,\beta} \colon \mathbb{C}[x, y] &\longrightarrow \mathbb{R} \\ \sum_{i,j\geq 0} a_{ij} x^i y^j &\longmapsto \min\{i\alpha + j\beta \mid a_{ij} \neq 0\} \end{aligned}$$

extends to a valuation in $K(X)$.

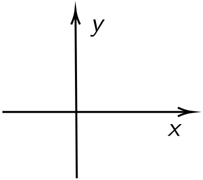
Monomial Valuations

Consider $X = \mathbb{A}^2 = \mathrm{Spec}\ \mathbb{C}[x, y]$ and fix $\alpha, \beta \in \mathbb{R}_{\geq 0}$. Then, the map

$$\begin{aligned} \nu_{\alpha,\beta} \colon \mathbb{C}[x, y] &\longrightarrow \mathbb{R} \\ \sum_{i,j \geq 0} a_{ij} x^i y^j &\longmapsto \min\{i\alpha + j\beta \mid a_{ij} \neq 0\} \end{aligned}$$

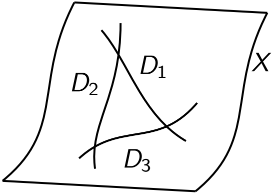
extends to a valuation in $K(X)$.
Example: If $\alpha = 1$, $\beta = 2$ and $f(x, y) = x^2 + 4xy + 2y$ then

$$\nu(f) = \min\{2\alpha, \alpha + \beta, \beta\} = 2$$



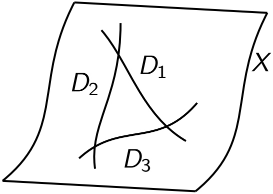
Quasi-monomial valuations

Let X be a variety and $D = \sum_{i=1}^{r} D_i$ a SNC divisor.



Quasi-monomial valuations

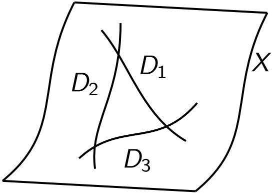
Let X be a variety and $D = \sum_{i=1}^{r} D_i$ a SNC divisor.



Quasi-monomial valuations

Let X be a variety and $D = \sum_{i=1}^{r} D_i$ a SNC divisor.

We can take local equations for each D_i to define monomial-like valuations.



Quasi-monomial valuations

Let X be a variety and $D = \sum_{i=1}^{r} D_i$ a SNC divisor.

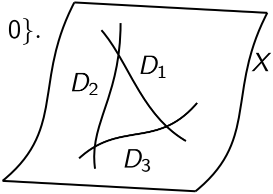
We can take local equations for each D_i to define monomial-like valuations.

Example: If X is a surface, $D_1 = V(x)$ and $D_2 = V(y)$ locally around their intersection p. Then, by the transversality

$$\widehat{\mathcal{O}}_{X,p} \cong \mathbb{C}[[x, y]].$$

So, given $\alpha, \beta \in \mathbb{R}_{\geq 0}$ we can consider $\nu_{\alpha,\beta}$ defined by

$$\nu_{\alpha,\beta}\left(\sum_{ij} a_{ij} x^i y^j\right) := \min\{i\alpha + j\beta \mid a_{ij} \neq 0\}.$$



Quasi-monomial valuations

Let X be a variety and $D = \sum_{i=1}^{r} D_i$ a SNC divisor.

We can take local equations for each D_i to define monomial-like valuations.

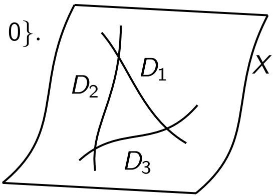
Example: If X is a surface, $D_1 = V(x)$ and $D_2 = V(y)$ locally around their intersection p. Then, by the transversality

$$\widehat{\mathcal{O}}_{X,p} \cong \mathbb{C}[[x, y]].$$

So, given $\alpha, \beta \in \mathbb{R}_{\geq 0}$ we can consider $\nu_{\alpha,\beta}$ defined by

$$\nu_{\alpha,\beta}\left(\sum_{ij} a_{ij} x^i y^j\right) := \min\{i\alpha + j\beta \mid a_{ij} \neq 0\}.$$

$\mathscr{M}(D) :=$ set of all quasi-monomial valuations w.r.t D.



Dual Cone Complex

Given an SNC divisor $D = \sum_{i=1}^{r} D_i$, we can construct a cone complex by

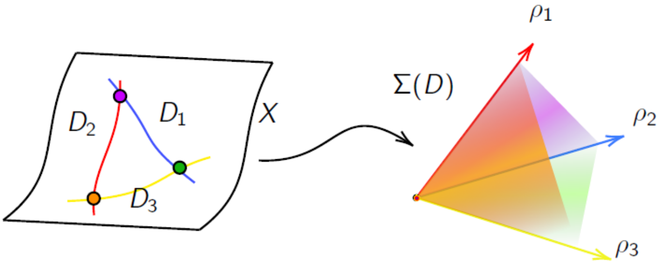
Dual Cone Complex

Given an SNC divisor $D = \sum_{i=1}^{r} D_i$, we can construct a cone complex by

1. Taking one ray for each D_i
2. Taking one face for each intersection of $D_i's$.

$$\Sigma(D) = \text{Dual cone complex of } D$$

Example:



Monomial Valuations Geometrically

Putting coordinates on each cone of $\Sigma(D)$ gives the equality

$$\mathscr{M}(D) \cong \Sigma(D)$$

So geometrically, the monomial valuations defined in terms of D with weights in $\mathbb{R}_{\geq 0}$ are parametrized by the dual complex!

Now, by changing the weights to $(\mathbb{R}^k)_{\geq_{\mathrm{lex}} 0}$ with its lexicographic order, we can construct monomial valuations of higher rank. Let us denote by $\mathscr{M}^k(D)$ the set of all such valuations.

→ Is there any way to relate $\mathscr{M}^k(D)$ geometrically to the dual cone complex $\Sigma(D)$?

Answer: Yes, they are given by tangent directions in $\Sigma(D)$.

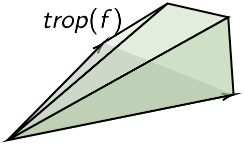
Tropicalization of rational functions

Given $f \in K(X)$ we define its tropicalization with respect to D as the map

$$\begin{aligned} \mathrm{trop}(f)\colon \Sigma(D) &\longrightarrow \mathbb{R} \\ p &\longmapsto \nu_p(f). \end{aligned}$$

In terms of coordinates, if $f = \sum_{ij} a_{ij} x^i y^j$ then

$$\mathrm{trop}(f)(x,y) = \min_{i,j}\{ix + jy \mid a_{ij} \neq 0\}.$$



Notice that $\mathrm{trop}(f)$ is continuous and piecewise linear.

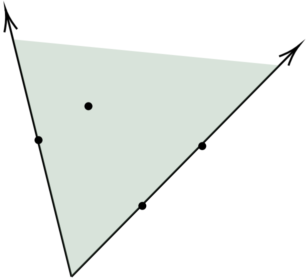
Theorem 1. Approximation Theorem (Amini - I '20/'21)

Any continuous piecewise linear function in $\Sigma(D)$ is the tropicalization of some rational function.

Tangent Cone Bundles

Given a polyhedral complex Σ, we define its tangent cone bundle of order k, denoted by $T\mathcal{C}^k \Sigma$ as the set of all elements $(x; w_1, \ldots, w_{k-1})$ such that:

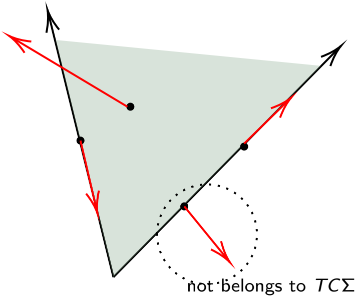
1. $x \in \Sigma$
2.
3.
4.



Tangent Cone Bundles

Given a polyhedral complex Σ, we define its tangent cone bundle of order k, denoted by $\mathcal{TC}^k\Sigma$ as the set of all elements $(x; w_1, \ldots, w_{k-1})$ such that:

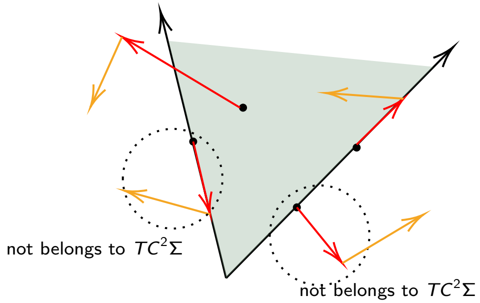
1. $x \in \Sigma$
2. $x + \varepsilon w_1 \in \Sigma$ for $\varepsilon > 0$ small.
3.
4.



Tangent Cone Bundles

Given a polyhedral complex Σ, we define its tangent cone bundle of order k, denoted by $\mathcal{TC}^k\,\Sigma$ as the set of all elements $(x; w_1, \ldots, w_{k-1})$ such that:

1. $x \in \Sigma$
2. $x + \varepsilon w_1 \in \Sigma$ for $\varepsilon > 0$ small.
3. $x + \varepsilon_1 w_1 + \varepsilon_2 w_2 \in \Sigma$ for $\varepsilon_1 > 0$ small and $\varepsilon_2 > 0$ small w.r.t ε_2.
4. ....



Partial Derivative Operators

With respect to each element $(x; \underline{w}) \in T\mathcal{C}^k\,\Sigma(D)$, we consider a partial derivative operator acting on tropical functions over $\Sigma(D)$. It is defined inductively as follows

$$
\begin{aligned}
D_{w_1}F(x) &= \lim_{h\to 0}\frac{F(x+hw_1)-F(x)}{h}\\
D_{w_1,w_2}F(x) &= \lim_{h\to 0}\frac{D_{w_1+hw_2}F(x)-D_{w_2}F(x)}{h}\\
&\vdots\\
D_{w_1,\dots,w_k}F(x) &= \frac{D_{w_1,\dots,w_{k-1}+hw_k}F(x)-D_{w_2}F(x)}{h}
\end{aligned}
$$

Duality Theorem

Theorem 2, Duality Theorem (Amini - I '20-'21)

There is an isomorphism of bundles over $\mathscr{M}(D) \simeq \Sigma(D)$

$$\begin{array}{ccc} \mathscr{M}^k(D) & \xrightarrow{\simeq} & T\mathcal{C}^{k-1}\Sigma(D) \\ \downarrow & & \downarrow \\ \mathscr{M}(D) & \xrightarrow{\simeq} & \Sigma(D) \end{array}$$

where $(x; \underline{w}) \in T\mathcal{C}^{k-1}\Sigma(D)$ corresponds to the valuation

$$\nu_{(x;\underline{w})}(f) = (\mathrm{trop}(f)(x), D_{w_1}\mathrm{trop}(f), \dots, D_{\underline{w}}\mathrm{trop}(f)(x))$$

which is a monomial valuation with respect to the weights

$$(x; \underline{w})^T = (\alpha_1, \dots, \alpha_k) \in (\mathbb{R}^k)_{\geq_{\mathrm{lex}} 0}.$$

Tropical Topology and Variations of Okounkov Bodies

We introduce the Tropical topology as the smallest topology of $\mathcal{TC}^k\,\Sigma(D)$ making all the evaluations maps $(x;\underline{w}) \mapsto \nu_{x;\underline{w}}(f)$ continuous for all $f \in K(X)$, where in $\mathbb{R}^k$ we put its Euclidean topology.

This is second countable topology finer than the Euclidean topology on $\mathcal{TC}^k\,\Sigma(D)$, which is not locally-compact in general. Although it is different from the Euclidean topology, they share the same dense sets.

Conjecture

For a big divisor E of X, the variation of Okounkov bodies $\nu \mapsto \Delta_\nu$ is continuous if we choose as domain $\mathcal{TC}^{k-1}\,\Sigma(D)$.

Space of all Valuations

Given a variety X it's birrational analytification of bounded rank k is the set

$$X^{\mathrm{bir}} := \{\nu \colon K(X)^* \to \mathbb{R}^k \mid \nu \text{ is a valuation}\}$$

together with the initial topology given by evaluations where in $\mathbb{R}^k$ we put the Euclidean topology. We consider two subsets $X^{\beth}$ and $X^{\daleth}$ of all valuations with center inside and respectively outside X.

The inclusion $T\mathcal{C}^{k-1}\Sigma(D) \to X^{\mathrm{bir}}$ is continuous and it has a retraction

$$r_D : X^{\beth} \longrightarrow T\mathcal{C}^{k-1}\Sigma(D).$$

→ The continuity turns out to be not obvious because the lexicographic order is not compatible with the Euclidean topology.

Log-smooth Pairs and Compactifications

- A log-smooth pair (Y, D) over X is a variety Y together with a proper birrational morphism $\varphi : Y \dashrightarrow X$ such that

$$\varphi|_{Y\setminus D} Y \setminus D \xrightarrow{\sim} X \setminus f(D)$$

  is an isomorphism.
- A log-smooth compactification of X is a variety Y containing X as an open subvariety such that $Y \setminus X$ is an SNC divisor on Y.

There are natural notions of morphism between log-smooth pairs and between log-smooth compactifications giving us the categories $\mathbf{LSP}_X$ and $\mathbf{LSC}_X$ respectively.

Limit Formulae

Theorem 3. Limit Formula (Amini - I, '20-'21)

For a log-smooth pair (Y, D) and a log-smooth compactifications Y there are retractions

$$r_{(Y,D)} : \mathcal{TC}^{k-1}(Y, D) \longrightarrow X^{\beth}$$
$$r_Y : \mathcal{TC}^{k-1}(Y, Y \setminus X) \longrightarrow X^{\daleth}$$

Wich are compatible and produce isomorphisms

$$\varprojlim_{Y \in \mathbf{LSP}_X} \mathcal{TC}^{k-1}(Y, D) \xrightarrow{\sim} X^{\beth}$$
$$\varprojlim_{Y \in \mathbf{LSC}_X} \mathcal{TC}^{k-1}(Y, Y \setminus X) \xrightarrow{\sim} X^{\daleth}$$

Polyhedral Geometry of Higher Rank

Higher Rank Generalizations

The theory just seen can be regarded as a higher rank version of Thuiller's tropicalization of toroidal varieties.

But what about higher rank versions of other instances of tropical geometry?

Convergent Hahn Series and Tropical Geometry of Higher Rank

Michael Joswig, Ben Smith

Tropical Geometry over Higher Dimensional Local Fields

S. Banerjee

Hahn analytification and connectivity of higher rank tropical varieties

Tyler Foster, Dhruv Ranganathan

Krull-tropical hypersurfaces

Fuensanta Aroca (Instituto de Matemáticas UNAM. México)

Higher Rank Generalizations

The theory just seen can be regarded as a higher rank version of Thuiller's tropicalization of toroidal varieties.

But what about higher rank versions of other instances of tropical geometry?

Convergent Hahn Series and Tropical Geometry of Higher Rank

Michael Joswig, Ben Smith

Tropical Geometry over Higher Dimensional Local Fields

S. Banerjee

Hahn analytification and connectivity of higher rank tropical varieties

Tyler Foster, Dhruv Ranganathan

Krull-tropical hypersurfaces

Fuensanta Aroca (Instituto de Matemáticas UNAM. México)

In the following, We present a theory of polyhedral geometry suitable for this generalizations and some applications.

The Ring of Generalized Dual Numbers

We work with $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which is an ordered ring under the lexicographic order, elements of $\mathbb{D}$ have the form

$$x = x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1} x^{(k-1)}.$$

The Ring of Generalized Dual Numbers

We work with $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which is an ordered ring under the lexicographic order, elements of $\mathbb{D}$ have the form

$$x = x^{(0)} + \varepsilon x^{(1)} + \cdots + \varepsilon^{k-1} x^{(k-1)}.$$

A polyhedra in $\mathbb{D}^n$ is a set of the form

$$P = \{x \in \mathbb{D}^n \mid f_1(x) \geq 0, \ldots, f_r(x) \geq 0\}$$

where

$$\begin{aligned} f_i : \mathbb{D}^n &\longrightarrow \mathbb{D} \\ x &\longmapsto \langle y_i \,, x \rangle + a_i \end{aligned}$$

are affine function with $y_i \in \mathbb{D}^n$ and $a_i \in \mathbb{D}$. If each $a_i = 0$ we call P a polyhedral cone.

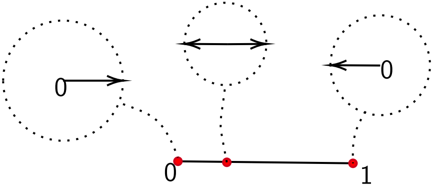
Examples

Let us fix $k = 2$.

Examples

Let us fix $k = 2$.

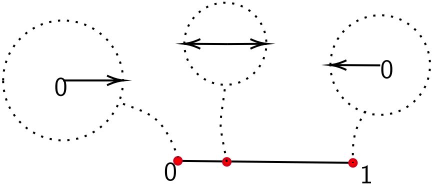
1. $P = \{x \in \mathbb{D} \mid 0 \leq x \leq 1\}$: In this case x has the form $x^{(0)} + \varepsilon x^{(1)}$. If we study the restriction on $x^{(0)}$ first and then on $x^{(1)}$ we obtain the fibration



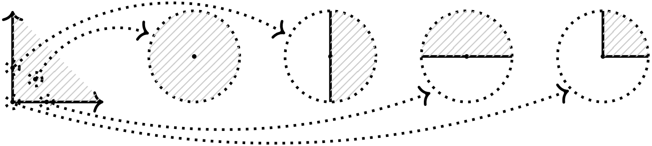
Examples

Let us fix $k = 2$.

1. $P = \{x \in \mathbb{D} \mid 0 \leq x \leq 1\}$: In this case x has the form $x^{(0)} + \varepsilon x^{(1)}$. If we study the restriction on $x^{(0)}$ first and then on $x^{(1)}$ we obtain the fibration



2. $P = \{(x, y) \in \mathbb{D}^2 \mid x \geq 0, y \geq 0\}$: Similarly as above



Real polyhedron and base change to $\mathbb{D}$

Theorem 4. Base change to $\mathbb{D}$ (I, '21)

Consider the real polyhedron

$$P = \{x \in \mathbb{R}^n \mid f_1(x) \geq 0, \dots, f_r(x) \geq 0\}$$

with $f_i : \mathbb{R}^n \to \mathbb{R}$ affine. If we interpret this inequalities over $\mathbb{D}$ we obtain

$$T\mathcal{C}^{k-1} P = \{x \in \mathbb{D}^n \mid f_1(x) \geq 0, \dots, f_r(x) \geq 0\}.$$

This analog to the equality $X\left(\frac{K[\varepsilon]}{(\varepsilon^2)}\right) = TX(K)$ for an algebraic variety X over K in algebraic geometry.

Faces

Let $P \subseteq \mathbb{D}^n$ be a polyhedron, a face of P is a subset of P which can be obtained as the minimizing set of a linear function over P.

Faces

Let $P \subseteq \mathbb{D}^n$ be a polyhedron, a face of P is a subset of P which can be obtained as the minimizing set of a linear function over P.

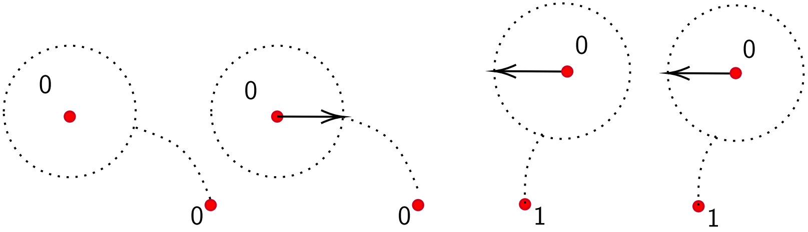
Remark: In general, it is not true that if a linear function is bounded below over P it achieves its minimum on P. As $f(x) = 1 - \varepsilon x$ which is bounded in $\{x \geq 0\}$ and does not achieves its minimum.

Faces

Let $P \subseteq \mathbb{D}^n$ be a polyhedron, a face of P is a subset of P which can be obtained as the minimizing set of a linear function over P.

Remark: In general, it is not true that if a linear function is bounded below over P it achieves its minimum on P. As $f(x) = 1 - \varepsilon x$ which is bounded in $\{x \geq 0\}$ and does not achieves its minimum.

Example: The faces of $P = \{x \in \mathbb{D} \mid x \geq 0, x \leq 1\}$ are P, the empty set and



Higher Rank Farkas' Lemma

Theorem 5. Higher Rank Farkas' Lemma (I. '21)

For a polyhedron

$$P = \{x \in \mathbb{D}^n \mid f_1(x) \geq 0, \ldots, f_r(x) \geq 0\}.$$

Any affine function achieving its minimum over P can be written in the form

$$f = \lambda_1 f_1 + \cdots + \lambda_r f_r + \min_P f$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$.

Higher Rank Farkas' Lemma

Theorem 5. Higher Rank Farkas' Lemma (I. '21)

For a polyhedron

$$P = \{x \in \mathbb{D}^n \mid f_1(x) \geq 0, \ldots, f_r(x) \geq 0\}.$$

Any affine function achieving its minimum over P can be written in the form

$$f = \lambda_1 f_1 + \cdots + \lambda_r f_r + \min_P f$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$.

As a corollary, any face of P is of the form

$$F = P \cap \{x \in \mathbb{D}^n \mid \varepsilon^{\alpha_i} f_i(x) = 0, \forall i\}$$

for some $\alpha_i \in \{0, 1, \ldots, k\}$.

Higher Rank Farkas' Lemma

Theorem 5. Higher Rank Farkas' Lemma (I. '21)

For a polyhedron

$$P = \{x \in \mathbb{D}^n \mid f_1(x) \geq 0, \ldots, f_r(x) \geq 0\}.$$

Any affine function achieving its minimum over P can be written in the form

$$f = \lambda_1 f_1 + \cdots + \lambda_r f_r + \min_P f$$

for some $\lambda_1, \ldots, \lambda_r \in \mathbb{D}_{\geq 0}$.

As a corollary, any face of P is of the form

$$F = P \cap \{x \in \mathbb{D}^n \mid \varepsilon^{\alpha_i} f_i(x) = 0, \forall i\}$$

for some $\alpha_i \in \{0, 1, \ldots, k\}$.

The proof of Farkas' Lemma relays on a Higher Rank version of the Fourier-Motzkin elimination algorithm to reduce the number of variables on a system of linear inequalities.

The Normal Fan

Consider the set

$$|\,\mathrm{NF}(P)| = \{y \in \mathbb{D}^n \mid \min_{x \in P} \langle y\,, x\rangle \mathit{exists}\}.$$

Then, each element $y \in |\,\mathrm{NF}(P)|$ defines a face

$$\mathrm{face}_y\, P = \mathrm{arg.\,min}_{x \in P} \langle y\,, x\rangle.$$

The Normal Fan

Consider the set

$$|\mathrm{NF}(P)| = \{y \in \mathbb{D}^n \mid \min_{x \in P} \langle y\,, x\rangle exists\}.$$

Then, each element $y \in |\mathrm{NF}(P)|$ defines a face

$$\mathrm{face}_y\, P = \mathrm{arg.\,min}_{x \in P} \langle y\,, x\rangle.$$

Theorem 6. Normal Fan Duality

The map

$$\begin{aligned} |\mathrm{NF}(P)| &\longrightarrow \text{Faces of } P \\ y &\longmapsto \mathrm{face}_y\, P \end{aligned}$$

is locally constant along a fan $\mathrm{NF}(P)$ supported on $|\mathrm{NF}(P)|$ called the Normal Fan of P.

Higher Rank Minkowski's Theorem

Over $|\mathrm{NF}(P)|$ we have the support function

$$\begin{aligned} h_P : |\mathrm{NF}(P)| &\longrightarrow \mathbb{D} \\ y &\longmapsto \min_{x \in P} \langle y\,, x \rangle \end{aligned}$$

Higher Rank Minkowski's Theorem

Over $|\mathrm{NF}(P)|$ we have the support function

$$h_P : |\mathrm{NF}(P)| \longrightarrow \mathbb{D}$$
$$y \longmapsto \min_{x \in P} \langle y \, , x \rangle$$

Theorem 7. Higher Rank Minkowski's Theorem (I. 21)

1. The map $P \mapsto (|\mathrm{NF}(P)|, h_p)$ gives a bijection between

$$\{\text{Polyhedra with convex normal fan}\} \leftrightarrow \left\{ \begin{matrix} \text{Polyhedral cones endowed with} \\ \text{piecewise linear concave functions.} \end{matrix} \right\}$$

   Remark: In general, $|\mathrm{NF}(P)|$ is not convex.

2. The bijection restricts to a bijection

$$\{\text{Polytopes in } \mathbb{D}^n\} \leftrightarrow \{\text{Piecewise linear concave functions } \mathbb{D}^n \to \mathbb{D}^n\}.$$

Some characterizations of polyhedra

Theorem 8. Characterization of Polytopes and Minkowski-Weil decompositions

1. For a polyhedron P, the set $|\mathrm{NF}(P)|$ is convex iff P admits a Minkoski-Weyl decomposition

$$P = Q + C \quad \text{with } Q \text{ a polytope and } C \text{ a polyhedral cone.}$$

2. A polyhedron is a polytope if and only if any linear function achieves its minimum on it.

Tropical Geometry of Higher Rank

Tropical Geometry

We consider the Tropical Semifield of rank k

$$\mathbb{T}_k = (\mathbb{D} \cup \{\infty\}, +, \min).$$

A tropical Laurent polynomial is an expression of the form

$$f = \text{“} \sum_{m \in \mathbb{Z}^n} a_m T^m \text{”} \in \mathbb{T}_k[T_1^{\pm}, \ldots, T_n^{\pm}]$$

with finite support, where “ ” means we are using tropical operations, that is, $+$ instead of multiplication and min instead of addition.
A polynomial $f = \text{“} \sum_{m \in \mathbb{Z}^n} a_m T^m \text{”} \in \mathbb{T}[T_1^{\pm}, \ldots, T_n^{\pm}]$ induce a function

$$\begin{aligned} f : \mathbb{D}^n &\longrightarrow \mathbb{D} \\ x &\longmapsto \min_{m \in \mathbb{Z}^n} \{\langle m \, , x \rangle + a_m\}. \end{aligned}$$

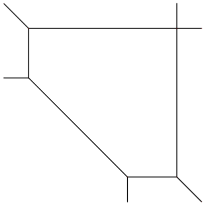
Tropical Geometry

A zero of f is an element $x \in \mathbb{D}^n$ for which the minimum is achieved at least twice.
The tropical hypersurface defined by f is the set $V(f)$ of all zeros of f.

Tropical Geometry

A zero of f is an element $x \in \mathbb{D}^n$ for which the minimum is achieved at least twice.
The tropical hypersurface defined by f is the set $V(f)$ of all zeros of f.

Example: Let us suppose $k = 1$, if
$f =$ "$4x^2y^2 + 4x^2y + 4xy^2 + 0xy + 2x + 2y + 4$" then $V(f)$ equals

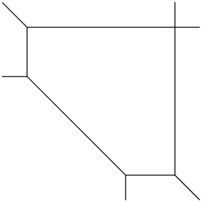


Tropical Geometry

A zero of f is an element $x \in \mathbb{D}^n$ for which the minimum is achieved at least twice.

The tropical hypersurface defined by f is the set $V(f)$ of all zeros of f.

Example: Let us suppose $k = 1$, if
$f =$ "$4x^2y^2 + 4x^2y + 4xy^2 + 0xy + 2x + 2y + 4$" then $V(f)$ equals

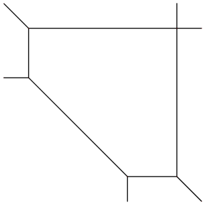


How to determine the drawing we get?

Tropical Geometry

A zero of f is an element $x \in \mathbb{D}^n$ for which the minimum is achieved at least twice.
The tropical hypersurface defined by f is the set $V(f)$ of all zeros of f.

Example: Let us suppose $k = 1$, if
$f =$ "$4x^2y^2 + 4x^2y + 4xy^2 + 0xy + 2x + 2y + 4$" then $V(f)$ equals



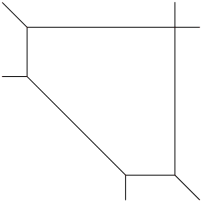
How to determine the drawing we get?

→ Brute force (or polymake).

Tropical Geometry

A zero of f is an element $x \in \mathbb{D}^n$ for which the minimum is achieved at least twice.
The tropical hypersurface defined by f is the set $V(f)$ of all zeros of f.

Example: Let us suppose $k = 1$, if
$f =$ "$4x^2y^2 + 4x^2y + 4xy^2 + 0xy + 2x + 2y + 4$" then $V(f)$ equals



How to determine the drawing we get?

- $\rightarrow$ Brute force (or polymake).
- $\rightarrow$ Use the hypersurface duality theorem.

Hypersurface Duality in Rank 1

Hypersurface Duality Theorem

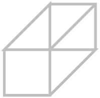
We can obtain the shape of $V(f)$ as follows:

1. Draw $\text{New}(f) = \text{conv}_{\mathbb{R}}(\text{Supp}(f))$ the Newton polytope of f.
2. Subdivide $\text{New}(f)$ with respect to the coefficients of f.
3. $V(f)$ is a polyhedral complex whose cells are dual to this subdivision followed by a rotation by 180° degrees.

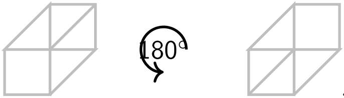
In particular, this result gives us a duality between tropical hypersurfaces and regular subdivisions of lattice polytopes.

Example

If $f = $ "$4x^2y^2 + 4x^2y + 4xy^2 + 0xy + 2x + 2y + 4$" then, the subdivision $\Delta(f)$ of $\mathrm{New}(f)$ looks like

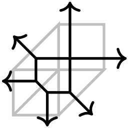
$\underline{\Delta}(f) =$ 

If we do a point reflection of it we get



.

Hence, the shape of the tropical hypersurface in this case is



.

Higher Rank Hypersurface Duality

Theorem (Higher Rank Hypersurface Duality I. '21)

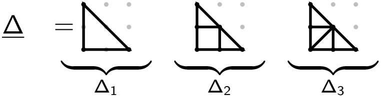
1. $V(f)$ is a locally constant iterated fibration of rank one tropical hypersurfaces.
2. The geometry of this fibration can be totally described in terms of a layered subdivision of $\mathrm{New}(f)$.
3. $V(f)$ can be endowed with a polyhedral structure over $\mathbb{D}$ compatible with the duality above.

Example of the Higher Rank Hypersurface Duality

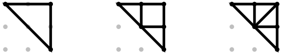
Consider $k = 3$, $M = \mathbb{Z}^2$ and the polynomial

$$f(x, y) = (0,1,2) + (0,1,1)x + (0,1,1)y + (0,1,2)xy + (0,0,0)x^2 + (0,0,0)y^2$$

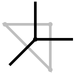
The Newton polytope of f is $\operatorname{New} f = \operatorname{conv}_{\mathbb{R}}((0,0),(2,0),(0,2))$ and its associated layered subdivision is the following:



After a point reflection it becomes

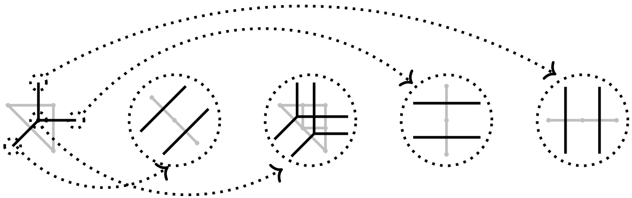


Therefore, the base of the fibration $V(f^{[1]})$ has the shape

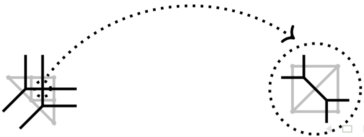


Example of Higher Rank Hypersurface Duality

And over each point of the base, there are 4 possible shapes for the fibers of $V(f^{[2]})$, represented in the following diagram:



Moreover, each of these fibers is the base for a fibration determined by $V(f^{[3]})$. As an example, the fiber corresponding to the subdivision of the square is:



Some Concluding Slogans

Some Concluding Slogans

1. The tropical analog of higher rank valuations on algebraic varieties are partial derivative operators.

Some Concluding Slogans

1. The tropical analog of higher rank valuations on algebraic varieties are partial derivative operators.
2. Higher rank analytifications can be approximated in terms of polyhedra of higher rank.

Some Concluding Slogans

1. The tropical analog of higher rank valuations on algebraic varieties are partial derivative operators.
2. Higher rank analytifications can be approximated in terms of polyhedra of higher rank.
3. There is a consistent theory of polyhedral geometry over the ring $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which help us work with valuations of higher rank.

Some Concluding Slogans

1. The tropical analog of higher rank valuations on algebraic varieties are partial derivative operators.
2. Higher rank analytifications can be approximated in terms of polyhedra of higher rank.
3. There is a consistent theory of polyhedral geometry over the ring $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which help us work with valuations of higher rank.
4. Higher rank objects are fibered and their combinatorics can be understood in terms of a layering.

Some Concluding Slogans

1. The tropical analog of higher rank valuations on algebraic varieties are partial derivative operators.
2. Higher rank analytifications can be approximated in terms of polyhedra of higher rank.
3. There is a consistent theory of polyhedral geometry over the ring $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^k)$ which help us work with valuations of higher rank.
4. Higher rank objects are fibered and their combinatorics can be understood in terms of a layering.
5. There duality between tropical hypersurfaces and regular subdivisions can be extended to a duality between higher rank tropical hypersurfaces and layered regular subdivisions.